1. Compute the following limits.

   (a) $\lim_{x\to 0} \frac{e^x - 1}{\sin 2x}$.

   $$\lim_{x\to 0} \frac{e^x - 1}{\sin 2x} = \lim_{x\to 0} \frac{e^x}{2\cos 2x} = \frac{e^0}{2\cos 0} = \frac{1}{2}.$$

   (We used L'Hôpital in the first step since limit was of form 0/0.)

   (b) $\lim_{x\to 0+} x^x$.

   $$\ln\left(\lim_{x\to 0+} x^x\right) = \lim_{x\to 0+} \ln(x^x) = \lim_{x\to 0+} x\ln x = \lim_{x\to 0+} \frac{\ln x}{1/x}.$$

   Now this is of the form ∞/∞ so apply L'Hôpital:

   $$= \lim_{x\to 0+} \frac{1/x}{-1/x^2} = -\lim_{x\to 0} x = 0.$$

   Thus, $\lim_{x\to 0+} x^x = \exp(0) = 1$.

   (c) $\lim_{x\to 1} \frac{1 - x + \ln x}{1 + \cos \pi x}$.
   This is of the form 0/0 so apply L'Hôpital:

   $$\lim_{x\to 1} \frac{1 - x + \ln x}{1 + \cos \pi x} = \lim_{x\to 1} \frac{-1 + 1/x}{-\pi \sin \pi x}.$$

   The above is again of the form 0/0 so L'Hôpital once more gives:

   $$= \lim_{x\to 1} \frac{-1/x^2}{-\pi^2 \cos \pi x} = \frac{-1/1^2}{-\pi^2 \cos \pi} = -\frac{1}{\pi^2}.$$

2. Consider the function $f(x) = \dfrac{x^3 - 4}{x^2}$.

   (a) Compute $f'(x)$ and analyze the regions where $f(x)$ is increasing or decreasing.
   First observe that the original function $f(x)$ is defined whenever $x \neq 0$. Now, $f'(x) = 1 + 8/x^3$ is positive if $x < -2$ or $x > 0$ and negative only when $-2 < x < 0$ so $f(x)$ is increasing in the two former intervals and decreasing in the latter.

   (b) Compute $f''(x)$ and analyze the concavity of $f(x)$.
   $f''(x) = -24/x^4$ which is negative whenever it is defined, so the graph of $f(x)$ is concave down.

   (c) Find all asymptotes (horizontal, vertical or oblique) to the graph of $y = f(x)$.
   $x = 0$ is the only point where $f(x)$ is undefined and $f(x)$ is otherwise continuous, so looking at the one-sided limits:

$$\lim_{x \to 0^\pm} \frac{x^3 - 4}{x^2} = -\infty$$

   (both one-sided limits are $-\infty$), hence $x = 0$ is a vertical asymptote approached by the graph of $y = f(x)$ "in the direction of $-\infty$" from both sides. Also, noting that $f(x)$ is a rational function with numerator of degree 1 more than the denominator, we separate the polynomial part:

$$f(x) = \frac{x^3 - 4}{x^2} = x - \frac{4}{x^2},$$

   and conclude that the diagonal line $y = x$ is an oblique asymptote to the graph. Consequently, there are no horizontal asymptotes.

   (d) Sketch the graph of $y = f(x)$.

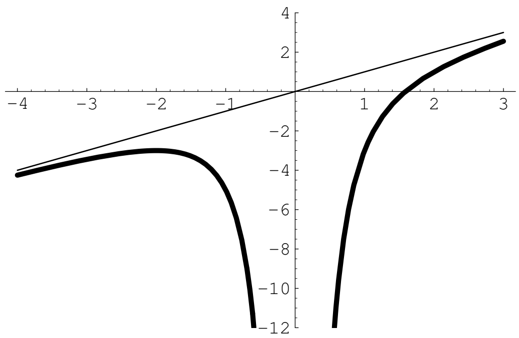

Figure 1: Graph of $y = x - 4/x^2$ showing the oblique asymptote $y = x$.

3. A wooden beam with rectangular cross section must be cut from a log with a circular cross section of diameter 2 feet. The strength of the beam is the product of its width w with the *square* of its height h. Find the optimal way to cut the beam from the log to maximize its resistance (aka, find the height and width of the most resistant beam.)

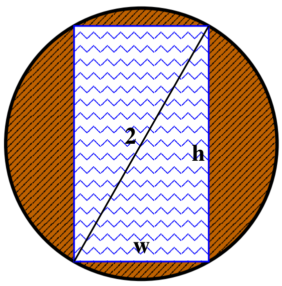

Figure 2: Cross section of the log and the beam to be cut from it.

The resistance R of the beam is $R = wh^2$ where, by Pythagoras' Theorem, $h^2 + w^2 = 2^2 = 4$, thus $h^2 = 4 - w^2$ and $R = w(4 - w^2)$ for $0 \le w \le 2$. Note that R is a continuous function defined on a closed interval so it must attain its maximum by the Extreme Value Theorem. At the endpoints $w = 0, 2$ we have $R = 0$. Checking for critical points we find $dR/dw = 4 - 3w^2$ is always well-defined, and vanishes only when $w = \sqrt{4/3}$. In this case we clearly have $R > 0$, so the value of $w = \sqrt{4/3}$ is necessarily the one attaining the maximal resistance. In this case, $h = \sqrt{4 - w^2} = \sqrt{4 - (4/3)} = \sqrt{8/3}$.

4. Find the area of the "boomerang" contained between the graphs of $f(x) = \frac{x}{3\pi}(3\pi - x)$ and $g(x) = -\sin x$ for $0 \le x \le 3\pi$.

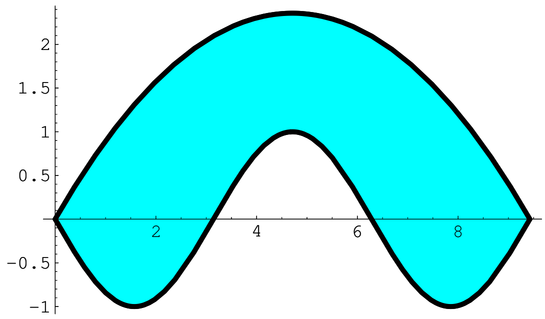

Figure 3: The "boomerang" $0 \le x \le 3\pi, g(x) \le y \le f(x)$.

$$\text{Area} = \int_0^{3\pi} \left[\frac{x}{3\pi}(3\pi - x) - (-\sin x)\right] dx = \int_0^{3\pi} \left(x - \frac{x^2}{3\pi} + \sin x\right) dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{9\pi} - \cos x\right]_0^{3\pi} = \frac{9}{2}\pi^2 - \frac{27\pi^3}{9\pi} + 2$$

$$= \frac{3}{2}\pi^2 + 2.$$