1. Find the following limits. Show your work!

(a) (10 points) $\lim_{u \to -3} \sqrt{u^3 + 3u^2 + 1}$

Observe that $\lim_{u\to -3} (u^3+3u^2+1) = (\lim_{u\to -3} u)^3 + 3(\lim_{u\to -3} u)^2 + 1$

$= (-3)^3 + 3(-3)^2 + 1 = 1 \geqslant 0$ so then

$\lim_{u\to -3} \sqrt{u^3+3u^2+1} = \sqrt{\lim_{u\to -3}(u^3+3u^2+1)} = \sqrt{1} = \underline{1.}$

(b) (10 points) $\lim_{x \to 1} \frac{1 - x^2}{x^2 + x - 2}$

First, $\lim_{x\to 1} (1-x^2) = 1-(1)^2 = 0$ & $\lim_{x\to 1} (x^2+x-2) = 1^2+1-2 = 0$

so limit is of type $\frac{0}{0}$ as $x-1$. However $\frac{1-x^2}{x^2+x-2} = -\frac{(x-1)(x+1)}{(x-1)(x+2)}$

$= -\frac{x+1}{x+2}$ provided $x \neq 1$ (which we may assume), hence

$\lim_{x\to 1} \frac{1-x^2}{x^2+x-2} = -\lim_{x\to 1} \frac{x+1}{x+2} = -\frac{1+1}{1+2} = \underline{-\frac{2}{3}}$

(c) (10 points) $\lim_{x \to +\infty} \frac{\exp(-x) + 3 - x^2}{4x^2 - 7}$

$\lim_{x\to\infty} \frac{\exp(-x)+3-x^2}{4x^2-7} = \lim_{x\to\infty} \frac{\frac{\exp(-x)}{x^2} + \frac{3}{x^2} - 1}{4 - \frac{7}{x^2}}$ (Dividing numerator + denominator by x^2)

Now, as $x \to \infty$, $\exp(-x) \to 0$ so $\exp(-x)/x^2 \to 0$ as well.

Also, $3/x^2$ & $7/x^2 \to 0$ as $x \to \infty$ so the limit is

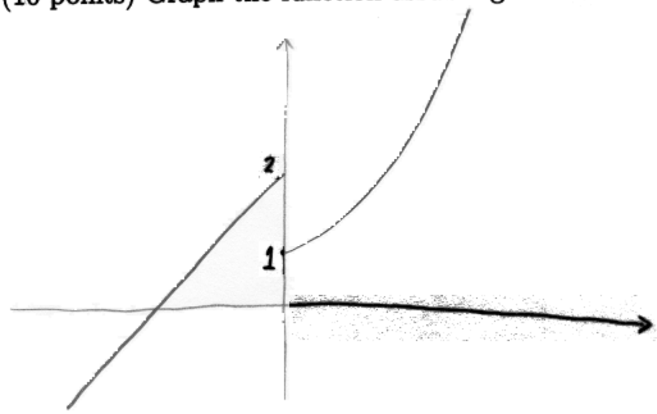
$\frac{0+0-1}{} = \underline{\frac{-1}{4}.}$

2. Consider the function

$$f(x) = \begin{cases} \exp(x), & \text{if } x \geq 0; \\ x + k, & \text{if } x < 0, \end{cases} \tag{1}$$

where k is a constant.

(a) (10 points) Graph the function assuming $k = 2$.



(b) (10 points) Find all possible values of k for which the function is continuous. *Justify your answer!*

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \exp(x) = \exp(0) = 1$, and

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x + k) = 0 + k = k$, and $f(0) = \exp(0) = 1$

Thus, f is continuous @ $x = 0$ when the left- and right-hand limits agree with each other and with $f(0)$.

This happens when $k = 1$ only.

3. Compute and simplify all answers as much as possible.

(a) (10 points) $\frac{d^2}{dx^2}\left(3x^4 - \sqrt{x^2+1} + \frac{3}{2x-1}\right)$

First derivative: $3(4x^3) - \frac{2x}{2\sqrt{x^2+1}} - \frac{3 \cdot 2}{(2x-1)^2} = 12x^3 - x(x^2+1)^{-1/2} - 6(2x-1)^{-2}$

Second derivative: $12(3x^2) - x\left(\frac{-1}{\cancel{2}}(x^2+1)^{-3/2}(\cancel{2}x)\right) - 1\cdot(x^2+1)^{-1/2} - 6(-2)(2x-1)^{-3}(2)$

$= 36x^2 + \frac{x^2}{\sqrt{(x^2+1)^3}} - \frac{1}{\sqrt{x^2+1}} + \frac{24}{(2x-1)^3} = \underline{36x^2 - \frac{1}{\sqrt{(x^2+1)^3}} + \frac{24}{(2x-1)^3}}$

(b) (10 points) $\frac{d}{dx}\ln(\sec x + \tan x)$.

$\frac{d}{dx}\ln(\sec x + \tan x) = \frac{1}{\sec x + \tan x}(\sec' x + \tan' x) = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$

$\frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} = \underline{\sec x.}$

(c) (10 points) $\frac{d}{dx}\ln\sqrt{\frac{x-1}{x+1}}$.

$\frac{d}{dx}\ln\sqrt{\frac{x-1}{x+1}} = \frac{d}{dx}\left[\frac{1}{2}\left(\ln(x-1) - \ln(x+1)\right)\right] = \frac{1}{2}\frac{d}{dx}\ln(x-1) - \frac{1}{2}\frac{d}{dx}\ln(x+1)$

$= \frac{1}{2}\cdot\frac{1}{x-1} - \frac{1}{2}\cdot\frac{1}{x+1} = \underline{\frac{1}{x^2-1}.}$

4. Consider the function

$$f(x) = \frac{x^2}{x-1}. \tag{2}$$

(a) (5 points) Show that the line $x = 1$ is an asymptote to the graph of $y = f(x)$ by computing $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$.

$$\lim_{x\to 1^+} \frac{x^2}{x-1} = \left(\lim_{x\to 1^+} x^2\right) \cdot \lim_{x\to 1^+} \frac{1}{x-1} = 1^2 \cdot (+\infty) = \underline{+\infty.}$$

$$\lim_{x\to 1^-} \frac{x^2}{} = \left(\lim_{x\to 1^+} x^2\right) \lim_{x\to 1^-} \frac{1}{x-1} = 1^2 \cdot (-\infty) = \underline{-\infty}$$

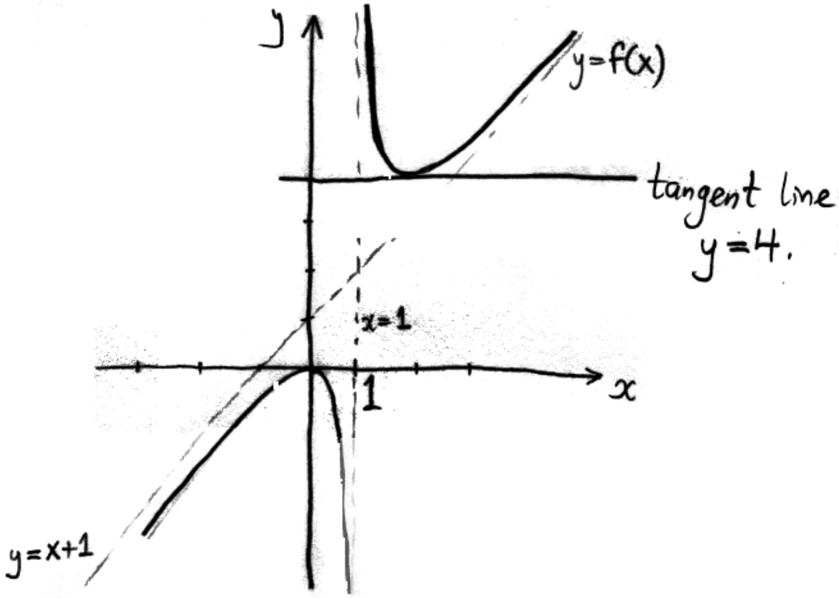
Thus, the graph of $y=f(x)$ approaches the vertical $x=1$ asymptotically when x approaches 1: from above when $x\to 1^+$ and from below when $x \to 1^-$.

(b) (5 points) Show that the line $y = x + 1$ is another asymptote by computing $\lim_{x\to+\infty} [f(x) - (x + 1)]$ and $\lim_{x\to-\infty} [f(x) - (x + 1)]$.

$$f(x) - (x+1) = \frac{x^2}{x-1} - (x+1) = \frac{1}{x-1}$$

Now, $\underline{\lim_{x\to\pm\infty} \frac{1}{x-1} = 0}$, so the graph of $y=f(x)$ approaches the line $y=x+1$ asymptotically as $x \to \pm\infty$

(c) (5 points) Sketch the graph of $y = f(x)$. (Hint: it's a hyperbola.)



(d) (5 points) Find the equation of the tangent line to the graph passing through the point $(2, 4)$ and draw this line in the graph above.

$$f'(x) = \frac{(2x)(x-1) - x^2(1)}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2}.$$

Thus, $f'(2) = \frac{2^2 - 2\cdot 2}{(2-1)^2} = 0$ so the tangent line has slope zero, hence its equation is

$y - 4 = 0$ or $\underline{y = 4.}$