Solutions, Midterm 1, 106
Fall 2001, Professor Sogge

1) (10 points, 2 parts)

a) Compute $\lim_{x\to+\infty}(\sqrt{x^2+3}-x)$

Solution:

$$\lim_{x\to+\infty}(\sqrt{x^2+3}-x) = \lim_{x\to+\infty}(\sqrt{x^2+3}-x)\frac{\sqrt{x^2+3}+x}{\sqrt{x^2+3}+x}$$
$$= \lim_{x\to+\infty}\frac{3}{\sqrt{x^2+3}+x} = 0.$$

b) Assign a value k so that the following function will be continuous

$$f(x) = \begin{cases} \frac{x+3}{x^2-x-12}, & x \neq -3 \\ k, & x = -3. \end{cases}$$

Solution: Since
$$\lim_{x\to-3}\frac{x+3}{x^2-x-12} = \lim_{x\to-3}\frac{1}{x-4} = -\frac{1}{7},$$
if $k = -1/7$ the above function will be continuous.

2) (20 points, 2 parts)

a) Write down the precise definition of f' (i.e., the one involving limits).

Solution: $f'(x) = \lim_{h\to0}\frac{f(x+h)-f(x)}{h}$.

b) Using the definition of the derivative from part a), compute the equation of the tangent line to $y = x^2 - 4x$ at the point $x = 2$.

Solution: The slope of the tangent line is
$$\lim_{h\to0}\frac{((2+h)^2-4(2+h))-(2^2-8)}{h} = \lim_{h\to0}\frac{4h+h^2-4h}{h} = 0$$
Since $y = 2^2 - 4\times2 = -4$ when $x = 2$, the tangent line must pass through the point $(2,-4)$. Hence the equation of the tangent line is $y = -4$.

3) (20 points, 2 parts) Compute the derivatives of the following functions

a) x^x

Solution: We first rewrite $x^x = e^{x \ln x}$. If we then use the chain rule we get that

$$\frac{d}{dx}x^x = e^{x \ln x}\frac{d}{dx}(x \ln x) = x^x(\ln x + 1).$$

b) $\cos(\sin x^2)$

Solution:

$$\begin{aligned}\frac{d}{dx}\cos(\sin x^2) &= -\sin(\sin x^2)\frac{d}{dx}(\sin x^2) \\ &= -\sin(\sin x^2)\cos(x^2)\frac{d}{dx}x^2 \\ &= -2x\sin(\sin x^2)\cos(x^2).\end{aligned}$$

4) (15 points) Consider the curve defined near $P = (0, 1)$ by the equation

$$y \cos x - e^{xy^2} = 0.$$

Find the slope of its tangent line through P.

Solution: We first compute the slope of the tangent line using implicit differentiation:

$$\cos x\frac{dy}{dx} - y \sin x - e^{xy^2}(y^2 + 2xy\frac{dy}{dx}) = 0.$$

Since $x = 0$ and $y = 1$ this equation simplifies to

$$\frac{dy}{dx} - 1 = 0,$$

and so $\frac{dy}{dx} = 1$. Since the tangent line must have this slope and also pass through $(0, 1)$, its equation is $y - 1 = x$.

5) (20 points) A stone is dropped into a pond, the ripples forming concentric circles which expand. At what rate is the area of one of these circles increasing when the radius is 4 m and increasing at the rate of 0.5 m/s? (Hint: The area of a disk of radius r is πr^2.)

Solution: The area is $A(r) = \pi r^2$, where $r = r(t)$ is a function of time, and we wish to determine $\frac{dA}{dt}$. We are given that $\frac{dr}{dt} = 1/2$ and $r = 4$ at the given time. If we differentiate the formula for the area we get

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}.$$

Since this is equal to $2\pi \times 4 \times \frac{1}{2} = 4\pi$ at the given time, we conclude that the area is increasing at the rate of 4π (meters^2/sec).

6) (15 points) The volume V of a ball of radius r is given by the $V(r) = \frac{4\pi}{3}r^3$. If you determine the radius within an accuracy of 3 %, how accurate is your calculation of the volume? Show your work.

Solution: The per cent error of the measurement of V is $100\Delta V/V$. By linear approximation this is approximately

$$100\frac{V'(r)\Delta r}{V} = 100\frac{4\pi r^2 \Delta r}{(4\pi/3)r^3} = 3(100\frac{\Delta r}{r}).$$

Thus, the error for the calculation of the volume is three times greater than that of the radius, and so the accuracy is 9 %.