5.2

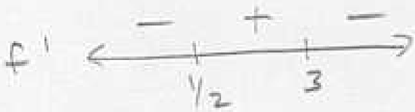
5. $f(x) = -\frac{2}{3}x^3 + \frac{7}{2}x^2 - 3x + 4$

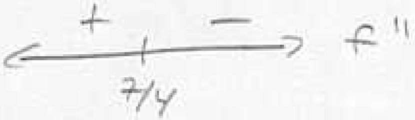
$f'(x) = -2x^2 + 7x - 3$

$= -(2x-1)(x-3)$

$= 0$ when $x = \frac{1}{2}, 3$

$f''(x) = -4x + 7 = 0$ when $x = 7/4$



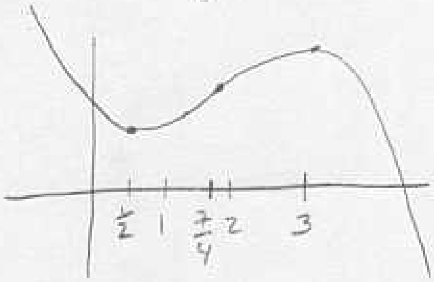


increasing = $(\frac{1}{2}, 3)$

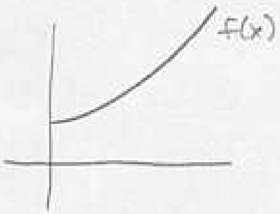
decreasing = $(-\infty, 1/2) \cup (3, \infty)$

Concave up = $(-\infty, 7/4)$

Concave down = $(7/4, \infty)$

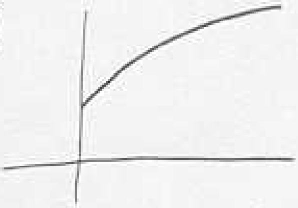


21. a)



incr. at an accelerating rate ⟺ increasing, concave up

b)



incr. at a decelerating rate ⟺ incr., concave down

c) need $f'(x) > 0$
$f''(x) > 0$
for part a,
need $f'(x) > 0$
$f''(x) < 0$
for part b

31. $f(p) = e^{-ap} \Rightarrow f'(p) = -ae^{-ap} < 0$
so f decreases with P

32. $f'(p) = -k\left(1 + \frac{ap}{k}\right)^{k-1}\left(\frac{a}{k}\right) < 0$, so f decreases with P

41. $y = bx^a$, so $\frac{dy}{dx} = abx^{a-1}$, $\frac{d}{dx}\left(\frac{y}{x}\right) = \frac{d}{dx}\left(bx^{a-1}\right) = (a-1)bx^{a-2}$

We need $ab > 0$ and $(a-1)b < 0$

b is positive, so $(a-1) < 0 \Longrightarrow 0 < a < 1$

$y'' = a(a-1)bx^{a-2}$ is then negative, so y is concave down

41 b. X = body length, Y = skull length, $0<a<1$

$\frac{d}{dx}\left(\frac{Y}{X}\right) < 0$ means $\frac{Y}{X}$ decreases as X increases

i.e. Ratio of Skull size to Body length decreases as body gets longer!

5.3

4. $y = \ln x$, $x>0$

$y' = \frac{1}{x} \Rightarrow \lim_{x\to\infty} y' = 0$ and $y'>0$ for $x>0$

$y'' = -\frac{1}{x^2} \Rightarrow y''<0$ for all x

$\Longrightarrow$ y increases and is concave down on $x>0$.

no max. or min.

14. $y = x^2(1-x) = x^2 - x^3$, $y' = 2x-3x^2$
$= x(2-3x)$
$= 0$
for $x = 0, 2/3$

y' — + — (0, 2/3)

y increasing $(0, 2/3)$

y decr. $(-\infty, 0) \cup (2/3, +\infty)$

$y'' = 2-6x$

$y''(0) = 2 > 0$

$\therefore$ 0 is a local min.

$y''(2/3) = 2 - 6(2/3) < 0$

$\therefore$ 2/3 is local max

27. $y = \frac{2}{3}x^3 - 2x^2 - 6x + 2$, $-2 \le x \le 5$

$y' = 2x^2 - 4x - 6$
$= 2(x^2-2x-3)$
$= 2(x-3)(x+1) = 0$
when $x = -1, 3$

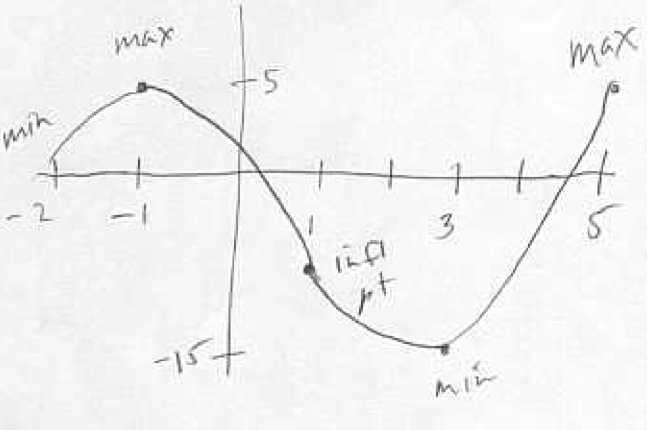
y' + — + (−1, 3)

incr. decr. incr.

$y'' = 4x - 4 = 0$
$x = 1$

— + y''

concave down | concave up



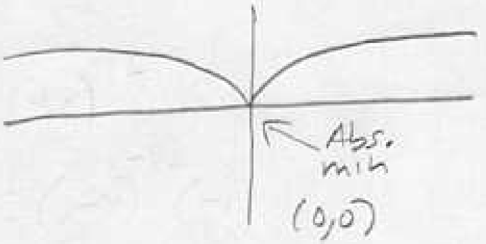
30. $y = \sqrt{|x|} = \begin{cases} \sqrt{x}, & x \geq 0 \\ \sqrt{-x}, & x < 0 \end{cases}$ (must do this so we can find y')

$y' = \begin{cases} \frac{1}{2\sqrt{x}}, & x > 0 \\ -\frac{1}{2\sqrt{-x}}, & x < 0 \end{cases}$

$y' = 0$ nowhere, but notice each piece is undefined at $x = 0$.

$y'' = \begin{cases} -\frac{1}{4x^{3/2}}, & x > 0 \\ \frac{-1}{4(-x)^{3/2}}, & x < 0 \end{cases}$

$y'' < 0$ concave down on $(-\infty, 0)$, $(0, \infty)$



y decreases $(-\infty, 0)$
increases $(0, \infty)$

33. $y = \frac{x^2-1}{x^2+1}$, $x \in \mathbb{R}$

$y' = \frac{(x^2+1)(2x) - (x^2-1)(2x)}{(x^2+1)^2} = \frac{2x^3 + 2x - 2x^3 + 2x}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2} = 0$ when $x = 0$

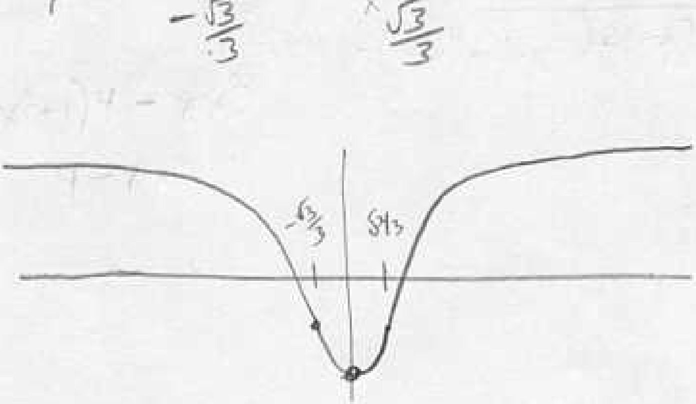
y': $-$ | $+$ (at 0)
decr. incr.

$y'' = \frac{(x^2+1)^2 4 - 4x \cdot 2(x^2+1)2x}{(x^2+1)^4} = \frac{4(x^2+1) - 16x^2}{(x^2+1)^3}$

$= \frac{4 - 12x^2}{(x^2+1)^3} = \frac{4(1-3x^2)}{(x^2+1)^3} = 0$

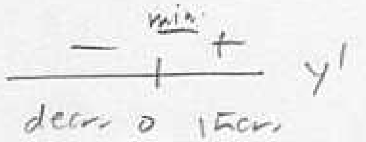
y'': $-$ | $+$ | $-$ (at $-\frac{\sqrt{3}}{3}$, $\frac{\sqrt{3}}{3}$)

when $1 - 3x^2 = 0$
$x^2 = 1/3$
$x = \pm \frac{\sqrt{3}}{3}$



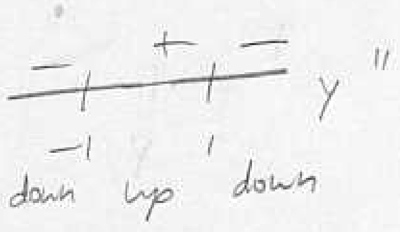
34. $y = \ln(x^2+1)$

$y' = \frac{2x}{x^2+1} = 0$ at $x=0$



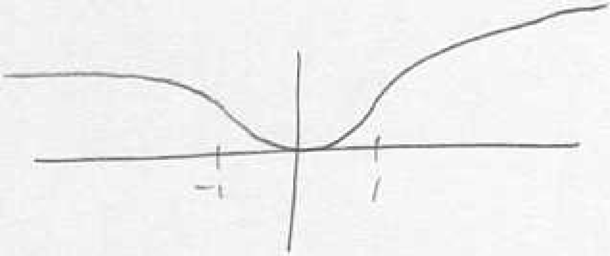
$$y'' = \frac{(x^2+1)2 - 2x(2x)}{(x^2+1)^2}$$

$$= \frac{2-2x^2}{(x^2+1)^2} = 0$$



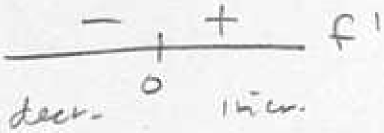
at $2-2x^2=0$

infl. pts. $x = \pm 1$



42. $f(x) = \frac{x^2}{x^2+a^2}$, $x \geq 0$

$$f'(x) = \frac{(x^2+a^2)2x - x^2(2x)}{(x^2+a^2)^2} = \frac{2xa^2}{(x^2+a^2)^2} = 0 \text{ when } x=0$$



$$f''(x) = 2\left[\frac{(x^2+a^2)^2 2a^2 - 2xa^2 \cdot 2(x^2+a^2)2x}{(x^2+a^2)^4}\right]$$

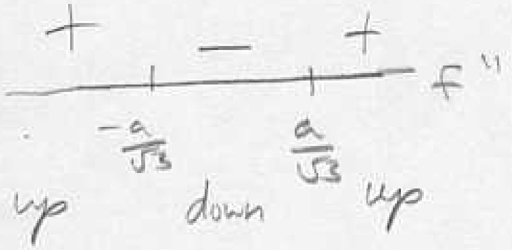
$$= 2\left[\frac{(x^2+a^2)2a^2 - 8x^2a^2}{(x^2+a^2)^3}\right]$$

$$= 2\left[\frac{-6a^2x^2 + 2a^4}{(x^2+a^2)^3}\right] = 0 \text{ when}$$

$-6a^2x^2 + 2a^4 = 0$

$x^2 = \frac{a^2}{3}$

$x = \pm \frac{a}{\sqrt{3}}$

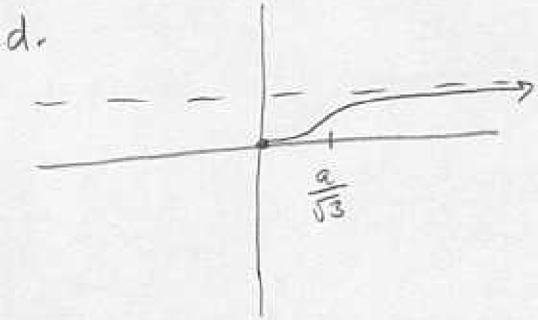


c. $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x^2}{x^2+a^2} = 1$

$y=1$ is H.A.

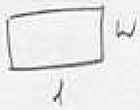
no V.A.

d.



5.4

2.



$A = lw$

$P = 2l + 2w$

$l = \frac{P-2w}{2}$

$A = \left(\frac{P-2w}{2}\right) w$

$= \frac{P}{2} w - w^2$

$\frac{dA}{dw} = \frac{P}{2} - 2w = 0$

$\frac{P}{2} = 2w$

$\boxed{\frac{P}{4} = w}$

$l = \frac{P - 2\cdot P/4}{2} = \boxed{\frac{P}{4}}$

endpts: $w=0$, $l=0$ $\Rightarrow A=0$

$+ \mid -$ A'

$P/4$

max

For max. area, $w = l = P/4$

∴ square!

5.

$L = 2x + y = 320$

$y = 320 - 2x$

$A = xy = x(320-2x)$

$= 320x - 2x^2$

$A' = 320 - 4x = 0$

$\boxed{\begin{array}{c} x = 80 \\ \hline y = 160 \end{array}}$

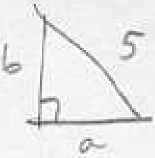
A' $+ \mid -$

80

max

endpts: $x=0$, $y=0$ $\Rightarrow A=0$

7.



$P = a + b + 5$

$P = \sqrt{25-b^2} + b + 5$

$a^2 + b^2 = 25$

$a = \sqrt{25-b^2}$

$$\frac{dP}{db} = \frac{1}{2}(25-b^2)^{-1/2}(-2b) + 1$$

$$= \frac{-b}{\sqrt{25-b^2}} + 1 = \frac{-b+\sqrt{25-b^2}}{\sqrt{25-b^2}} = 0$$

then $\sqrt{25-b^2} = b$

$25 - b^2 = b^2$

$25 = 2b^2$

$b^2 = \frac{25}{2}$

$b = \pm\frac{5}{\sqrt{2}}$ (throw out $-$)

$$a = \sqrt{25-\left(\frac{25}{2}\right)}$$

$$= \sqrt{\frac{50-25}{2}}$$

$$= \frac{5}{\sqrt{2}}$$

$$P = \frac{5}{\sqrt{2}} + \frac{5}{\sqrt{2}} + 5 = 5(1+\sqrt{2})$$

$$\frac{d}{db}\frac{dP}{db} = \frac{-\sqrt{25-b^2} + b\cdot\frac{1}{2}(25-b^2)^{-1/2}(-2b)}{25-b^2}$$

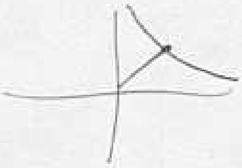
< 0

∴ <u>minimum</u>

endpts: $a = 0$, $b = 0$ ⟹ $P = 0$

make no sense!

12. $y = \frac{1}{x}$



$D = \sqrt{x^2+y^2}$

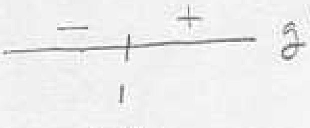
$= \sqrt{x^2 + \frac{1}{x^2}}$

$g = D^2 = x^2 + \frac{1}{x^2}$

$$\frac{dg}{dx} = 2x - \frac{2}{x^3} = \frac{2x^4-2}{x^3} = 0$$

$x^4 - 1 = 0$

$x = \pm 1$



<u>min</u>

$$\boxed{D = \sqrt{1+1} = \sqrt{2}}$$

16.

$$S = \pi r^2 + 2\pi r h = \pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right)$$

$$= \pi r^2 + \frac{2000}{r}$$

$V = 1L = 1000\text{ cm}^3$

$1000 = \pi r^2 h$

$h = \frac{1000}{\pi r^2}$

$$S' = 2\pi r - \frac{2000}{r^2} = 0$$

$$\frac{2\pi r^3 - 2000}{r^2} = 0$$

$$r^3 = \frac{1000}{\pi}$$

$$\boxed{\begin{array}{l} r = \frac{10}{\sqrt[3]{\pi}} \\ h = \frac{10}{\sqrt[3]{\pi}} \end{array}}$$

min

$-$ $+$ S'

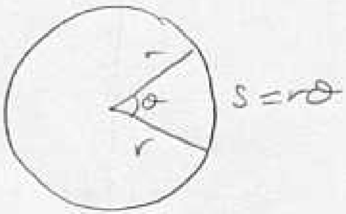
$\frac{10}{\sqrt[3]{\pi}}$

endpts: $r=0$, $h=0$ $\Rightarrow S=0$

but then no can!

17. $A = \frac{1}{2}r^2\theta$

$P = r\theta + 2r$



When $A=2$

$\theta = \frac{4}{r^2}$

$$P = r\left(\frac{4}{r^2}\right) + 2r = \frac{4}{r} + 2r$$

$$P' = \frac{-4}{r^2} + 2 = 0$$

$$\frac{-4+2r^2}{r^2} = 0$$

$$r^2 = 2$$

$$\boxed{\begin{array}{l} r = \sqrt{2} \\ \theta = \frac{4}{2} = 2 \end{array}}$$

min

$-$ $+$

$\sqrt{2}$

endpts: $r=0$, $\theta=\infty$

$r=\infty$, $\theta=0$

$\Rightarrow P=0$