Calculus I Homework #2

Section 3.2

(6) $f(x) = \begin{cases} \dfrac{2x^2+x-6}{x+2} & \text{if } x \neq -2 \\ -7 & \text{if } x = -2 \end{cases}$

$\lim_{x\to -2} f(x) = \lim_{x\to -2} \dfrac{2x^2+x-6}{x+2} = \lim_{x\to -2} \dfrac{(x+2)(2x-3)}{x+2} = \lim_{x\to -2}(2x-3) = -4-3 = -7$

$\therefore \lim_{x\to -2} f(x) = f(-2)$ so the function is continuous at $x=-2$.

(8)

$f(x) = \begin{cases} \dfrac{x^2+x-2}{x-1} & \text{if } x \neq 1 \\ a & \text{if } x = 1 \end{cases}$

$\lim_{x\to 1} f(x) = \lim_{x\to 1} \dfrac{x^2+x-2}{x-1} = \lim_{x\to 1} \dfrac{(x-1)(x+2)}{x-1} = \lim_{x\to 1}(x+2) = 3$

$a=3$ in order to make $f(x)$ be continuous at $x=1$.

(10) $f(x) = \begin{cases} \dfrac{1}{x^2-1} & \text{for } x \neq -1, 1 \\ 0 & \text{for } x = -1 \text{ or } 1 \end{cases}$ is discontinuous at $x=-1, x=1$

since $\lim_{x\to 1} f(x)$, $\lim_{x\to -1} f(x)$ do not exist

$\left(\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \dfrac{1}{x^2-1} = \infty, \quad \lim_{x\to 1^-} f(x) = -\infty \right.$

$\left. \lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \dfrac{1}{x^2-1} = -\infty, \quad \lim_{x\to -1^-} f(x) = \infty \right)$

(12) $f(x) = \begin{cases} x^2-1 & \text{for } x \leq 0 \\ x & \text{for } x > 0 \end{cases}$

is discontinuous at $x=0$;

pf) $\begin{cases} \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x = 0 \\ \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2-1 = -1 \end{cases}$ so $\lim_{x\to 0} f(x)$ does not exist.

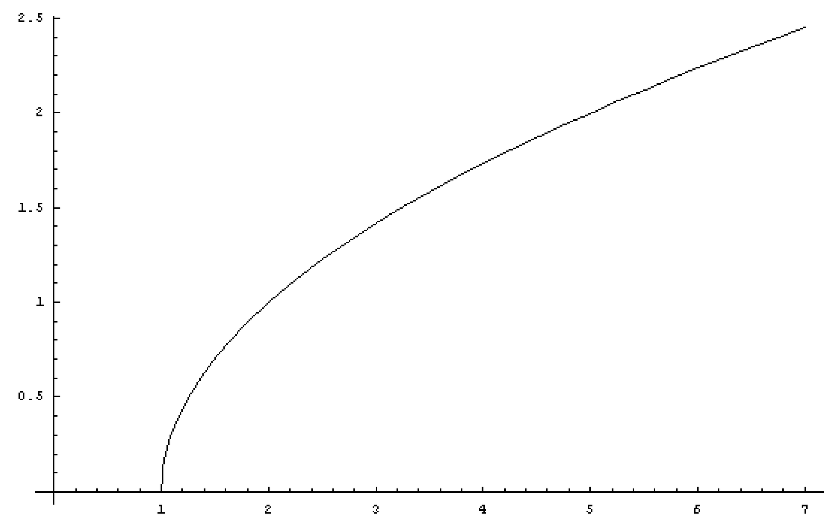
hence $f(x)$ is discontinuous at $x=0$

(15) (a) $f(x)=\sqrt{x-1}$, $x \geq 1$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \sqrt{x-1} = 0 = f(1)$

$f(x)$ is right continuous at $x=1$

(b)



(c) $f(x)$ is not defined for $x<1$, so we cannot think $\lim_{x \to 1^-} f(x)$.
It doesn't make sense to look at continuity from the left at $x=1$.

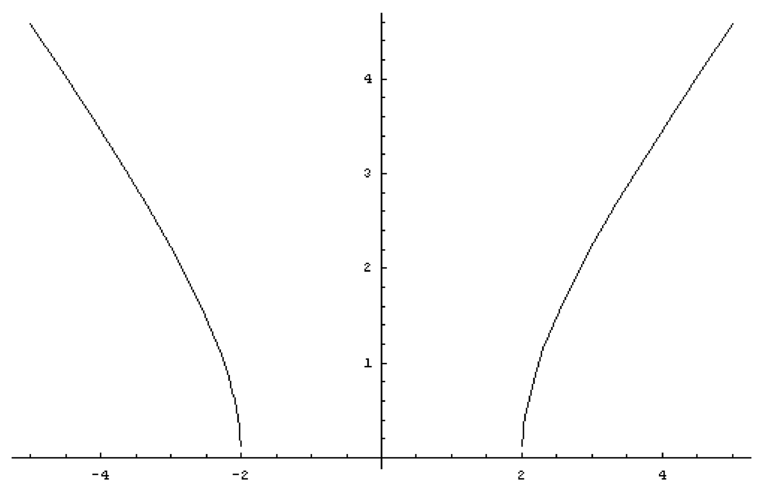
(16) (a) $f(x)=\sqrt{x^2-4}$, $|x| \geq 2$

$\lim_{x \to 2^+} f(x) = 0 = f(2)$

$\lim_{x \to -2^-} f(x) = 0 = f(-2)$

so $f(x)$ is continuous from the right at $x=2$,
continuous from the left at $x=-2$.

(b)



(c) $f(x)$ is not defined for $-2<x<2$, so we cannot think
$\lim_{x \to -2^+} f(x)$ and $\lim_{x \to 2^-} f(x)$.
It doesn't make sense to look at continuity from the left at $x=2$
and at continuity from the right at $x=-2$.

(18) $f(x)=\sqrt{x^2-1}$

$f(x)$ is continuous at $x \Leftrightarrow f(x)$ is defined at x

$x^2-1 \geq 0 \Leftrightarrow x^2 \geq 1 \Leftrightarrow |x| \geq 1$.

(22) $f(x)=\ln(x-2)$

$x-2>0 \Leftrightarrow x>2$

(40) $\lim_{x\to -1} e^{x/2 - 1} = e^{\frac{1}{2}-1} = e^{-\frac{1}{2}} \quad \left(= \frac{1}{\sqrt{e}}\right)$

(41) $\lim_{x\to 0} \frac{e^{2x}-1}{e^{x}-1} = \lim_{x\to 0} \frac{(e^{x}-1)(e^{x}+1)}{e^{x}-1} = \lim_{x\to 0} (e^{x}+1) = e^{0}+1 = 1+1 = 2$

(42) $\lim_{x\to 0} \frac{e^{-x}-e^{x}}{e^{-x}+1} = \frac{e^{0}-e^{0}}{e^{0}+1} = \frac{0}{2} = 0$

(43) $\lim_{x\to -2} \frac{1}{\sqrt{5x^2-4}} = \frac{1}{\sqrt{5\cdot 4-4}} = \frac{1}{\sqrt{16}} = \frac{1}{4}$

Section 3.3

(1) $\lim_{x\to\infty} \frac{2x^2-3x+5}{x^4-2x+1} = \lim_{x\to\infty} \frac{\frac{2}{x^2}-\frac{3}{x^3}+\frac{5}{x^4}}{1-\frac{2}{x^3}+\frac{1}{x^4}} = \frac{0-0+0}{1-0+0} = 0$

$\lim_{x\to 0} \frac{2x^2-3x+5}{x^4-2x+1} = \frac{0-0+5}{0-0+1} = 5$

(2) $\lim_{x\to\infty} \frac{x^2+3}{5x^2-2x+1} = \lim_{x\to\infty} \frac{1+\frac{3}{x^2}}{5-\frac{2}{x}+\frac{1}{x^2}} = \frac{1}{5}$

$\lim_{x\to 0} \frac{x^2+3}{5x^2-2x+1} = 3$

(3) $\lim_{x\to\infty} \frac{x^3+3}{x-2} = \lim_{x\to\infty} \frac{1+\frac{3}{x^3}}{\frac{1}{x^2}-\frac{2}{x^3}} = \infty \quad \left(\frac{1}{x^2}-\frac{2}{x^3} \to 0 \text{ and } \frac{1}{x^2}-\frac{2}{x^3} > 0 \text{ for large } x\right)$

$\lim_{x\to 0} \frac{x^3+3}{x-2} = -\frac{3}{2}$

(4) $\lim_{x\to -\infty} \frac{2x-1}{3-4x} = \lim_{x\to -\infty} \frac{2-\frac{1}{x}}{\frac{3}{x}-4} = \frac{2-0}{0-4} = -\frac{1}{2}$

$\lim_{x\to 0} \frac{2x-1}{3-4x} = -\frac{1}{3}$

(5) $\lim_{x\to\infty} \frac{3x^4-x^3+1}{x^4+2x^2} = \lim_{x\to\infty} \frac{3-\frac{1}{x}+\frac{1}{x^4}}{1+\frac{2}{x^2}} = \frac{3-0+0}{1+0} = 3$

$\lim_{x\to 0} \frac{3x^4-x^3+1}{x^4+2x^2} = \infty \quad \left(3x^4-x^3+1 \to 1,\ x^4+2x^2 \to 0 \text{ and } x^4+2x^2 \geq 0\right)$

(16) $\lim_{x\to\infty} \frac{e^{x}}{2-e^{x}} = \lim_{x\to\infty} \frac{1}{2e^{-x}-1} = \frac{1}{0-1} = -1$

(17) $\lim_{x\to -\infty} \exp(x) = \lim_{x\to -\infty} e^{x} = \lim_{x\to\infty} e^{-x} = 0$

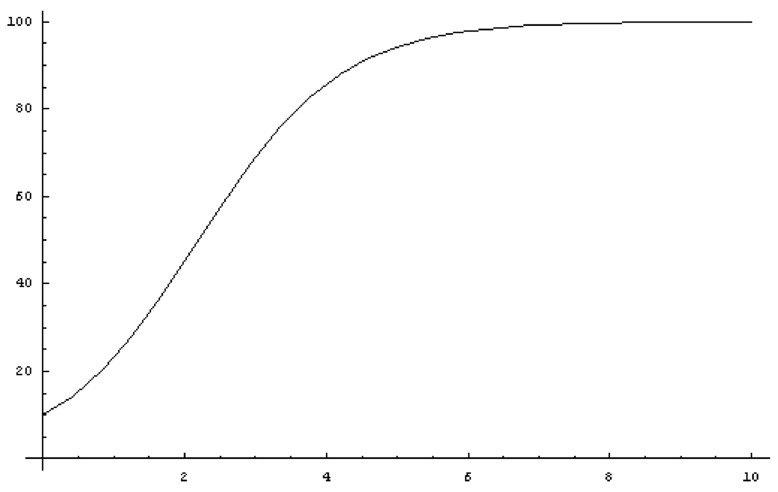
(18) $\lim_{x\to\infty} \exp(-\ln x) = \lim_{x\to\infty} e^{-\ln x} = \lim_{x\to\infty} \frac{1}{e^{\ln x}} = \lim_{x\to\infty} \frac{1}{x} = 0$

(19) $\lim_{x\to\infty} e^{-x}\sin x = 0$ $\quad$ ($|\sin x|\le 1$ and $e^{-x}\to 0$)

(20) $\lim_{x\to\infty} e^{-x}\cos x = 0$ $\quad$ ($|\cos x|\le 1$ and $e^{-x}\to 0$)

(28) $N(t) = \dfrac{100}{1+9e^{-t}}$ $\quad (t\ge 0)$

(a)



(b) $\lim_{t\to\infty} N(t) = \lim_{t\to\infty} \dfrac{100}{1+9e^{-t}} = \dfrac{100}{1+9\cdot 0} = 100$

Section 4.1

* $f(x) = x^2$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$
$$= \lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h}$$
$$= \lim_{h\to 0} \frac{2xh+h^2}{h}$$
$$= \lim_{h\to 0} (2x+h)$$
$$= 2x$$

$f'(1) = 2 \rightarrow$ slope of tangent line at $(1, f(1)) = (1,1)$

$\therefore$ tangent line : $y = 2(x-1)+1$

$y = 2x-1$

$g(x) = x^3$

$$g'(x) = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$
$$= \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}$$
$$= \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3}{h}$$
$$= \lim_{h\to 0} (3x^2+3xh+h^2)$$
$$= 3x^2$$

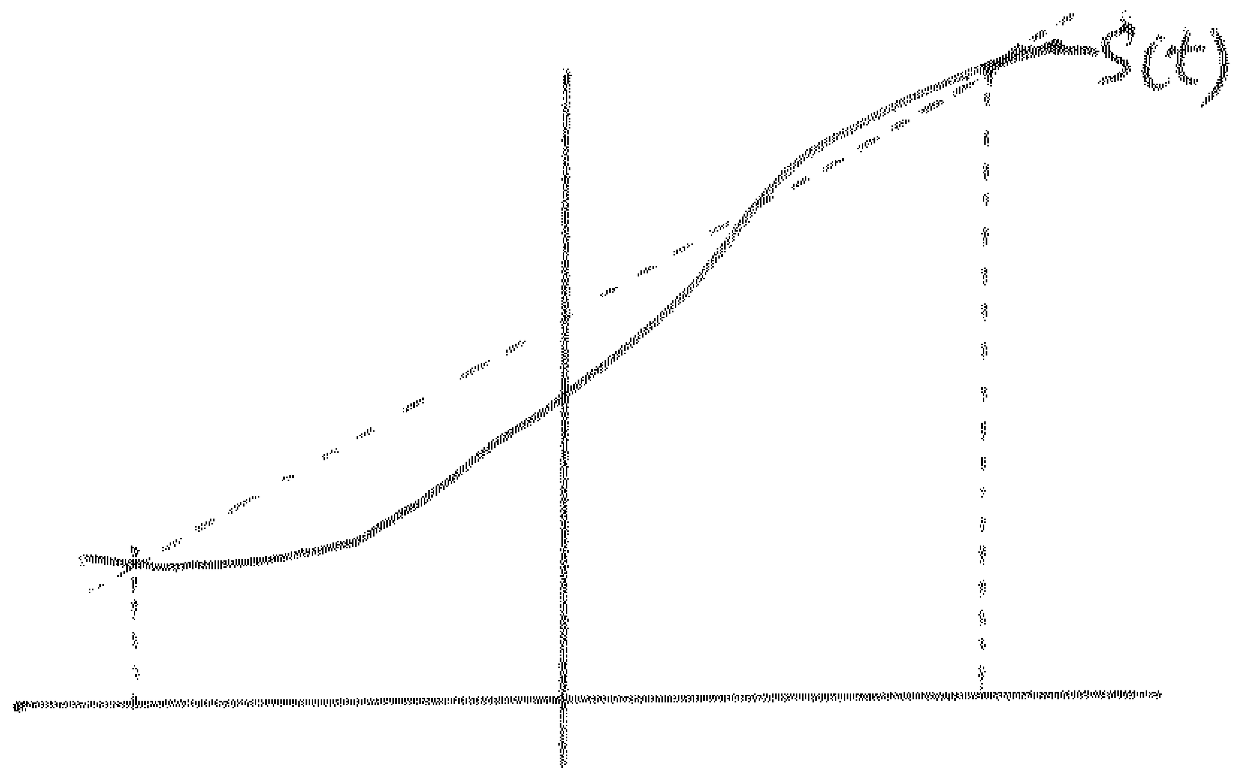
$g'(1) = 3 \rightarrow$ slope of tangent line at $(1, g(1)) = (1,1)$

$\therefore$ tangent line : $y = 3(x-1)+1$

$y = 3x-2$

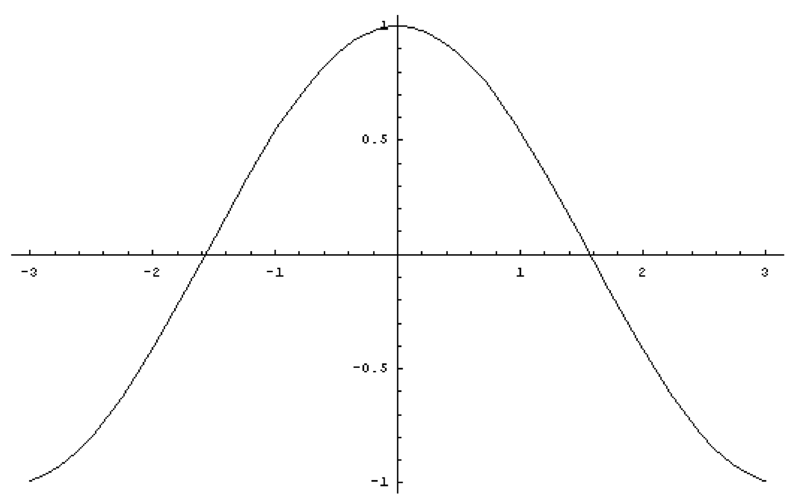
* $S(t)$: position of a person (in meter) t : time (in sec)

average velocity on $[-1, 1]$ > instantaneous velocity at $t=-1$ and at $t=1$



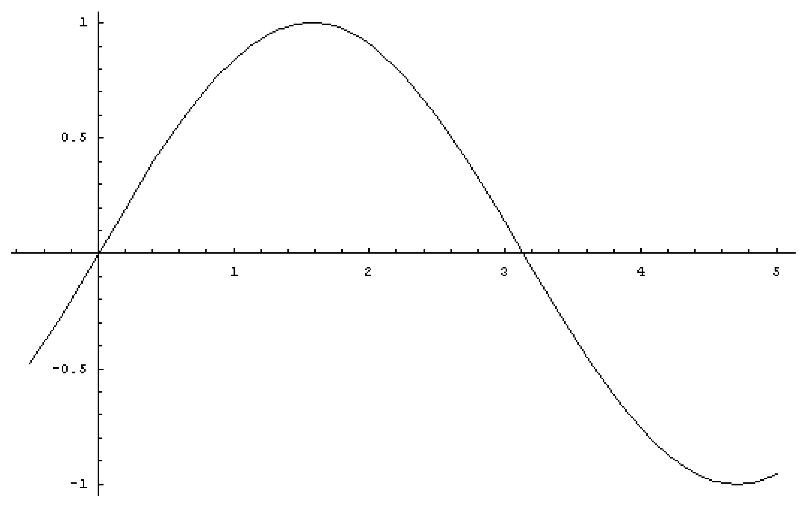
(7) $f(x) = \cos x$; $x = 0$

$f'(0) = 0$



(8) $f(x) = \sin x$; $x = \frac{\pi}{2}$

$f'(\frac{\pi}{2}) = 0$



(27) $$f'(a) = \lim_{h \to 0} \frac{2(a+h)^2 - 2a^2}{h} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$f(a+h) - f(a) = 2(a+h)^2 - 2a^2$

$f(x) = 2x^2 + c$ (c : constant)

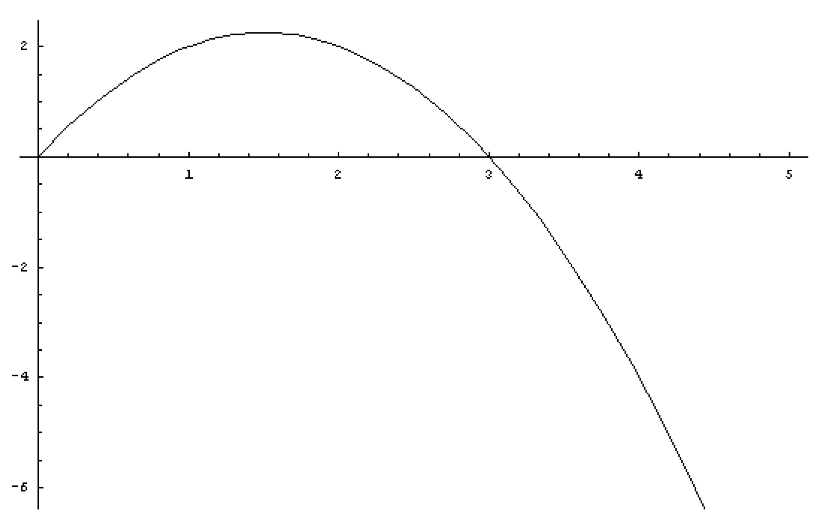
(28) $$f'(a) = \lim_{h \to 0} \frac{4(a+h)^3 - 4a^3}{h} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$f(a+h) - f(a) = 4(a+h)^3 - 4a^3$

$f(x) = 4x^3 + C$ (C : constant)

(34) $s(t) = 3t - t^2, \quad t \geq 0$; position of particle (meters), t: time (second)

(a)



(b) (i) $t = 0 \Rightarrow s(t) = 0$

(ii) $s(t) = 0 \Rightarrow 3t - t^2 = 0 \Rightarrow t(3-t) = 0 \Rightarrow t = 0, 3$

$\therefore t = 3$

(iii) maximal value of $s(t)$, $t \geq 0$;

$s(t) = \frac{9}{4}$, when $t = \frac{3}{2}$

(iv) minimal value of $s(t)$, $t \geq 0$; $-\infty$

(v) velocity positive : $0 \leq t < \frac{3}{2}$

negative : $t > \frac{3}{2}$

0 : $t = \frac{3}{2}$

(c) $s'(t) = 3 - 2t$

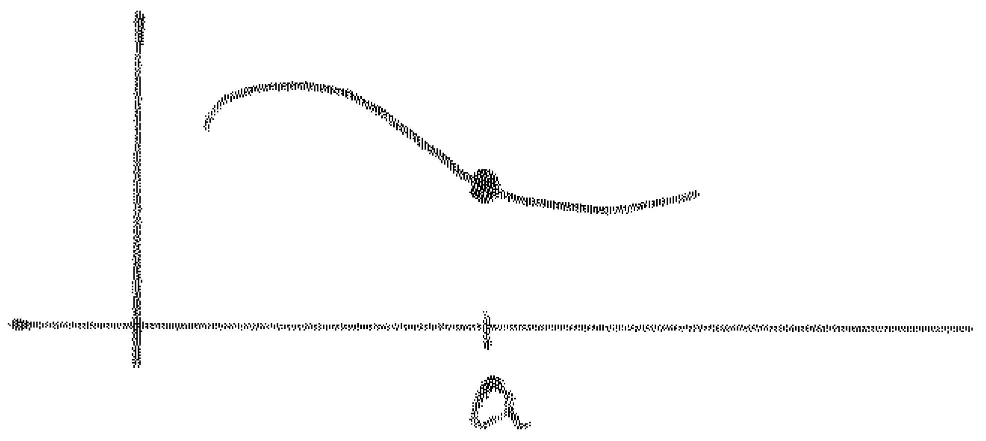
(d) $s'(t) = 1 \Leftrightarrow 3 - 2t = 1 \Leftrightarrow t = 1$

(60) $f(x) = \begin{cases} f_1(x) & \text{for } x \leq a \\ f_2(x) & \text{for } x > a \end{cases}$

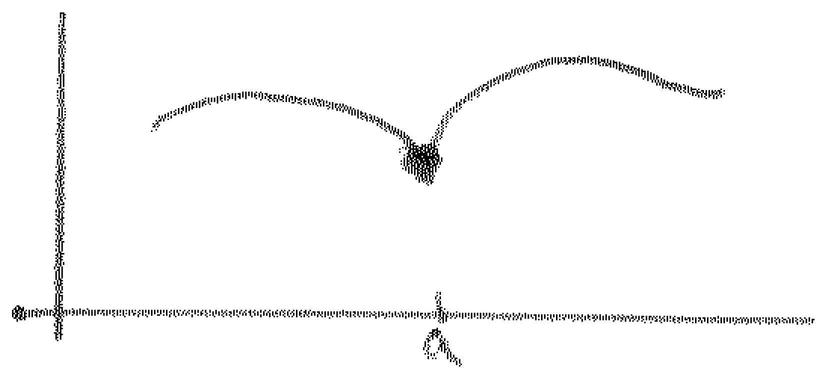
$f_1(x)$: continuous, differentiable for $x < a$

$f_2(x)$: continuous, differentiable for $x > a$

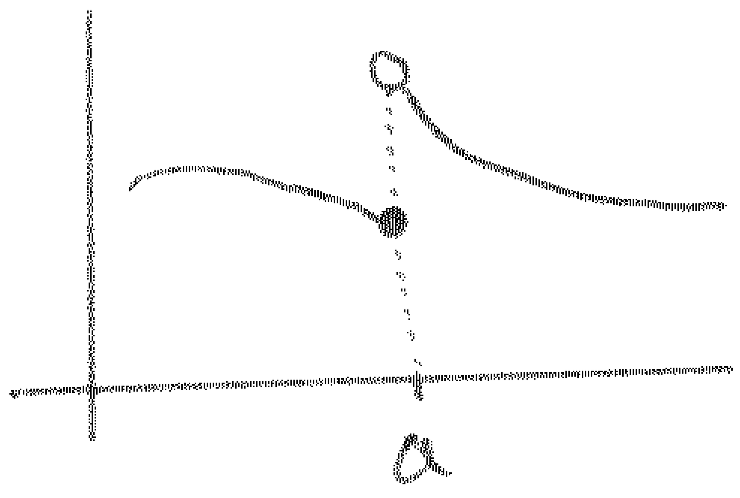
(a) $f(x)$ is continuous and differentiable at $x = a$



(b) $f(x)$ is continuous but not differentiable at $x = a$



(c) $f(x)$ is neither continuous nor differentiable at $x=a$



or