# MATH 106 — FINAL EXAM

DEPARTMENT OF MATHEMATICS
Johns Hopkins University

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NAME: ____________________________ SIGNATURE: ___________________________

SECTION NUMBER:_____________
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1. This exam has eleven pages including this cover. There are twelve questions.
2. Use of books, notes, or scratch paper is not allowed. You may certainly use a calculator (but not its manual).
3. **Show all of your work!** Partial credit is available for many problems but can only be given if the graders understand your work. Be sure to explain your reasoning carefully. Include units in your answers whenever appropriate. Showing your work will show graders that you are not using your calculator inappropriately, especially whenever you antidifferentiate functions. In general, you will not get credit for work done on your calculator if we expect you to do it by hand.
4. Read directions carefully. For some problems, a brief answer is sufficient, but others require you to show all work or give explanations.

| PROBLEM | POINTS | SCORE |
|---|---|---|
| 1 | 6 | |
| 2 | 8 | |
| 3 | 16 | |
| 4 | 10 | |
| 5 | 4 | |
| 6 | 4 | |
| 7 | 8 | |
| 8 | 12 | |
| 9 | 8 | |
| 10 | 8 | |
| 11 | 8 | |
| 12 | 8 | |
| TOTAL | 100 | |

**1.** (6 points) State each of the fundamental theorems of Calculus. Please use brief, symbolic, mathematical notation (as opposed to words and phrases) where appropriate.

**2.** (8 points) A cube ($3$ x $3$ x $3$ inches) of extra-soft tofu is placed on the countertop. It begins to deform under its own weight so that the square base grows outward in both directions (remaining square), and the height of the block decreases accordingly. After a while, you measure the height to be $2$ inches, and decreasing at $\frac{1}{30}$ inches per second. How fast is the width of the base increasing?

**3.** (4 points each) Calculate these definite integrals by hand. Show all of your work, step by step.

**a.** $\int \frac{(\ln(x))^3}{x} dx$

**b.** $\int \frac{(x^2+1)^2}{\sqrt{x}} dx$

**c.** $\int_c^6 \frac{ax^2}{b} dx$

**d.** $\int xe^x dx$

**4.** (10 points) A $12$ fl. oz. can has a volume of $21.6$ cubic inches. Assuming the can is cylindrical, what shape minimizes its surface area? (Hint: The surface area of a cylinder of height $h$ and radius $r$ is $2\pi r^2 + 2\pi rh$.)

**5.** (4 points) You have $320$ dollars, all of which you plan to spend on Jelly Beans and Gummy Bears. Gummi Bears cost 30 cents each, and Jelly Beans cost twice that much. Let $x$ be the number of Gummi Bears purchased, and let $y$ be the number of Jelly Beans purchased. Calculate $\frac{dy}{dx}$, the derivative of $y$ with respect to $x$.

**6.** (4 points) A herd of wild termites has an exponentially growing population: $P(t) = 100e^{0.1t}$, where $t$ is time in days and $P(t)$ is the number of termites. Each devours three grams of wood per day. How much wood is eaten in the week starting at $t = 0$?

**7.** (8 points) Of course, the universal law of gravitation for a $50$ kilogram person attracted to the earth's mass $M$ states that

$$F = \frac{GMm}{r^2}$$

where $m = 50$, $M = 5.98 * 10^{24}$, and $G = 6.67 * 10^{-11}$ are constants, $r$ is the distance (in meters) from the center of the earth, and $F$ is the resulting force (in Newtons, which may be regarded in this context as a unit of *weight*). In all questions below, you may leave your answer in terms of these constants.

**a.** Find $\frac{dF}{dr}$ at the point $r_0 = 6400000$. (This value of $r_0$ is the radius of the earth, and therefore your current approximate position.)

**b.** Explain the significance of the answer from part a., in real world terms, without using technical calculus jargon.

**c.** Considering $F$ to be a function of $r$, find the tangent line approximation to this function at the point $r_0 = 6400000$ meters.

**8.** (12 points) Write the Taylor series for each function. Include enough terms to express the pattern in each series. You may be able to do some without work. No need to show work!

**a.** $f(x) = \sin(x^3)$

**b.** $g(x) = x^2e^x$

**c.** $h(x) = (1+x)^{\frac{-1}{2}}$

**9.** (8 points) Calculate $\int_{-3}^{4} \frac{1}{(1-2x)^2} dx$. Cautiously show work.

**10.** (8 points) The functions $\cosh(x) = \frac{e^x+e^{-x}}{2}$ and $\sinh(x) = \frac{e^x+e^{-x}}{2}$ have properties roughly similar to $\sin(x)$ and $\cos(x)$:

$$\frac{d}{dx}\sinh(x) = \cosh(x), \qquad \frac{d}{dx}\cosh(x) = \sinh(x), \qquad \cosh^2(x) - \sinh^2(x) = 1$$

Eerie, isn't it?

Using these properties, find the length of the curve $\cosh(x)$ between $x = -2$ and $x = 2$. This curve (and *not* a parabola) is the exact shape of a loose hanging rope. Show all work, of course.

**11.** (8 points) Consider the area above the $x$-axis, right of the $y$-axis, and below the curve $y = \cos(x)$ between $x = 0$ and the first place $\cos(x)$ hits the $x$-axis. If the area is rotated around the $y$-axis, a hill will be created. Find its volume. Show all your work.

**12.** (8 points) Calculate the integral. You must show work for credit.

$$\int \sin(2x)\sin(x)dx$$