MATH 106 — FINAL EXAM

DEPARTMENT OF MATHEMATICS
Johns Hopkins University

December 12, 2003

NAME: ____________________________ SIGNATURE: ___________________________

SECTION NUMBER:_____________

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1. This exam has nine pages including this cover. There are twelve questions.
2. Use of books, notes, or scratch paper is not allowed. You may certainly use a calculator (but not its manual).
3. **Show all of your work!** Partial credit is available for many problems but can only be given if the graders understand your work. Be sure to explain your reasoning carefully. Include units in your answers whenever appropriate.
4. Read directions carefully. For some problems, a brief answer is sufficient, but others require you to show all work or give explanations.

1. (6 points) State each of the fundamental theorems of Calculus. Please use brief, symbolic, mathematical notation (as opposed to words and phrases) where appropriate.

Order is irrelevant. The "first" fundamental theorem of Calculus states that $\frac{d}{dx}\int_a^x f(t)dt = f(x)$. The "second" says that $\int_a^b f(x)dx = F(b) - F(a)$, where F is an antiderivative of f. Precise hypotheses were not required for credit, but one should state the assumption that f is continuous.

2. (8 points) To show off your talents as a computer programmer, you have written a program to estimate integrals using the trapezoid rule, and you plan to apply it to approximate the integral $\int_0^3 x^4 dx$. As you can check, the exact value of this integral is 48.6. How many subdivisions of $[0, 3]$ are required to guarantee a trapezoid-rule approximation between 48.5 and 48.7?

The required accuracy requires that the error be less than or equal to 0.1, so take the error formula $\frac{K(b-a)^3}{12n^2}$ and set it equal to 0.1. Here K is a bound for the second derivative. Since the second derivative of x^4 is $12x^2$, $K = 12 \cdot 3^2 = 108$ is the best bound. Therefore

$$\frac{108(3-0)^3}{12n^2} = 0.1$$

Solving for n gives $n = 49.3$. Since n must be at least this large, choose $n = 50$.

3. (4 points) Circle **True** or **False**: The function below is continuous for *all* values of x. (No explanation necessary.)

$$f(x) = \sin(e^{x^2+x+4}) \cos(\cos((x-1)(x-2)(x-3)))$$

True, because the function is created by adding, multiplying, and composing continuous functions. There is no way to get a discontinuous function by combining continuous functions in this way.

4. (8 points) Find a constant c so that the function

$$f(x) = \begin{cases} \frac{x}{2e^x-2} & \text{if } x > 0, \\ x + c & \text{if } x \leq 0. \end{cases} \quad (1)$$

is continuous. Show your work carefully.

In order for $f(x)$ to be continuous, the left-hand limit and right-hand limit at $x = 0$ must match. Calculating the left-hand limit, using the first formula is hard: $\lim_{x\to 0^+} \frac{x}{2e^x-2} = \lim_{x\to 0^+} \frac{1}{2e^x} = 1/2$ (by L'hopital's rule–the first is an indeterminate form 0/0). This means that the right hand limit and the value of the function at $x = 0$ must both be $1/2$, which can be arranged by choosing $c = 1/2$.

5. (8 points) Find the equation of the tangent line through the point $(1, e^2)$ to the curve defined by the equation $\ln(xy) = 2x$.

Implicit differentiation gives: $\frac{1}{xy}(y + x\frac{dy}{dx}) = 2$. Solving for $\frac{dy}{dx}$ gives $\frac{dy}{dx} = (2xy - y)/x$. Plugging in $x = 1$ and $y = e^2$ yields $\frac{dy}{dx} = 2e^2 - e^2 = e^2$. So this is the slope of the tangent line at the given point. To get the equation, use the point-slope formula: $(y - e^2) = e^2(x - 1)$.

6. (8 points) Compute the Taylor polynomial of degree three about $a = 0$ for the function $f(x) = \sqrt{1+x}$. Show your work.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1+x)^{\frac{1}{2}}$	1
1	$\frac{1}{2}(1+x)^{\frac{-1}{2}}$	$1/2$
2	$\frac{-1}{4}(1+x)^{\frac{-3}{2}}$	$-1/4$
3	$\frac{3}{8}(1+x)^{\frac{-5}{2}}$	$3/8$

So the Taylor polynomial is $P_3(x) = 1 + \frac{1}{2}x + \frac{\frac{-1}{4}x}{2!} + \frac{\frac{3}{8}x}{3!}$, which can of course be simplified, but need not be.

7. (4 points) Simplify:

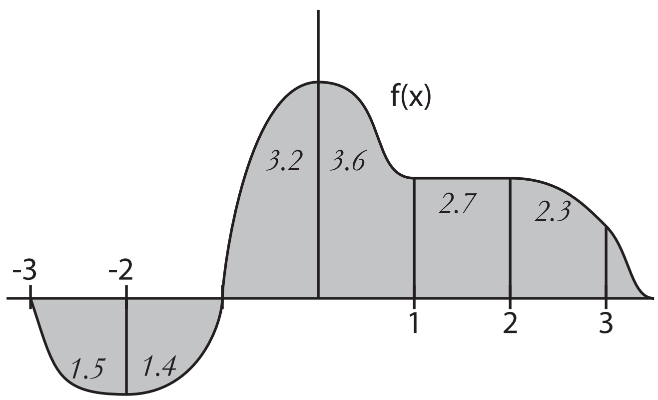
$$1 + \frac{3}{1} + \frac{9}{2!} + \frac{27}{3!} + \frac{81}{4!} + \ldots$$

This is the Taylor series for e^x, with 3 substituted for x. Therefore it is e^3, or roughly 20.085536923187667741.

8. (10 points) Below is a graph of $f(x)$ with some areas marked in italics. Compute

$$\int_0^2 xf(x^2-2)dx$$

as accurately as possible.



When you see a "wildcard" problem like this, which looks nothing like anything you've seen on homework, try to remain calm and keep your wits about you. Have faith that it tests something you understand.

But what does it test? Well, the integrand is complicated, and we have a few techniques (substitution, integration by parts, and partial fractions) to deal with complicated integrals. Certainly partial fractions is out. Integration by parts is tempting, because differentiating that x will simplify things. But can we antidifferentiate $f(x^2-2)$? No chance without a formula for f.

That leaves only substitution, and indeed the integrand looks agreeable: There is an obvious candidate $u = x^2-2$, and its derivative $2x$ appears as a factor, except for the 2 of course, but we've learned not to worry about the constants. So this must be a substitution problem. Performing the substitution gives

$$\int_{x=0}^{x=2} f(u)\frac{1}{2}du$$

as you can check. It might be convenient to have endpoints in terms of u. When $x = 0$, $u = -2$, and when $x = 2$, $u = 2$. So we can modify slightly:

$$\frac{1}{2}\int_{u=-2}^{u=2} f(u)du$$

I also factored out the $1/2$. Now that integral can be read off the graph as 8.1. So the answer is 4.05.

9. (14 points) The elite Hopkins Medical Center Sneezologists, through diligent measurement and experimentation, have recorded and modeled the exhalation rate (in cc's per second) during a typical human sneeze. Their approximate formula for the exhalation rate $E(t)$ is

$$E(t) = \frac{1-t}{t^2+2t},$$

where t is time in seconds after the "sneeze initiation event."

a. (10 points) Find the definite integral of this function, from $t = 0$ to $t = 1$.

$$\begin{aligned}
\int_0^1 \frac{1-t}{t^2+2t}dt &= \int_0^1 \frac{1-t}{t(t+2)}dt \\
&= \int_0^1 \frac{A}{t} + \frac{B}{t+2}dt \\
&= \int_0^1 \frac{1/2}{t} + \frac{-3/2}{t+2}dt \\
&= \lim_{a\to 0^+}\int_a^1 \frac{1/2}{t} + \frac{-3/2}{t+2}dt \\
&= \lim_{a\to 0^+} \frac{1}{2}\ln(t) + \frac{-3}{2}\ln(t+2)|_a^1 \\
&= \lim_{a\to 0^+} [\frac{1}{2}\ln(1) + \frac{-3}{2}\ln(1+2)] - [\frac{1}{2}\ln(a) + \frac{-3}{2}\ln(a+2)] \\
&= 0 + \frac{-3}{2}\ln(3) - \lim_{a\to 0^+}\ln(a) - \frac{-3}{2}\ln(2)] \\
&= \infty
\end{aligned}$$

Above, A and B are calculated by the cover-up method. The left endpoint a is introduced because the integrand has a pole at $t = 0$. It diverges because $\ln(a)$ approaches $-\infty$ as a goes to 0.

b. (2 points) Assuming the Sneezologists' formula is correct, interpret your answer to part (a), in one plain English sentence.

Infinitely many cc's of air are exhaled in the first second of one sneeze.

c. (2 points) The Sneezologists model is [**Reasonable** or **Unreasonable**]. (Circle one. No explanation necessary)

Unreasonable, because that's too much air!

10. (10 points) Calculate the integral by hand. Of course, show all work.

$$\int e^{2x}\sin(x)dx$$

We will integrate by parts twice and use the circle trick. We will have to be careful with the constants because chain rule constants will appear when we differentiate u each time.

$$\begin{aligned}\int e^{2x}\sin(x)dx &= -e^{2x}\cos(x) - \int -2e^{2x}\cos(x)dx\\ &= -e^{2x}\cos(x) + \int 2e^{2x}\cos(x)dx\\ &= -e^{2x}\cos(x) + [2e^{2x}\sin(x) - \int 4e^{2x}\sin(x)dx]\\ &= -e^{2x}\cos(x) + 2e^{2x}\sin(x) - \int 4e^{2x}\sin(x)dx\end{aligned}$$

The circle trick applies:

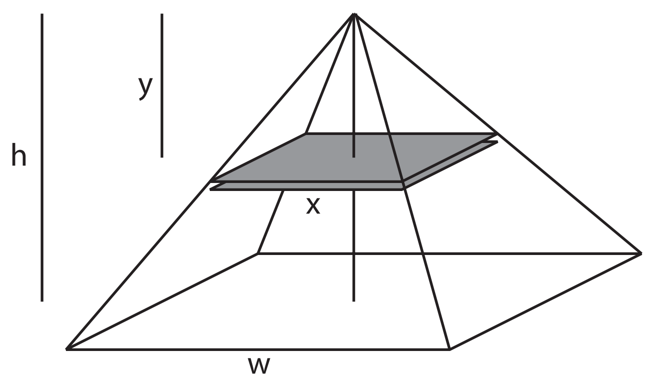
$$5\int e^{2x}\sin(x)dx = -e^{2x}\cos(x) + 2e^{2x}\sin(x)$$

which gives, finally,

$$\int e^{2x}\sin(x)dx = \frac{-e^{2x}\cos(x) + 2e^{2x}\sin(x)}{5} + C$$

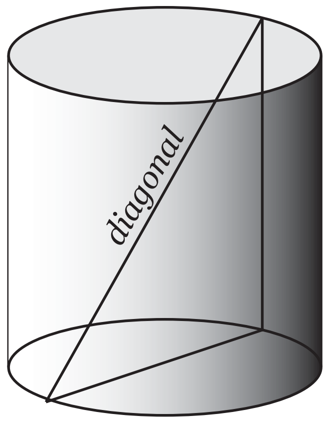
11. (10 points) Find the volume of a pyramid whose height is h and whose base is a square of side length w. (You must show how to use techniques of integration to solve the problem — a memorized formula for this volume will not be worth credit.)

Always start with a nice picture:



We have sliced the pyramid horizontally, of course, according to the value of y. Note: y is not h, and x is not w! The volume of one slice of the pyramid is x^2dy, so the volume of the whole pyramid is $\int_{y=0}^{y=h} x^2 dy$. This integral is not correctly expressed–it should be written in terms of y only. The relationship between x and y comes from the similarity of certain triangles in the picture (which is why you need a good picture). The relationship is $\frac{x}{y} = \frac{w}{h}$. So $x = \frac{wy}{h}$. This gives the integral $\int_0^h \frac{w^2y^2}{h^2} dy = \frac{w^2}{h^2} \int_0^h y^2 dy = \frac{w^2}{h^2} \frac{y^3}{3}|_0^h = \frac{w^2}{h^2} \frac{h^3}{3} = \frac{hw^2}{3}$.

12. (10 points) If you join a point on the base of a cylinder to an opposite point on the top, you get a line through the center of the cylinder which could be called a "diagonal" (see picture). Many different cylinders have diagonal length 6 inches. Which has the largest volume? (Hints: If you choose wisely, you can avoid ugly square roots in your formulas. Also, $V_{\text{cylinder}} = \pi r^2 h$)



Clearly this is an optimization problem, because we're asked to compute the *largest* possible volume. We already have a formula for the volume, but we must write it in terms of a single variable. To do this, we need a relationship between r and h, which can be taken from the Pythagorean theorem: $(2r)^2 + h^2 = 6^2$. If on your test you "simplified" this equation to $2r + h = 6$, then you must write "Square roots do not pass across plus" 200 times on the chalkboard after class.

Given the equation $4r^2 + h^2 = 36$, we could solve for h ($h = \sqrt{36 - 4r^2}$) and plug this in for h in the expression $V = \pi r^2 h$. But this would create ugly square roots. It's more convenient to solve for r^2 ($r^2 = (36 - h^2)/4$) and plug this in for the r^2 in the expression $\pi r^2 h$, yielding $V = \pi \frac{36-h^2}{4} h$, which is a polynomial, and therefore more convenient.

Now $V = \pi \frac{36-h^2}{4} h$, or rather $V = \frac{\pi}{4}(36h - h^3)$. Setting the derivative equal to zero ($0 = 36 - 3h^2$) and solving gives $h = \sqrt{12}$ (discard the negative solution, because h is a length). Since $r^2 = (36 - h^2)/4$, $r = \sqrt{6}$.

Now the big finish.... "Among cylinders with diagonal measure 6 units, the one with largest volume is the one with height $\sqrt{12}$ units and radius $\sqrt{6}$ units."

Thank you and good night. -jason