

MATH 106 — SECOND EXAM

DEPARTMENT OF MATHEMATICS
Johns Hopkins University

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NAME: _____

SIGNATURE: _____

SECTION NUMBER: _____

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1. This exam has seven pages including this cover. There are six questions.
2. Use of books, notes, or scratch paper is not allowed. You may certainly use a calculator (but not its manual).
3. **Show all of your work!** Partial credit is available for many problems but can only be given if the graders understand your work. Be sure to explain your reasoning carefully. Include units in your answers whenever appropriate.
4. Read directions carefully. For some problems, a brief answer is sufficient, but others require you to show all work or give explanations.

PROBLEM	POINTS	SCORE
1	20	20
2	20	20
3	16	16
4	15	15
5	14	14
6	15	15
TOTAL	100	100

1. Three unrelated questions.

a. **TRUE** or **FALSE**: (4 pts) Every continuous function $f(x)$ has two or more antiderivatives.

(Show no work on True/False. Only the answer is graded!)

True. Every continuous function has an antiderivative. But “+C” gives us infinitely many more. So certainly “two or more.”

b. (8 pts) State each of the fundamental theorems of Calculus. Please use brief, symbolic, mathematical notation (as opposed to words and phrases) where appropriate.

First Fundamental Theorem of Calculus:

$$\frac{d}{dx} \int_a^x f(t)dt = f(x).$$

Second Fundamental Theorem of Calculus:

$$\int_a^b f(x)dx = F(b) - F(a),$$

where $F' = f$. (Equivalently, F is the antiderivative of f .)

c. (8 pts) Find the approximate value of the sum below, by any method other than adding up all the terms on your calculator! *Show your work to indicate your method.*

$$\frac{1}{1000} \left[\cos\left(\frac{1}{1000}\right) + \cos\left(\frac{2}{1000}\right) + \cos\left(\frac{3}{1000}\right) + \cos\left(\frac{4}{1000}\right) + \dots + \cos\left(\frac{1000}{1000}\right) \right]$$

Tricky. This is a big giant Riemann sum, with 1000 subdivisions. Note the factor of $\frac{1}{1000}$ out front, which is the width of each rectangle. The function involved is $\cos(x)$, and the endpoints seem to be 0 and 1, judging from the first and last evaluations. Therefore, this is a Riemann sum approximation to $\int_0^1 \cos(x)dx$. But we can easily just calculate the integral, which must be very close indeed to the Riemann sum. The integral is $\sin(1) - \sin(0) \approx .84$.

2. (20 pts) You are planning your next modern art masterpiece, titled “Red with Frame.” You will paint a red rectangle of area 1200 square inches, and attach a frame. The frame will be four inches wide along the sides of the red rectangle, but only three inches wide on the top and bottom. What is the smallest possible area of the whole project, including frame? *Show all work—be thorough!*

Let’s call the width and height of the red rectangle x and y . Then we get the quick equation $xy = 1200$. This is the constraint equation. We want to minimize the total area (with frame), so we have to write an equation for this area. It is $A = (x + 8)(y + 6)$. This is the priority equation, which, in typical inconvenience, depends on both x and y . Eliminating $y = 1200/x$ yields

$$A = (x + 8)\left(\frac{1200}{x} + 6\right).$$

Now we must optimize this equation of x . Before taking the derivative, it's convenient to multiply out the polynomial: $A = 1200 + \frac{9600}{x} + 6x + 48$. Now we differentiate and set the derivative equal to zero, in order to locate the critical points: $0 = 0 + \frac{-9600}{x^2} + 6 + 0$. Solving for x gives $x = \sqrt{9600/6} = 40$. Circle this as the "answer" if you don't need the last four points.

In all probability, $x = 40$ and $y = 1200/40 = 30$ are the correct dimensions, and the answer to the problem. But to be careful, we must do the second derivative test and check endpoints. We have the first derivative, above: $dA/dx = \frac{-9600}{x^2} + 6$. Differentiating once more gives $19200/x^3$, which is positive. This indicates that A is concave up, as a function of x , which in turn causes our critical point ($x = 40$) to be a local min. That's strong evidence that it's the global min we're looking for.

Just to be sure, let's check endpoints. As x approaches 0 (the smallest conceivable value), $A = (x + 8)(\frac{1200}{x} + 6)$ approaches $+\infty$, which rules out that endpoint (remember we're looking for a global *min*, so $+\infty$ is not a candidate). As x approaches ∞ (the largest conceivable value), A approaches infinity again, so that's no good. We have ruled out the endpoints, and can confidently say...

The smallest possible total area of the project occurs when $x = 40$, $y = 30$, and this area is $48 * 36 = 1728$ square inches.

3. (16 pts) In this problem, you will "set up" but not completely solve several problems related to one curve with given endpoints. For each part, write one ("grammatically correct") integral representing the quantity sought. You *need not* compute the values of these integrals. There is no need to show any work.

$$\text{The Curve: } y = g(x) = x^3, \quad 1 \leq x \leq 2$$

a. (4 pts) The area under the curve (and above the x -axis).

$$\int_1^2 x^3 dx$$

b. (4 pts) The volume enclosed when the curve is rotated around the x -axis.

$$\int_1^2 \pi(x^3)^2 dx$$

c. (4 pts) The volume enclosed when the curve is rotated around the y -axis.

$$\int_{y=1}^{y=8} (y^{1/3})^2 dy$$

d. (4 pts) The length of the curve.

$$\int_1^2 \sqrt{1 + (3x^2)^2} dx$$

4. Calculate these limits. Show your work, because credit will not be given for calculator shortcuts.

a. (5 pts) $\lim_{x \rightarrow 0} x \ln(x)$

First, plug in $x = 0$ to find out whether this is an indeterminate form. Since $\lim_{x \rightarrow 0} (\ln(x)) = -\infty$ this produces $0 \cdot \infty$, a bona fide indeterminate form. We *must* convert this to fractional form to use L'Hospital's rule, and the easiest way to do that is to move the x to the denominator, thus inverting it:

$$= \lim_{x \rightarrow 0} \frac{\ln(x)}{1/x}$$

Differentiate top and bottom, yielding $\lim_{x \rightarrow 0} \frac{1/x}{-1/x^2}$, which mess can be simplified to $\lim_{x \rightarrow 0} -x$, which is 0.

b. (5 pts) $\lim_{x \rightarrow \pi} \frac{\sin(x)}{x}$

This is a shameless trick. It looks like a L'Hospital's rule problem, but it's not because plugging in $x = \pi$ doesn't give an indeterminate form. Actually, it gives $\frac{0}{\pi}$, which is simply 0. So that's the answer.

c. (5 pts) $\lim_{x \rightarrow 0} \frac{1}{x}(1 - 2^x)$

The $\frac{1}{x}$ out front threw some people off, but it means the same thing as $\frac{1-2^x}{x}$. Plugging in $x = 0$ gives $\frac{1-1}{0}$, which is the $0/0$ form, so L'Hospital's rule applies, and gives:

$$\lim_{x \rightarrow 0} \frac{-\ln(2)2^x}{1}$$

Plugging in $x = 0$ once more gives $-\ln(2)$, which is not indeterminate, so it's the answer. Box it, and move on!

5. (14 pts) Sketch a graph of a single function $f(x)$ for which all of the following statements are true:

i. For $x > 2$, $f'(x) > 0$, but for $x < 2$, $f'(x) < 0$.

ii. For $x > 4$, $f''(x) < 0$, but for $x < 4$, $f''(x) > 0$.

iii. $\lim_{x \rightarrow \infty} f(x) = 0$.

It's difficult to doodle electronically, but here's what the graph should look like. Based on the data above, it should start out decreasing and concave up, hit a local min at $x = 2$, turn around and increase, still concave up, go through an inflection point at $x = 4$ (still increasing), then continue increasing, but with a concave down trend, as it asymptotically approaches the x -axis. In other words, this picture:

6. (15 pts) Even after an organism's death, its cells continue to burn oxygen at a decreasing rate. Assume that the rate of oxygen consumption after t hours is $3 - t/4$ grams per hour. How much oxygen is consumed in the three hours immediately following death?

The key idea here is that the rate of oxygen consumption can be *integrated* to produce the total amount of oxygen consumed during the time period. The integral of a rate gives the total change!

Therefore, setting up this problem is as simple as writing $\int_0^3 (3 - \frac{t}{4}) dt$. Solving the integral gives $(3t - \frac{t^2}{8})|_0^3 = 9 - \frac{9}{8} = 7.875$. The units are *grams*. This is the total number of grams of oxygen consumed in the first three hours.