DRP Writeup

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What is the calculus of variations?

When we are first introduced to some of the most fundamental PDE, such as the heat equation or wave equation, we can use the structure of those PDE to develop techniques to solve them, such as separation of variables, Fourier transform methods, or others. However, most PDE are much more complicated. Consider, for example, a PDE of the form:

$$A[u] = 0$$

$$A : W^{1,p}(U) \to L^2(U)$$

where A is a nonlinear, partial differential operator from some function space to another function space. There is no evident way of solving a PDE of such a form, so the goal is to transform this complicated PDE problem into one that we can solve.

The inspiration for the calculus of variations comes from single variable calculus. If we are trying to find a minimizer of a function $f(x)$ on some domain $D \subset \mathbb{R}$, we know how to solve this - we find the critical points of f by checking for where the first derivative is 0. Furthermore, if our domain D is compact, or if our function f satisfies further requirements (such as a second derivative test or other asymptotic behavior), we know that this local minimum is a global minimizer for f.

This fundamental technique is the underpinning of calculus of variations - we will assume that our nonlinear PDE operator $A[u] = I'[u]$ is the "derivative" in some sense of some functional I, which we will call the energy functional. Places where $I'[u] = 0 \implies A[u] = 0$, so if u minimizes I, one would hope it would be a solution to the PDE.

As an example, consider the Laplace equation on a domain U:

$$\Delta u = 0, u = g \text{ on } \partial U$$

One can define the energy functional of the Laplace equation (i.e. how much the function fluctuates) by:

$$I[w] := \int_U \frac{1}{2}|Dw|^2 dx$$

Using integration by parts and Cauchy-Schwarz, one can actually prove Dirichlet's principle, which states that a function u satisfies Laplace's equation if and only if it is a minimizer of the functional I. This was my first introduction to variational calculus problems - similar functionals exist for the heat and wave equation.

Now, the energy functional of the Laplace can be derived from the Euler-Lagrange equations above, and it turns out that minimizing the functional provides us with a solution. However, this is not always the case - any energy functional is not guaranteed to have a minimizer - and thus we will need to put conditions on I.

Thinking back to calculus 1, we might have a function that does not attain its infimum. For example, consider $f(x) = \frac{1}{1+x^2}$ on the whole real line. $\inf_{x\in\mathbb{R}} f = 0$, but f never attains 0. One way we could deal with this is by demanding that our domain is compact, but this is not going to always be the case for a general non-linear PDE. Thus, we introduce a "coercivity," or rapid growth, condition. That is, we want our functional $I[w] \to \infty$ as $\|Dw\|_{L^p} \to \infty$ for all functions w in the space we are interested in. This means our functionals all blow up as our derivative gets large. The equivalent calculus 1 formulation would be forcing $f'(x) \to \infty$ as $|x| \to \infty$.

This condition is not enough, because we need a sense of "compactness" to guarantee convergence to an actual minimizer (since one can think about a minimizer as the limit of a sequence). This will happen in a compact subspace of $\mathbb{R}$, but it will not happen in an infinite dimensional space such as $W^{1,p}(U)$. To combat this, we require that I satisfies a lower semi-continuity property - that is:

$$I[u] \leq \liminf_{k\to\infty} I[u_k] \text{ if } u_k \rightharpoonup u \text{ weakly}$$

That is, if we have weak convergence of a sequence to a minimzer u, $I[u]$ eventually is below $I[u_k]$. This is a difficult condition to prove - a sufficient condition is convexity of I in the derivative $|Dw|$.

If a functional I satisfies both coercivity and lower semi-continuity conditions, it is guaranteed to have a minimizer. This is proven in large part by using Sobolev inequalities, including Poincare's. To get uniqueness of a minimizer, we need a few additional constraints - our functional cannot depend on w (only can depend on Dw and x) and we need strict convexity instead of convexity.

An Example

We can't always use the techniques above to prove a minimizer exists. For example, the minimial surface equation has the energy functional:

$$I[w] = \int_U (1 + |Dw|^2)^{\frac{1}{2}} \, dx$$

This only satisfies the coercivity equation for the $L^1(U)$ space. To see this, note that we need:

$$I[w] \geq \alpha \|Dw\|^q_{L^q(U)} .$$

But this becomes a problem quickly by asking what happens as $|Dw| \to \infty$:

$$\lim_{|Dw| \to \infty} \frac{\left(1 + |Dw|^2\right)^{\frac{1}{2}}}{|Dw|^q} = 0$$

if $q > 1$. However, we need $\alpha > 0$, so the minimal surface equation does not satisfy the coercivity requirements except in the L^1 norm sense.