$SL_2(\mathbb{Z})$

Sally(Tingting) Bao

Mentor: Xiyuan Wang

May 1st, 2019

Table of Contents

1. Basic Definition
2. Group Action of $\mathrm{SL}_2(\mathbb{Z})$ on the upper half plane
3. Generators of $\mathrm{SL}_2(\mathbb{Z})$
4. Fundamental Domain
5. Summary

Basic Definitions

Definition (Group Action)

A **group action** of G on a set S is a map $\mathrm{G} \times \mathrm{S} \longrightarrow \mathrm{S}$ which satisfies the following axioms:

a. For any $s \in S$, $e \cdot s = s$.

b. For any $g, g^{'} \in G$ and $s \in S$, $(gg') \cdot s = g \cdot (g' \cdot s)$.

Definition (Orbit)

Let group G act on a set S. For any $x \in X$, the **orbit** of x is the set

$$\mathrm{O}_x = \{y \in X \mid \exists\, g \in G \ s.t. \ g \cdot x = y\}.$$

Definition (Stabilizer)

Let group G act on a set S. For any $x \in X$, the **stabilizer** of x is the subgroup of G

$$\mathrm{G}_x = \{g \in G \mid g \cdot x = x\}.$$

Group Action of $\mathrm{SL}_2(\mathbb{Z})$ on the upper half plane

- The **special linear group**,

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 \right\}.$$

Group Action of $\mathrm{SL}_2(\mathbb{Z})$ on the upper half plane

- The **special linear group**,

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 \right\}.$$

- The **'upper half plane'**,

$$\bar{\mathcal{H}} = \mathcal{H} \cup \mathbb{Q} \cup \{\infty\},\ \text{where } \mathcal{H} = \{x + iy \mid y > 0\}.$$

Group Action of $\mathrm{SL}_2(\mathbb{Z})$ on the upper half plane

- The **special linear group**,

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 \right\}.$$

- The **'upper half plane'**,

$$\bar{\mathcal{H}} = \mathcal{H} \cup \mathbb{Q} \cup \{\infty\},\ \text{where } \mathcal{H} = \{x + iy \mid y > 0\}.$$

- Now, we can define an action of $\mathrm{SL}_2(\mathbb{Z})$ on $\bar{\mathcal{H}}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az + b}{cz + d},\ z \in \bar{\mathcal{H}}.$$

Group Action of $\mathrm{SL}_2(\mathbb{Z})$ on the upper half plane

- To check this is a group action,
  - (a) If $\mathrm{Im}(z) > 0$, then $\mathrm{Im}\left(\frac{az+b}{cz+d}\right) = \frac{ad-bc}{|cz+d|^2}\,\mathrm{Im}(z) = \frac{1}{|cz+d|^2}\,\mathrm{Im}(z) > 0$.
  - (b) $\mathrm{SL}_2(\mathbb{Z}) \times \bar{\mathcal{H}} \longrightarrow \bar{\mathcal{H}}$ is a group action.

    Let $z \in \bar{\mathcal{H}}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.
    - (i) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot z = z$
    - (ii) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} e & f \\ g & h \end{pmatrix} \cdot z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \left(\frac{ez+f}{gz+h}\right) =$
      $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \left(\begin{pmatrix} e & f \\ g & h \end{pmatrix} \cdot z\right)$

Generators of $\mathrm{SL}_2(\mathbb{Z})$

- $$\mathrm{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathrm{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

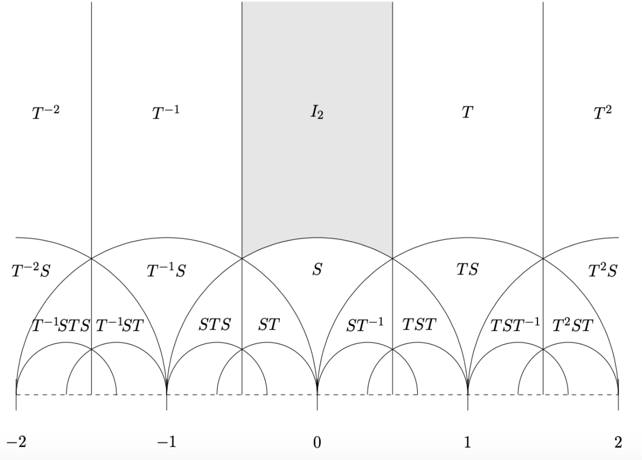
Generators of $\mathrm{SL}_2(\mathbb{Z})$

- $$\mathrm{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathrm{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
- The order of S is **4** and the order of T is **infinite**.

Generators of $\mathrm{SL}_2(\mathbb{Z})$

Theorem

$$\mathrm{SL}_2(\mathbb{Z}) = \langle \mathrm{S}, \mathrm{ST} \mid \mathrm{S}^4 = (\mathrm{ST})^6 = \mathbf{e} \rangle$$



Generators of $\mathrm{SL}_2(\mathbb{Z})$

Example

Express $\mathrm{A} = \begin{pmatrix} -5 & -7 \\ 3 & 4 \end{pmatrix}$ in terms of S and T.

Fundamental Domain $\mathcal{F}$

Suppose we have a group action of G on S.

Definition

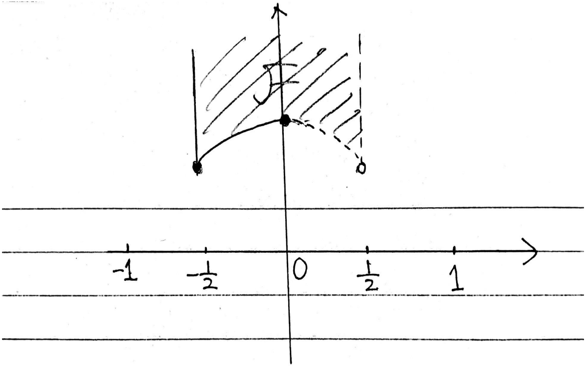
A **fundamental domain** $\mathcal{F}$ is a subset of S which contains **exactly** one point of each these orbits.

Fundamental Domain $\mathcal{F}$

Back to our example that the group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\overline{\mathcal{H}}$, we claim

$$\mathcal{F} = \{\infty\} \cup \{z \in \bar{\mathcal{H}} \mid -\frac{1}{2} \leq \mathrm{Re}(z) < \frac{1}{2} \text{ if } |z| > 1;\ \mathrm{Re}(z) \leq 0 \text{ if } |z| = 1\}$$

is a fundamental domain.



Fundamental Domain $\mathcal{F}$

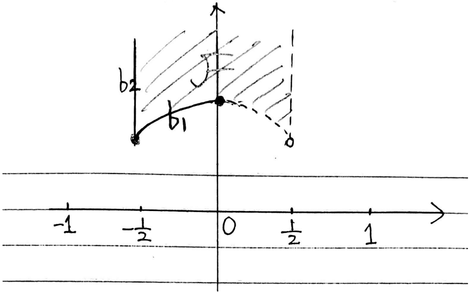
- **Q:** How to prove $\mathcal{F}$ is a fundamental domain?
  - (a) Show that, for any $z \in \overline{\mathcal{H}}$, there exists a point $x \in \mathrm{O_z}$, such that $x \in \overline{\mathcal{H}}$.
  - (b) Show that, for any $z_1, z_2 \in \mathcal{F}$, $\mathrm{O_{z_1}} \neq \mathrm{O_{z_2}}$.

Let us sketch the proof of part (b).

Goal: show that **no** such $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ s.t.

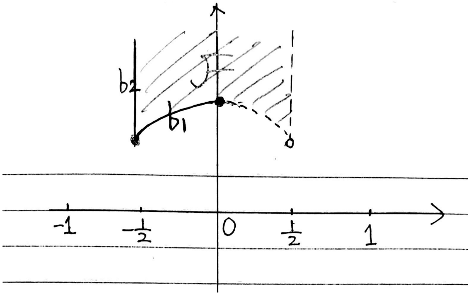
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z_1 = z_2 \text{ where } z_1, z_2 \in \mathcal{F}$$

Fundamental Domain $\mathcal{F}$



- **Case 1:** $z_1, z_2 \in \mathcal{F} \setminus (b_1 \cup b_2)$
- **Case 2:** $z_1 \in \mathcal{F},\ z_1 \in b_1$
- **Case 3:** $z_1 \in \mathcal{F},\ z_1 \in b_2$
- **Case 4:** $z_1 \in b_1,\ z_2 \in b_2$

Fundamental Domain $\mathcal{F}$



- **Case 1:** $z_1, z_2 \in \mathcal{F} \setminus (b_1 \cup b_2)$
- **For sake of Contradiction:**

$$\mathrm{Re}(z_1) - \mathrm{Re}(z_2) \geq 1$$

Cool Topological Result