Tensors In 10 Minutes Or Less

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Under the esteemed mentorship of Apurva Nakade

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JHU Math Directed Reading Program

Why tensors are cool, kids

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Theorem

If there is a relationship between two tensor fields in one coordinate system, that relationship holds in certain other coordinate systems

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If there is a relationship between two tensor fields in one coordinate system, that relationship holds in certain other coordinate systems

This means the laws of physics can be expressed as relationships between tensors!

What tensors look like

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Example

Einstein's field equations govern general relativity,

$$\underbrace{G_{\mu\nu}}_{\binom{0}{2}\text{ tensor}} = \frac{8\pi G}{c^4} \underbrace{T_{\mu\nu}}_{\binom{0}{2}\text{ tensor}}$$

The Dual Space

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Covectors form the *dual space* V^*

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$$\begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 7 \\ 11 \\ 13 \end{pmatrix} = (0)7 + 1(11) + (0)13 = 11$$

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A *multilinear* map is linear in each argument, i.e.

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$$t(..., u + \lambda w, ...) = t(..., u, ...) + \lambda t(..., u, ...)$$

We've already seen multilinear maps:

$$\begin{aligned} \text{Covectors, } \alpha : V &\mapsto \mathbb{R} \\ u &\rightarrow \langle \alpha, u \rangle \\ \text{Vectors, } v : V^* &\mapsto \mathbb{R} \\ \alpha &\rightarrow \langle \alpha, u \rangle \\ \text{(Think of scalars like } a : \emptyset &\mapsto \mathbb{R} \\ &\rightarrow a) \end{aligned}$$

Examples of tensors

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A *tensor* of type $\binom{p}{q}$ is a multilinear map

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- a $\binom{0}{1}$ tensor $t : V \to \mathbb{R}$ is a covector
- a $\binom{0}{0}$ tensor $t : \to \mathbb{R}$ is a scalar

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Definition

These values are called *components*

$$t^{c,\dots,d}_{a,\dots,b} := t(e_a, \dots, e_b; e^c, \dots, e^d)$$

Representing tensors with components

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$\binom{1}{2}$ tensor ???

Tensor fields

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A *tensor field* of type $\binom{p}{q}$ assigns a tensor of type $\binom{p}{q}$ to every point in a manifold (more general kind of mathematical space than Euclidean space)[1]

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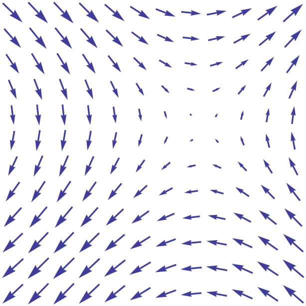

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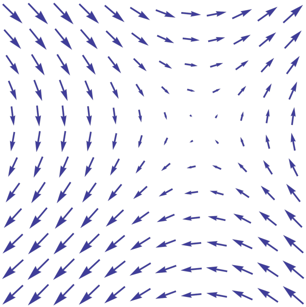

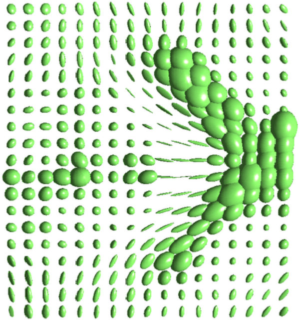

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References

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- https://en.wikipedia.org/wiki/Vector_field
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