An Introduction to Nonstandard Analysis

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Directed Reading Program Fall 2018
December 5, 2018

Motivation

- Developing/Understanding Differential and Integral Calculus using infinitely large and small numbers
- Provide easier and more intuitive proofs of results in analysis

Filters

Definition

Let I be a nonempty set. A filter on I is a nonempty collection $F \subseteq P(I)$ of subsets of I such that:

- If $A, B \in F$, then $A \cap B \in F$.
- If $A \in F$ and $A \subseteq B \subseteq I$, then $B \in F$.

F is proper if $\emptyset \notin F$.

Definition

An ultrafilter is a proper filter such that for any $A \subseteq I$, either $A \in F$ or $A^c \in F$. $F^i = \{A \subseteq I : i \in A\}$ is called the principal ultrafilter generated by i.

Filters

Theorem

Any infinite set has a nonprincipal ultrafilter on it.

Pf: Zorn's Lemma/Axiom of Choice.

The Hyperreals

Let $\mathbb{R}^{\mathbb{N}}$ be the set of all real sequences on $\mathbb{N}$, and let F be a fixed nonprincipal ultrafilter on $\mathbb{N}$. Define an (equivalence) relation on $\mathbb{R}^{\mathbb{N}}$ as follows:

$\langle r_n \rangle \equiv \langle s_n \rangle$ iff $\{n \in \mathbb{N} : r_n = s_n\} \in F$.

One can check that this is indeed an equivalence relation. We denote the equivalence class of a sequence $r \in \mathbb{R}^{\mathbb{N}}$ under $\equiv$ by $[r]$. Then

$$^*\mathbb{R} = \{[r] : r \in \mathbb{R}^{\mathbb{N}}\}.$$

Also, we define

$$[r] + [s] = [\langle r_n + s_n \rangle]$$
$$[r] * [s] = [\langle r_n * s_n \rangle]$$

The Hyperreals

We say $[r] = [s]$ iff $\{n \in \mathbb{N} : r_n = s_n\} \in F$. $<$ is defined similarly. A subset A of $\mathbb{R}$ can be enlarged to a subset ${}^*A$ of ${}^*\mathbb{R}$, where

$$[r] \in {}^*A \iff \{n \in \mathbb{N} : r_n \in A\} \in F.$$

Likewise, a function $f : \mathbb{R} \to \mathbb{R}$ can be extended to ${}^*f : {}^*\mathbb{R} \to {}^*\mathbb{R}$, where

$${}^*f([r]) := [\langle f(r_1), f(r_2), ...\rangle]$$

The Hyperreals

A hyperreal b is called:

- limited iff $|b| < n$ for some $n \in \mathbb{N}$.
- unlimited iff $|b| > n$ for all $n \in \mathbb{N}$.
- infinitesimal iff $|b| < \frac{1}{n}$ for all $n \in \mathbb{N}$.
- appreciable iff $\frac{1}{n} < |b| < n$ for some $n \in \mathbb{N}$.

Transfer Principle

Statement: A defined $\mathcal{L}_{\mathcal{R}}$ sentence ϕ is true iff ${}^*\phi$ is true.

Examples:

$$\forall x, y \in \mathbb{R}, x < y \Rightarrow \exists q \in \mathbb{Q}(x < q < y).$$

gets transferred to

$$\forall x, y \in {}^*\mathbb{R}, x < y \Rightarrow \exists q \in {}^*\mathbb{Q}(x < q < y).$$

$$\forall x, y \in \mathbb{R}, \sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

gets transferred to

$$\forall x, y \in {}^*\mathbb{R}, {}^*\sin(x + y) = {}^*\sin(x){}^*\cos(y) + {}^*\cos(x){}^*\sin(y)$$

Shadows and Halos

We say a hyperreal b is infinitely close to hyperreal c if $b - c$ is infinitesimal and denote this by $b \simeq c$. One can show that $\simeq$ is an equivalence relation. We define

$$hal(b) = \{c \in {}^*\mathbb{R} : b \simeq c\}.$$

Theorem

Every limited hyperreal b is infinitely close to exactly one real number, called the shadow of b, denoted by $sh(b)$.

Convergence

Note that a real-valued sequence is a function from $\mathbb{N} \to \mathbb{R}$, so it extends to a hypersequence mapping ${}^*\mathbb{N} \to {}^*\mathbb{R}$.

Theorem

A real valued sequence $\langle s_n \rangle$ converges to $L \in \mathbb{R}$ iff $s_n \simeq L$ for all unlimited n.

Theorem

A real valued sequence $\langle s_n \rangle$ is Cauchy in $\mathbb{R}$ iff for all m,n unlimited hypernaturals, $s_m \simeq s_n$.

Using these concepts, we can prove that a real-valued sequence s convergent in $\mathbb{R} \Rightarrow s$ is Cauchy.

Convergence

Pf: Suppose $\langle s_n \rangle$ converges in $\mathbb{R}$. Then by the first theorem, $s_n \simeq L$ for all unlimited n. So for all l, m unlimited hypernaturals, $s_l \simeq L \simeq s_m \Rightarrow s_l \simeq s_m$ because $\simeq$ an equivalence relation. Then by the second theorem, $\langle s_n \rangle$ is Cauchy.

Continuity

Theorem

f is continuous at $c \in \mathbb{R}$ iff ${}^*f(x) \simeq {}^*f(c)$ for all $x \in {}^*\mathbb{R}$ such that $x \simeq c$.

Example: $f(c) = c^2$. Let c be real and $x \simeq c$. Then $x = c + \epsilon$ for some infinitesimal ϵ, and

$$\begin{aligned} f(x) - f(c) &= x^2 - c^2 \\ &= (c+\epsilon)^2 - c^2 \\ &= c^2 + 2\epsilon c + \epsilon^2 - c^2 \\ &= 2\epsilon c + \epsilon^2 \end{aligned}$$

which is infinitesimal because c is a real number and so it is limited. Thus, c^2 is continuous.

Another Application:

Theorem

Let f be a real function defined on some open neighborhood of $c \in \mathbb{R}$, and let ${}^*f$ be constant on $hal(c)$. Then f is constant on some open interval $(c - \epsilon, c + \epsilon) \subseteq \mathbb{R}$.

Pf: Note that for some positive infinitesimal d, we have the statement $\forall x \in {}^*\mathbb{R}$ such that $(|x - c|) < d$, ${}^*f(x) = {}^*f(c) = L$ for some L. This implies that $\exists y \in {}^*\mathbb{R}^+$, $\forall x \in {}^*\mathbb{R}$ such that $(|x - c|) < y$, ${}^*f(x) = {}^*f(c) = L \in {}^*\mathbb{R}$. By transfer, we have the sentence $\exists y \in \mathbb{R}^+$, $\forall x \in \mathbb{R}$ such that $(|x - c|) < y$, $f(x) = f(c) = L \in \mathbb{R}$. Thus, f is constant on the interval $(c - y, c + y) \subseteq \mathbb{R}$.

Differentiation

Theorem

If f is defined at $x \in \mathbb{R}$, then $L \in \mathbb{R}$ is the derivative of f at x iff for every nonzero infinitesimal ϵ, ${}^*f(x+\epsilon)$ is defined and $\frac{{}^*f(x+\epsilon) - {}^*f(x)}{\epsilon} \simeq L$.

Example: Consider the real-valued function $\sin(x)$, where $x \in \mathbb{R}$. Now consider $\frac{\sin(x+\epsilon) - \sin(x)}{\epsilon}$ for some ϵ an infinitesimal. Then by sum of sines, we get

$$\frac{\sin(x+\epsilon) - \sin(x)}{\epsilon} = \frac{\sin(x)\cos(\epsilon) + \cos(x)\sin(\epsilon) - \sin(x)}{\epsilon}$$

$\cos(x)$ is continuous, so $\cos(\epsilon) \simeq \cos(0) = 1$ and so $\sin(x)\cos(\epsilon) \simeq \sin(x)$. Thus,

$$\frac{\sin(x)\cos(\epsilon) + \cos(x)\sin(\epsilon) - \sin(x)}{\epsilon} \simeq \frac{\cos(x)\sin(\epsilon)}{\epsilon}$$

Also, $\sin(x)$ is continuous, so $\sin(\epsilon) \simeq \sin(0) = 0$, so $\sin(\epsilon) \simeq \epsilon$ and

$$\frac{\cos(x)\sin(\epsilon)}{\epsilon} \simeq \cos(x).$$

By the theorem, this implies that the derivative of $\sin(x)$ at $x \in \mathbb{R}$ is $\cos(x)$.

Overview

- The Transfer Principle is key.
- Nonstandard Analysis makes analysis easier!