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Multiplicative structure on Real Johnson-Wilson theory

Nitu Kitchloo, Vitaly Lorman, and W. Stephen Wilson

Abstract. We prove that the Real Johnson-Wilson theories $ER(n)$ are homotopy associative and commutative ring spectra up to phantom maps. We further show that the $ER(n)$ represent associatively and commutatively multiplicative cohomology theories on the category of (possibly non-compact) spaces. We also revisit a result of the first and third authors concerning the $MR(n)$-orientation of $\mathrm{MO}[2^{n+1}]$.

1. Introduction

At the prime 2, Johnson-Wilson theory $E(n)$ [**JW73**] is a complex-oriented cohomology theory which has a C_2-equivariant refinement, $\mathbb{E}(n)$ as a genuine C_2-equivariant spectrum, where the action of C_2 stems from complex conjugation. This was first constructed in [**HK01**], and Real Johnson-Wilson theory $ER(n)$ is defined to be the C_2-fixed points of $\mathbb{E}(n)$. The underlying nonequivariant spectrum of $\mathbb{E}(n)$ is Johnson-Wilson theory $E(n)$, and it is a homotopy associative, commutative, and unital ring spectrum. The goal of this note is to investigate whether the same properties hold for $\mathbb{E}(n)$ and $ER(n)$.

Interest in this problem comes from the fact that $ER(n)$ is quickly becoming a useful and computable cohomology theory. For $n = 1$ and 2, it reproduces familiar cohomology theories, $ER(1) = KO_{(2)}$ and $ER(2) = TMF_0(3)$ (the latter after suitable completion, see [**HM17**]). The $ER(n)$-cohomology of a large (and growing) collection of spaces has been computed: real projective spaces and their products [**KW08a**, **KW08b**, **Ban13**] (for $n = 2$), complex projective spaces [**Lor16**, **KLW17**], BO and some of its connective covers [**KW14**, **KLW17**], and half of all Eilenberg MacLane spaces [**KLW16**, **KLW17**]. Furthermore, these computations have applications. In [**KW08a**, **KW08b**], the first and third authors used computations in $ER(2)$-cohomology to prove new nonimmersion results for real projective spaces.

The existence of a multiplicative structure on Real Johnson-Wilson theory has been suggested in a comment in [**HK01**] (Comment 5 following the proof of Theorem 2.28) which claims that $\mathbb{E}(n)^\star(\mathbb{E}(n)\wedge\mathbb{E}(n))$ may be calculated and from this it may be shown that $\mathbb{E}(n)$ is a (homotopy) associative, commutative, and unital ring spectrum. The results in this note were born in the attempt to verify the above

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claim. Unfortunately, we were unsuccessful in doing so. However, we show that $\mathbb{E}(n)$ represents an $\mathbb{MU}$-algebra which is homotopy unital, associative, and commutative *up to phantom maps*. By a phantom map, we mean a map $f : X \longrightarrow Y$ which has trivial restriction to any finite CW complex mapping into X. In addition, we show that the $\mathbb{E}(n)$-cohomology of an equivariant topological space is canonically a commutative ring.

THEOREM 1.1. *$\mathbb{E}(n)$ is a homotopy commutative, homotopy associative, unital $\mathbb{MU}$-algebra up to phantom maps. In other words, there exist unit and multiplication maps:*

$$1 : \mathbb{MU} \longrightarrow \mathbb{E}(n), \qquad \hat{\mu} : \mathbb{E}(n) \wedge \mathbb{E}(n) \longrightarrow \mathbb{E}(n),$$

such that all the obstructions to $\hat{\mu}$ being a homotopy associative and homotopy commutative $\mathbb{MU}$-algebra structure are phantom maps. Differently said, all the corresponding structure diagrams commute up to phantom maps. Furthermore, the forgetful map:

$$\rho : \mathbb{E}(n)^0(\mathbb{E}(n) \wedge \mathbb{E}(n)) \longrightarrow E(n)^0(E(n) \wedge E(n)),$$

maps $\hat{\mu}$ to the canonical product μ on the non-equivariant Johnson-Wilson spectrum $E(n)$.

Theorem 1.1 tells us that the Real Johnson-Wilson theory is valued in commutative rings when applied to *finite* CW complexes. Our second result extends this to the category of all spaces.

THEOREM 1.2. *With any choice of multiplication $\hat{\mu}$ as above, the spectrum $\mathbb{E}(n)$ represents a multiplicative cohomology theory on the category of C_2-spaces valued in (bigraded) commutative rings. There are natural transformations of ring-valued cohomology theories $\mathbb{MU}^\star(-) \longrightarrow \mathbb{E}(n)^\star(-)$ and $\mathbb{E}(n)^\star(-) \longrightarrow E(n)^*(-)$.*

The results of this document justify the assumption of commutativity in the computations of the $ER(n)$-cohomology of topological spaces made by the authors in previous work. We conclude by revisiting a result of the first and third authors concerning the $MR(n)$-orientation of $\mathrm{MO}[2^{n+1}]$ to correct an error in the proof of [**KW14**, Theorem 1.4].

In the course of proving Theorem 1.2 we show that the infinite loop space underlying $ER(n)$, $\underline{ER(n)}_0$, is a homotopy commutative, associative, and unital H-ring space (Lemma 5.1). Applying the Bousfield-Kuhn functor shows that the $K(n)$-localization, $L_{K(n)}ER(n)$, is in fact a homotopy commutative, associative and unital ring spectrum (not just up to phantom maps). The authors are grateful to Tyler Lawson for pointing this out.

Hahn and Shi [**HS17**] have recently proved that Johnson-Wilson theory $E(n)$ admits an A_∞ ring structure for which the C_2-action is given by an A_∞ involution (but not necessarily an A_∞ ring automorphism). The question of (homotopy) commutativity is not readily addressed by their techniques. They show that these issues resolve on $K(n)$-localization giving rise to an E_∞ ring spectrum structure on $L_{K(n)}ER(n)$.

The present paper shows a (homotopy) commutative and associative ring structure up to phantom maps, but to the authors' present knowledge, ER(n) is not (yet) known to be a homotopy commutative and associative ring spectrum.

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2. Background

In this section, we recall a few background definitions and theorems from [**KW07b**] and [**KLW16**] that we use in subsequent sections.

A genuine C_2-equivariant spectrum $\mathbb{E}$ is a family of C_2-spaces $\underline{\mathbb{E}}_{a+b\alpha}$, indexed over elements $a + b\alpha \in RO(C_2)$ where α denotes the sign representation, together with a compatible system of equivariant homeomorphisms

$$\underline{\mathbb{E}}_{a-r+(b-s)\alpha} \xrightarrow{\simeq} \Omega^{r+s\alpha}\underline{\mathbb{E}}_{a+b\alpha}$$

where the right hand side denotes the space of pointed maps (endowed with the conjugation action) from the one point compactification of the representation $r + s\alpha$. The reader may refer to [**HK01**] for more details on C_2-spectra. We will denote by ER the homotopy fixed point spectrum of the C_2-action on $\mathbb{E}$ and by E the underlying nonequivariant spectrum given by forgetting the C_2-action. An example of a C_2- spectrum of interest to us is $\mathbb{MU}$, whose underlying nonequivariant spectrum is complex cobordism, MU, studied first by Landweber [**Lan68**], Fujii [**Fuj76**], Araki and Murayama [**AM78**], and more recently by Hu-Kriz [**HK01**]. The action of C_2 is induced by the complex conjugation action on the pre-spectrum representing $\mathbb{MU}$ in the usual way (see, e.g. [**HK01**]).

The $p = 2$ Johnson-Wilson theory $E(n)$ lifts to a C_2-spectrum, $\mathbb{E}(n)$, defined as an $\mathbb{MU}$-module by coning off certain equivariant lifts of the Araki generators v_i for $i > n$, and then inverting the lift of v_n. We shall call these equivariant lifts by the same names, v_i. The Real Johnson-Wilson theories, $ER(n)$, are defined as the fixed points of $\mathbb{E}(n)$.

Working with cohomological grading, let Y be a C_2-space and let $\mathbb{E}^{*(1+\alpha)}(Y)$ denote the subgroup of diagonal elements in the equivariant $\mathbb{E}$-cohomology of Y i.e.

$$\mathbb{E}^{*(1+\alpha)}(Y) := \pi_0 \mathrm{Maps}^{C_2}(Y, \underline{\mathbb{E}}_{*(1+\alpha)}).$$

Consider the group homomorphism given by forgetting the C_2-action:

$$\rho : \mathbb{E}^{*(1+\alpha)}(Y) \longrightarrow E^*(Y).$$

Notice that the image of ρ belongs to the graded sub-group of elements in even degree. We recall the following definitions from [**KW07b**].

DEFINITION 2.1. A C_2-space Y is said to have the weak projective property with respect to a C_2-spectrum $\mathbb{E}$ if the map

$$\rho : \mathbb{E}^{*(1+\alpha)}(Y) \longrightarrow E^{2*}(Y),$$

is an isomorphism of graded abelian groups.

Note that in the case that the C_2-spectrum $\mathbb{E}$ is an $\mathbb{MU}$-module spectrum, the map ρ in the definition above is a map of $\mathbb{MU}^{*(1+\alpha)} \cong MU^{2*}$-modules by virtue of the fact that the $\mathbb{MU}$-module structure on $\mathbb{E}$ forgets to an MU-module structure on E.

We will make extensive use of spaces with the weak projective property. In order to recognize such spaces, we need some auxiliary definitions.

DEFINITION 2.2. A pointed C_2-space X is said to be projective if

(1) $H_*(X;\mathbb{Z})$ is of finite type.
(2) X is homeomorphic to $\bigvee_I (\mathbb{C}P^\infty)^{\wedge k_I}$ for some weakly increasing sequence of integers k_I, with the C_2 action given by complex conjugation.

By a C_2-equivariant H-space, we shall mean an H-space whose multiplication map is C_2-equivariant.

DEFINITION 2.3. A C_2-equivariant H-space Y is said to have the projective property if there exists a projective space X, along with a pointed C_2-equivariant map $f : X \longrightarrow Y$, such that $H_*(Y;\mathbb{Z}/2)$ is generated as an algebra by the image of f.

The following theorem, proved in [**KLW16**, Theorem 2.6], establishes that spaces with the projective property have the weak projective property with respect to certain $\mathbb{MU}$-module spectra (in particular, $\mathbb{E}(n)$).

THEOREM 2.4. *Let Y be a C_2-equivariant H-space with the projective property. Let $\mathbb{E}$ denote any complete $\mathbb{MU}$-module spectrum with underlying spectrum E, satisfying the property that the forgetful map: $\rho^* : \mathbb{E}^{*(1+\alpha)} \longrightarrow E^*$, is an isomorphism. Then the space Y has the weak-projective property with respect to $\mathbb{E}$. In other words, the following map is an isomorphism of $\mathbb{MU}^{*(1+\alpha)}$-modules:*

$$\rho : \mathbb{E}^{*(1+\alpha)}(Y) \longrightarrow E^*(Y).$$

REMARK 2.5. The smash product of a finite collection of spaces with the projective property is an example of a space that has the weak projective property with respect to any $\mathbb{E}$ as in Theorem 2.4, but not the projective property (since it is not an H-space). This follows from writing $Y_1 \wedge Y_2 = (Y_1 \times Y_2)/(Y_1 \vee Y_2)$ and the Five Lemma.

There are many examples of spaces with the projective property. The ones of interest in this document will be $\underline{\mathbb{E}(n)}_0$ and its products.

LEMMA 2.6. $\underline{\mathbb{E}(n)}_0^{\times j} \times \underline{\mathbb{MU}}_{i(2^n-1)(1+\alpha)}^{\times s}$ *has the weak projective property with respect to $\mathbb{E}(n)$ for all $i, j, s \geq 1$.*

PROOF. By [**KW13**, Theorem 1-4], the spaces $\underline{\mathbb{E}(n)}_0$ and $\underline{\mathbb{MU}}_{(i(2^n-1)(1+\alpha)}$ have the projective property. Thus $\underline{\mathbb{E}(n)}_0^{\times j} \times \underline{\mathbb{MU}}_{i(2^n-1)(1+\alpha)}^{\times s}$ is a restricted product of a family of spaces with the projective property (i.e. it is the colimit of finite products of spaces with projective property). A finite product of spaces with projective property evidently has the weak projective property, and since $\rho : \mathbb{E}(n)^{*(1+\alpha)}(-) \longrightarrow E(n)^{2*}(-)$ is an isomorphism at each stage, it follows that it is an isomorphism in the limit. $\square$

3. A stable multiplicative structure

We begin with the observation that the spaces $\underline{\mathbb{E}(n)}_0$ are $|v_n| = (2^n-1)(1+\alpha)$-periodic. That is, the adjoint of the multiplication-by-v_n map on the spectrum $\mathbb{E}(n)$ induces, on the 0-space level, an equivalence

$$\underline{\mathbb{E}(n)}_0 \xrightarrow{\simeq} \Omega^{|v_n|}\underline{\mathbb{E}(n)}_0$$

We now express the spectrum $\mathbb{E}(n)$, and its products as colimits of shifted suspension spectra

$$\mathbb{E}(n) = \text{colim }_m \Sigma^{-m(2^n-1)(1+\alpha)}\underline{\mathbb{E}(n)}_0, \quad \mathbb{E}(n)^{\wedge k} = \text{colim }_m \Sigma^{-mk(2^n-1)(1+\alpha)}\underline{\mathbb{E}(n)}_0^{\wedge k},$$

where the maps are given by successive $v_n^{\wedge k}$ multiplications

$$\Sigma^{-mk|v_n|}\underline{\mathbb{E}(n)}_0^{\wedge k} \longrightarrow \Sigma^{-(m+1)k|v_n|}\underline{\mathbb{E}(n)}_0^{\wedge k}.$$

Applying $\mathbb{E}(n)$ cohomology, Milnor's $\lim^1$-sequence gives us

$$0 \longrightarrow \lim{}^1\mathbb{E}(n)^{-1}(\Sigma^{-mk|v_n|}\underline{\mathbb{E}(n)}_0^{\wedge k}) \longrightarrow \mathbb{E}(n)^0(\mathbb{E}(n)^{\wedge k}) \longrightarrow \lim{}^0\mathbb{E}(n)^0(\underline{\mathbb{E}(n)}_0^{\wedge k}) \longrightarrow 0.$$

Using the $|v_n|$-periodicity of $\mathbb{E}(n)$ noted above to identify $\Sigma^{-|v_n|}\underline{\mathbb{E}(n)}_0$ with $\underline{\mathbb{E}(n)}_0$, we have the sequence

$$0 \longrightarrow \lim{}^1\mathbb{E}(n)^{-1}(\underline{\mathbb{E}(n)}_0^{\wedge k}) \longrightarrow \mathbb{E}(n)^0(\mathbb{E}(n)^{\wedge k}) \longrightarrow \lim{}^0\mathbb{E}(n)^0(\underline{\mathbb{E}(n)}_0^{\wedge k}) \longrightarrow 0.$$

We may now invoke the weak projective property of the spaces $\underline{\mathbb{E}(n)}_0^{\wedge k}$ (see Remark 2.5) and identify the last term with $\lim^0 E(n)^0(\underline{E(n)}_0^{\wedge k})$. The space $\underline{E(n)}_0$ has evenly graded cohomology (since it has the projective property) and homotopy; it follows from the Atiyah-Hirzebruch spectral sequence that the $E(n)$-cohomology of $\underline{E(n)}_0$ is evenly graded as well (as are its k-fold smash products). Thus, in the analogous non-equivariant Milnor sequence

$$0 \longrightarrow \lim{}^1 E(n)^{-1}(\underline{E(n)}_0^{\wedge k}) \longrightarrow E(n)^0(E(n)^{\wedge k}) \longrightarrow \lim{}^0 E(n)^0(\underline{E(n)}_0^{\wedge k}) \longrightarrow 0$$

the $\lim^1$ term vanishes. Consequently, we identify $\lim^0 E(n)^0(\underline{E(n)}_0^{\wedge k})$ with $E(n)^0(\underline{E(n)}_0)$ and obtain our short exact sequence of interest

$$0 \longrightarrow \lim{}^1\mathbb{E}(n)^{-1}(\underline{\mathbb{E}(n)}_0^{\wedge k}) \longrightarrow \mathbb{E}(n)^0(\mathbb{E}(n)^{\wedge k}) \longrightarrow E(n)^0(E(n)^{\wedge k}) \longrightarrow 0, \tag{3.1}$$

with the last map being ρ.

By writing $\mathbb{E}(n)^{\wedge k}$ as a colimit of v_n^k-multiplication maps as above and $\mathbb{MU}^{\wedge s}$ as a colimit of suspension spectra

$$\mathbb{MU}^{\wedge s} = \text{colim }_m \Sigma^{\infty - ms|v_n|}\underline{\mathbb{MU}}^{\wedge s}_{m|v_n|}$$

the above proof readily extends to show the existence of a short exact sequence

(3.2)

$$\begin{aligned} 0 \longrightarrow \lim{}^1\mathbb{E}(n)^{-1}(\underline{\mathbb{E}(n)}_0^{\wedge k} \wedge \underline{\mathbb{MU}}^{\wedge s}_{m|v_n|}) &\longrightarrow \mathbb{E}(n)^0(\mathbb{E}(n)^{\wedge k} \wedge \mathbb{MU}^{\wedge s}) \\ &\longrightarrow E(n)^0(E(n)^{\wedge k} \wedge MU^{\wedge s}) \longrightarrow 0. \end{aligned}$$

where we have used Lemma 2.6 above to identify the right hand term as before.

We are now ready to construct the $\mathbb{MU}$-algebra structure on $\mathbb{E}(n)$ that will be shown to be homotopy commutative and homotopy associative up to phantom maps.

Definition 3.1. Define $\hat{\mu}$ to be any element in $\mathbb{E}(n)^0(\mathbb{E}(n)^{\wedge 2})$ that lifts the canonical ring structure of $E(n)^0(E(n)^{\wedge 2})$ along ρ. Define the unit map $1 : \mathbb{MU} \longrightarrow \mathbb{E}(n)$ to be the canonical map expressing $\mathbb{E}(n)$ as a localized quotient of $\mathbb{MU}$.

As a formal consequence of the short exact sequence constructed above, we obtain Theorem 1.1 from the introduction:

THEOREM 1.1. *The class $\hat{\mu}$ defines a homotopy commutative, homotopy associative, and unital $\mathbb{MU}$-algebra structure on $\mathbb{E}(n)$ up to phantom maps.*

PROOF. By construction, $\hat{\mu}$ maps to the canonical ring structure on $E(n)$ under the forgetful map, ρ. It follows that any obstruction to the homotopy associativity, commutativity, or unitality of $\hat{\mu}$, viewed as a class in $\mathbb{E}(n)^0(\mathbb{E}(n)^{\wedge k} \wedge \mathbb{MU}^{\wedge s})$, maps to zero under ρ. We claim that the only elements of the kernel of ρ are phantoms.

To see this, recall the short exact sequence (3.2) above:

$$0 \to \lim{}^1 \mathbb{E}(n)^{-1}(\underline{\mathbb{E}(n)}_0^{\wedge k} \wedge \underline{\mathbb{MU}}^{\wedge s}_{m|v_n|}) \to \mathbb{E}(n)^0(\mathbb{E}(n)^{\wedge k} \wedge \mathbb{MU}^{\wedge s}) \to E(n)^0(E(n)^{\wedge k} \wedge MU^{\wedge s}) \to 0$$

where the right hand side was identified via

$$\lim{}^0 \mathbb{E}(n)^0(\underline{\mathbb{E}(n)}_0^{\wedge k} \wedge \underline{\mathbb{MU}}^{\wedge s}_{m|v_n|}) \cong \lim{}^0 E(n)^0(\underline{E(n)}_0^{\wedge k} \wedge \underline{MU}^{\wedge s}_{m|v_n|}) \cong E(n)^0(E(n)^{\wedge k} \wedge MU^{\wedge s}).$$

Notice that by exactness, an element in the kernel of ρ is in the image of the $\lim^1$-term in $\mathbb{E}(n)^0(\mathbb{E}(n)^{\wedge k} \wedge \mathbb{MU}^{\wedge s})$ and so restricts trivially to all the terms $\mathbb{E}(n)^0(\underline{\mathbb{E}(n)}_0^{\wedge k} \wedge \underline{\mathbb{MU}}^{\wedge s}_{m|v_n|})$ in the inverse system. Any map from a finite CW complex into $\mathbb{E}(n)^{\wedge k} \wedge \mathbb{MU}^{\wedge s}$ must factor through a finite stage of the colimit, that is, through $\underline{\mathbb{E}(n)}_0^{\wedge k} \wedge \underline{\mathbb{MU}}^{\wedge s}_{m|v_n|}$ for some m. It follows that any element in the image of the $\lim^1$ term is zero upon restriction to any finite CW-complex. In other words, the kernel of ρ consists entirely of phantoms. □

REMARK 3.2. One may attempt to compute the group of phantom maps $\lim{}^1 \mathbb{E}(n)^{-1}(\underline{\mathbb{E}(n)}_0^{\wedge k})$ explicitly by identifying it with the vector space $E(n)^{-1}(E(n)^{\wedge k}) \otimes \mathbb{Z}/2$ (suitably extended by a $\mathbb{Z}/2$-agebra). This is an open problem, but it appears to the authors that this vector space is trivial for $n = 1$, but may fail to be so for $n > 1$. Hence we at present have no general way of ensuring that the ring structure we have constructed is rigid up to homotopy for the spectra $\mathbb{E}(n)$, $n > 1$.

4. Multiplicative structure on $\mathbb{MU}_{(2)}[v_n^{-1}]$

We would like to show that the multiplication $\hat{\mu}$ constructed above naturally induces a commutative algebra structure on the $\mathbb{E}(n)$-cohomology of *any* (not necessarily finite-dimensional) space. An essential ingredient in our construction will be the multiplication on $\mathbb{MU}_{(2)}[v_n^{-1}]$. We pause to describe it in this section. The ingredient we need is the following proposition, which appears as Proposition 9.15 in [**HHR16**] and is proved in [**HH14**].

PROPOSITION 4.1. [**HH14**, Corollary 4.11] *Let R be a G-equivariant commutative ring with $D \in \pi_\star^G(R)$. If D has the property that for every $H \subset G$, $N_H^G i_H^* D$ divides a power of D, then the spectrum $D^{-1}R$ has a unique commutative algebra structure such that the map $R \longrightarrow D^{-1}R$ is a map of commutative rings.*

We begin by constructing a C_2-equivariant associative and commutative ring (in the highly structured sense) that lifts $\mathbb{E}(n)$. We begin with the fact that $\mathbb{MU}$ has this structure (see [**HHR16**] or [**HK01**]). We localize at $p = 2$. The spectrum $\mathbb{MU}_{(2)}$ is a C_2-equivariant commutative ring, as shown in [**HH14**]. By [**HK01**], the forgetful map

$$\rho : \pi_{*(1+\alpha)}(\mathbb{MU}_{(2)}) \longrightarrow \pi_{2*}(MU_{(2)})$$

is an isomorphism. The classes v_i (Araki, Hazewinkel, or others) in $\pi_{2*}(MU_{(2)})$ may now be lifted via ρ^{-1} to equivariant classes, $\rho^{-1}(v_i)$. While in the rest of the manuscript, we abuse notation by denoting $\rho^{-1}(v_i)$ by v_i, in the following lemma, we will distinguish between the nonequivariant v_i and the equivariant $\rho^{-1}(v_i)$. Note that for $i \leq n$ and the Araki v_i, the images of these lifts $\rho^{-1}(v_i)$ in the coefficients of $\mathbb{E}(n)$ are exactly the equivariant v_i we have been working with throughout.

Our next step is to invert $\rho^{-1}(v_n)$.

LEMMA 4.1. *The spectrum $(\rho^{-1}(v_n))^{-1}\mathbb{MU}_{(2)}$ is a C_2-equivariant commutative ring.*

PROOF. We apply Proposition 4.1 above (quoted from [**HHR16**]). The map i_H^* is exactly ρ, and so $\rho(\rho^{-1}(v_n)) = v_n \in \pi_{2(2^n-1)}(MU_{(2)})$. We need to show that $N_{\{e\}}^{C_2}(v_n)$ divides a power of $\rho^{-1}(v_n)$. In fact, we claim that $N_{\{e\}}^{C_2}(v_n) = -[\rho^{-1}(v_n)]^2$. To see this, we apply the isomorphism ρ to both sides. Let c denote the action of the generator of C_2. Using the fact that $c(v_n) = -v_n$, the double coset formula (see e.g. Proposition 10.9(v) in [**Sch**] or [**May96**]) reduces in our case to

$$\rho \circ N_{\{e\}}^{C_2}(v_n) = v_n \cdot c(v_n) = -v_n^2,$$

which completes the proof. □

REMARK 4.2. Let us again denote the spectrum $(\rho^{-1}(v_n))^{-1}\mathbb{MU}_{(2)}$ by $\mathbb{MU}[v_n^{-1}]$. This spectrum serves as a commutative proxy for $\mathbb{E}(n)$. Indeed, essentially all prior results of the authors that hold for $\mathbb{E}(n)$ extend verbatim to this spectrum.

5. Unstable properties of the multiplicative structure

We now address the question of the (unstable) multiplicative structure on $\mathbb{E}(n)$. We begin with the following lemma:

LEMMA 5.1. *There is a unique (homotopy) commutative, associative, and unital equivariant H-ring structure on the infinite loop space of $\mathbb{E}(n)$ that lifts any fixed H-ring structure on $\underline{E(n)}_0$:*

$$\hat{\mu}_0 : \underline{\mathbb{E}(n)}_0 \times \underline{\mathbb{E}(n)}_0 \longrightarrow \underline{\mathbb{E}(n)}_0.$$

PROOF. Recall that Lemma 2.6 shows that $\underline{\mathbb{E}(n)}_0^{\times j}$ has the weak projective property. As $E(n)$ is a (homotopy) associative and commutative ring spectrum, we may define the map $\hat{\mu}_0 : \underline{\mathbb{E}(n)}_0 \times \underline{\mathbb{E}(n)}_0 \longrightarrow \underline{\mathbb{E}(n)}_0$ as the preimage of the multiplication on $\underline{E(n)}_0$ along the isomorphism

$$\mathbb{E}(n)^0(\underline{\mathbb{E}(n)}_0 \times \underline{\mathbb{E}(n)}_0) \xrightarrow[\cong]{\rho} E(n)^0(\underline{E(n)}_0 \times \underline{E(n)}_0)$$

The unity, commutativity, and associativity of $\hat{\mu}_0$ are similarly verified by applying the isomorphism ρ, as the desired relations all hold in $E(n)$-cohomology. Likewise, given a choice of H-ring structure on $\underline{E(n)}_0$, the uniqueness of the equivariant lift follows from the fact that ρ is an isomorphism. □

REMARK 5.2. The unstable multiplication $\hat{\mu}_0$ we constructed in the previous lemma and the stable multiplication $\hat{\mu}$ we constructed in Definition 3.1 are compatible in the sense that $\hat{\mu}$ restricts to $\hat{\mu}_0$ on the zero space of $\mathbb{E}(n)$. To see this, apply the isomorphism ρ and note that this claim is true nonequivariantly by construction.

We now prove our second main result, Theorem 1.2.

PROOF. (of Theorem 1.2) Let X be a space and consider $f \in \mathbb{E}(n)^V(X)$ and $g \in \mathbb{E}(n)^W(Y)$. We define the product $fg \in \mathbb{E}(n)^{V+W}(X)$ as follows. First, note that since $\mathbb{E}(n)$ is an $\mathbb{MU}[v_n^{-1}]$-module, we may multiply f and g by classes in the coefficients $\mathbb{MU}[v_n^{-1}]_\star$. Let k and l be the minimal integers such that

$$v_n^k f \in \mathbb{E}(n)^{V'}(X), \qquad v_n^l g \in \mathbb{E}(n)^{W'}(X)$$

with $V', W' \leq 0$ (by this we mean that when we express each representation as a combination of irreducibles, each coefficient should be nonpositive). These classes are represented by maps

$$X \xrightarrow{v_n^k f} \underline{\mathbb{E}(n)}_{V'} = \Omega^{-V'}\underline{\mathbb{E}(n)}_0\,, \qquad X \xrightarrow{v_n^l g} \underline{\mathbb{E}(n)}_{W'} = \Omega^{-W'}\underline{\mathbb{E}(n)}_0\,.$$

We adjoin the loops over to form classes

$$\Sigma^{-V'}X \xrightarrow{v_n^k f} \underline{\mathbb{E}(n)}_0\,, \qquad \Sigma^{-W'}X \xrightarrow{v_n^l g} \underline{\mathbb{E}(n)}_0\,.$$

Note that since $-V', -W' \geq 0$, these are positive suspensions and so the sources of these maps are *spaces*. We may thus smash them together, precompose with the diagonal on X and postcompose with the multiplication on the zero space (which factors through the smash product) constructed in Lemma 5.1:

$$\Sigma^{-V'-W'}X \xrightarrow{\Delta} \Sigma^{-V'}X \wedge \Sigma^{-W'}X \xrightarrow{v_n^k f \wedge v_n^l g} \underline{\mathbb{E}(n)}_0 \wedge \underline{\mathbb{E}(n)}_0 \xrightarrow{\hat{\mu}_0} \underline{\mathbb{E}(n)}_0$$

This produces a class in $\mathbb{E}(n)^{V'+W'}(X)$. Finally, we multiply by v_n^{-k-l} to define the product $f \cdot g \in \mathbb{E}(n)^{V+W}(X)$.

The unit of this multiplication comes from the unit on $\mathbb{MU}[v_n^{-1}]$. The unity, associativity, and commutativity of this multiplication follow from the corresponding properties of $\underline{\mathbb{E}(n)}_0$.

We have shown that for any space X, $\mathbb{E}(n)^\star(X)$ is a graded associative and commutative ring. It remains to show that this is compatible with the multiplication on $\mathbb{MU}[v_n^{-1}]$-cohomology. If we carry out the above construction to define multiplication on $\mathbb{MU}[v_n^{-1}]^\star(X)$, it is evident that this agrees with the multiplication coming from the ring spectrum structure on $\mathbb{MU}[v_n^{-1}]$. To see that this multiplication agrees with the one defined on $\mathbb{E}(n)$-cohomology, it suffices to check this fact on zero spaces. To see that this diagram

$$\begin{array}{ccc} \underline{\mathbb{MU}}_0 \times \underline{\mathbb{MU}}_0 & \xrightarrow{\hat{\mu}_{\mathbb{MU}}} & \underline{\mathbb{MU}}_0 \\ \downarrow & & \downarrow \\ \underline{\mathbb{E}(n)}_0 \times \underline{\mathbb{E}(n)}_0 & \xrightarrow{\hat{\mu}_{\mathbb{E}(n)}} & \underline{\mathbb{E}(n)}_0 \end{array}$$

commutes, we may use the fact that $\underline{\mathbb{MU}}_0$ and $\underline{\mathbb{MU}}_0 \times \underline{\mathbb{MU}}_0$ are spaces with weak projective properties to map the diagram isomorphically along ρ where its commutativity is apparent. □

6. The MR(n) orientation for MO[2^{n+1}] revisited

Let $n > 0$ and let MO[2^{n+1}] denote the Thom spectrum for the virtual bundle over BO given by multiplication by 2^{n+1} seen as a self-map of BO. In other words, MO[2^{n+1}] is the spectrum that represents real vector bundles ξ endowed with an isomorphism $\xi \to 2^{n+1}\zeta$ for some bundle ζ.

In [**KW14**], the first and third authors showed that MO[2^{n+1}] admits an orientation with respect to $ER(n)$. However, that proof requires the homotopy commutativity of $\mathbb{E}(n)$ which is unclear in light of this document. Consequently, in this section, we reproduce the argument in [**KW14**] to show that MO[2^{n+1}] admits a canonical orientation with respect to the commutative ring spectrum MR(n) defined as the homotopy fixed points of the spectrum $\mathbb{MU}[v_n^{-1}]$, where it is understood that $\mathbb{MU}$ is 2-local. This orientation descends to an orientation with respect to $ER(n)$, generalizing the $\hat{A}$-genus for real K-theory. We also take this opportunity to fix an error in the proof of this result given in [**KW14**]; see Remark 6.4.

THEOREM 6.1. *The spectrum* MO[2^{n+1}] *supports a canonical orientation* u_{n+1} *with respect to* MR(n). *Furthermore, given a real vector bundle of the form* $2^{n+1}\zeta$, *this orientation is uniquely determined by the property that the image of* u_{n+1} *in* MU[v_n^{-1}] *is given by*

$$u_{n+1}(2^{n+1}\zeta) = \mu(2^n\zeta \otimes \mathbb{C}) \cup \psi(\zeta \otimes \mathbb{C})^{2^{n-1}},$$

where μ *is the usual Thom class from usual complex orientation, and* $\psi(\zeta \otimes \mathbb{C})$ *is the series in* $\mathrm{MU}[v_n^{-1}]^*(BU)$ *generated from line bundles by:*

$$\psi(x) = \frac{[-1]_{\mathrm{MU}}(x)}{-x}.$$

First, we lay some groundwork. Consider the Real orientation of $\mathbb{MU}$ given by a $\mathbb{Z}/2$-equivariant map:

$$\mu_1 : \mathbb{MU}(1) \longrightarrow \Sigma^{(1+\alpha)}\mathbb{MU}, \tag{6.1}$$

where $\mathbb{MU}(1) \simeq \mathbb{CP}^\infty$ denotes the C_2-space in the usual prespectrum defining $\mathbb{MU}$. We view $MU^*(MU(1))$ as a rank one free module on μ_1 over $MU^*(BU(1))$. We need the following fact regarding the C_2-action on μ_1.

LEMMA 6.2. *Let x denote the first Chern class in $MU^2(BU(1))$ and let $\psi(x)$ denote the series defined in Theorem* 6.1 *above. Then*

$$c(\mu_1) = \mu_1 \frac{[-1]_{\mathrm{MU}}(x)}{-x} = \mu_1\, \psi(x).$$

PROOF. The action of complex conjugation on $\Sigma^{1+\alpha}\mathbb{MU}$ can be identified with $-c$ (the c from $\mathbb{MU}$ and the -1 from the orientation reversing action on the two sphere). On the other hand, complex conjugation on MU(1) is induced by the (complex anti-linear) self map of the universal line bundle γ_1 over BU(1) that sends a vector to its complex conjugate. This map can be seen as a (complex linear) isomorphism from $\overline{\gamma_1}$ to γ_1, where $\overline{\gamma_1}$ is the opposite complex structure on the real bundle underlying γ_1. Since $\overline{\gamma}_1$ is isomorphic to the dual bundle γ_1^*, we see that the action of complex conjugation on MU(1) sends the Thom class $\mu_1 \in \mathrm{MU}^2(\mathrm{MU}(1))$ to the class $[-1]_{\mathrm{MU}}(\mu_1)$. MU(1) and BU(1) are homotopy equivalent and the Thom isomorphism is an $\mathrm{MU}^*(\mathrm{BU}(1))$ module map so that $\mu_1^2 = \mu_1\, x$, where $x = c_1(\gamma_1)$.

From this we have $\mu_1^k = \mu_1 x^{k-1}$, so any power series $\sum a_i \mu_1^{i+1}$ can be rewriten as $\mu_1 \sum a_i x^i$. Hence

$$[-1]_{\mathrm{MU}}(\mu_1) = \mu_1 \frac{[-1]_{\mathrm{MU}}(x)}{x}.$$

Incorporating this observation into the $\mathbb{Z}/2$ equivariance of μ_1 from Equation (6.1) by computing on the left and the right, this translates to the equality:

$$c(\mu_1) = \mu_1 \frac{[-1]_{\mathrm{MU}}(x)}{-x} = \mu_1\, \psi(x).$$

□

Outline of proof: (of Theorem 6.1) Before beginning the proof of Theorem 6.1 in earnest, we give a brief overview. We will make significant use of the Bockstein spectral sequence constructed from the fibration in [**KW07a**, Theorem 1.6]. Though the results in [**KW07a**] are stated for $ER(n)$, since the class v_n has been inverted, they will apply verbatim to $MR(n)$. In particular, the proof of [**KW07a**, Theorem 1.6] shows that there is a Bockstein spectral sequence $E_r(\mathrm{MO}[4])$, starting with the $\mathrm{MU}[v_n^{-1}]$-cohomology of the spectrum MO[4] and converging to the $MR(n)$-cohomology of MO[4]. The proof of Theorem 6.1 is somewhat technical and assumes familiarity with the properties of this spectral sequence, which may be found summarized in [**KW14**, Theorem 2.1]. In particular, note that the spectral sequence collapses at the $E_{2^{n+1}}$ page. The final part of the proof will make use of the structure of the spectral sequence for a point and for BO. These are described in Section 3 and 5 of [**KW14**], respectively.

The orientation u_{n+1} will be constructed inductively, starting with a class $u_2 \in \mathrm{MU}[v_n^{-1}]^*(\mathrm{MO}[4])$. Once the class u_2 is constructed, we will inductively define u_{k+1} in terms of u_k (see Equation 6.2 below) and begin our analysis of the Bockstein spectral sequence differentials. We will show that for $1 < k < n+1$ that the class u_k survives to the E_{2^k-1} page and that $E_{2^k-1}(\mathrm{MO}[2^k])$ is a rank one free module over the $E_{2^k-1}(\mathrm{BO})$ (the E_{2^k-1} page of the spectral sequence for BO) on the distinguished generator u_k. Continuing in this way, we will conclude that u_{n+1} survives to the last stage $E_{2^{n+1}-1}$ where there is one last possible differential left in the spectral sequence. We will show that this differential on u_{n+1} must be zero, and thus $MR(n)^*(\mathrm{MO}[2^{n+1}])$ is a rank one free module over $MR(n)^*(BO)$, which will establish the theorem.

Proof. (of Theorem 6.1) We start by defining a class u_2. Let ζ denote the universal real vector bundle over BO, $\zeta_{\mathbb{C}}$ the universal complex vector bundle over BU, and $\overline{\zeta_{\mathbb{C}}}$ its conjugate. Consider the composite

$$\mathrm{BU} \xrightarrow{\Delta} \mathrm{BU} \times \mathrm{BU} \longrightarrow \mathrm{BU}$$

classifying the bundle $\zeta_{\mathbb{C}} \oplus \overline{\zeta_{\mathbb{C}}}$ over BU. Precomposing with the complexification map $\mathrm{BO} \longrightarrow \mathrm{BU}$, we have that $\zeta_{\mathbb{C}} \oplus \overline{\zeta_{\mathbb{C}}}$ pulls back to 4ζ. Taking Thom spectra and mapping into $\mathrm{MU}[v_n^{-1}]$ yields the composite

$$\mathrm{MO}[4] \longrightarrow \mathrm{MU} \wedge \mathrm{MU} \longrightarrow \mathrm{MU}[v_n^{-1}]$$

where the second map is induced by the twisted multiplication map (the counit of the norm-forgetful adjunction)

$$m \circ (id \wedge c) : \mathrm{MU} \wedge \mathrm{MU} \longrightarrow \mathrm{MU}\,.$$

This is an $\mathrm{MU}[v_n^{-1}]$-Thom class for 4ζ, and we define it to be $u_2(4\zeta)$. We have

$$u_2(4\zeta) = \mu((\zeta \otimes \mathbb{C}) \oplus (\overline{\zeta \otimes \mathbb{C}})) = \mu(\zeta \otimes \mathbb{C})\mu(\overline{\zeta \otimes \mathbb{C}}).$$

Reasoning from line bundles using Lemma 6.2 and applying the splitting principle shows that

$$\mu(\overline{\zeta \otimes \mathbb{C}}) = \mu(\zeta \otimes \mathbb{C})\psi(\zeta \otimes \mathbb{C}).$$

It follows that

$$u_2(4\zeta) = \mu(2\zeta \otimes \mathbb{C}) \cup \psi(\zeta \otimes \mathbb{C}))$$

and that u_2 is c-invariant.

We now begin our investigation of the Bockstein spectral sequence. Since u_2 is c-invariant, it follows that $d_1(u_2) = 0$ in the Bockstein spectral sequence converging to $\mathrm{MR}(n)^*(\mathrm{MO}[4])$. We also have $d_2(u_2) = 0$ for degree reasons, so that $d_r(u_2) = 0$ for $r < 3$ will begin our induction. If some Thom class u_r (to be defined inductively in Equation 6.2 below) survives to E_{r+1}, notice that E_{r+1} for the Thom space is a rank one free module over $\mathrm{E}_{r+1}(\mathrm{BO})$ on the Thom class. We will show for $1 < k < n+1$ that $E_{2^k-1}(\mathrm{MO}[2^k])$ is a rank one free module over $E_{2^k-1}(\mathrm{BO})$ on a distinguished generator u_k.

By induction, assume this is true for $k > 1$. Writing $2^{k+1}\zeta$ as $2(2^k\zeta)$, we have a factorization:

$$[2^{k+1}] = ([2^k] + [2^k]) \circ \Delta : \mathrm{BO} \longrightarrow \mathrm{BO} \times \mathrm{BO} \longrightarrow \mathrm{BO},$$

which induces a map of Thom spectra:

$$\tau = m \circ \Delta : \mathrm{MO}[2^{k+1}] \longrightarrow \mathrm{MO}[2^k] \wedge \mathrm{MO}[2^k] \longrightarrow \mathrm{MO}[2^k].$$

This induces a map of spectral sequences $\tau^* : \mathrm{E}_{2^k-1}(\mathrm{MO}[2^k]) \longrightarrow \mathrm{E}_{2^k-1}(\mathrm{MO}[2^{k+1}])$. Define

$$u_{k+1} := \tau^*(u_k). \tag{6.2}$$

Observe that since $c(u_{k+1}) = u_{k+1}$ by naturality and the fact that $c(u_k) = u_k$ by induction (the observation that u_2 is c-invariant above started the induction). By the properties of the Bockstein spectral sequence, we have:

$$\begin{aligned} d_{2^k-1}(u_{k+1}) &= \Delta^*\mu^* d_{2^k-1}(u_k) = \Delta^* d_{2^k-1}\mu^*(u_k) \\ &= \Delta^*(d_{2^k-1}(u_k) \wedge u_k + u_k \wedge d_{2^k-1}(u_k)). \end{aligned}$$

But notice that $\Delta : \mathrm{MO}[2^{k+1}] \longrightarrow \mathrm{MO}[2^k] \wedge \mathrm{MO}[2^k]$ is invariant under the swap map on $\mathrm{MO}[2^k] \wedge \mathrm{MO}[2^k]$. Therefore $\Delta^*(d_{2^k-1}(u_k) \wedge u_k) = \Delta^*(u_k \wedge d_{2^k-1}(u_k))$. It follows that $d_{2^k-1}(u_{k+1})$ is a multiple of 2 and must consequently be zero since $\mathrm{E}_{2^k-1}(\mathrm{MO}[2^{k+1}])$ is a $\mathbb{Z}/2$-module for external degrees greater than zero [1]. It follows that u_{k+1} survives to $\mathrm{E}_{2^k}(\mathrm{MO}[2^{k+1}])$.

We now need to show that $d_{2^k+r}(u_{k+1}) = 0$ by induction on r, for $0 \le r < 2^k - 1$. Here we use a degree argument that relies on the structure of the Bockstein spectral

[1] Proposition 6.3 below shows that $d_{2^k-1}(u_k)$ is in fact non-trivial for $k \le n$

sequence as described in Section 5 of [**KW14**] and we import our notation from there, in particular the 'hatted' classes below. We write the Bockstein spectral sequence for BO as a tensor product of a ring of permanent cycles and the Bockstein spectral sequence for a point, as in [**KW14**]:

$$E_r(\mathrm{BO}) = \hat{\mathrm{MU}}[\hat{v}_n^{-1}]^*(\mathrm{BO}) \otimes_{\hat{\mathrm{MU}}[\hat{v}_n^{-1}]^*} E_r(\mathrm{pt}).$$

By construction, all classes in $\hat{\mathrm{MU}}[v_n^{-1}]^*(\mathrm{BO})$ have internal degree divisible by 2^{n+2}. On the other hand, the differentials longer than d_{2^k-1} have external degree larger than 2^k-1, and hence represent elements divisible by x^{2^k-1}.

Using the structure of the spectral sequence for a point, the domain of these differentials is generated by the classes $y, \hat{v}_{i,l} := \hat{v}_i v_n^{l2^{i+1}}$ and $v_n^{\pm 2^k}$ for $i > k-1$, where y is the permanent cycle representing the nilpotent class x. All of these classes have internal degree divisible by 2^{k+1}. Therefore, for dimensional reasons, there can be no differentials in this spectral sequence that land in internal degree between 2^k and 2^{k+1}, until we reach the differential $d_{2^{k+1}-1}$.

Continuing in this way, we notice that u_{n+1} survives until the last stage: $\mathrm{E}_{2^{n+1}-1}(\mathrm{MO}[2^{n+1}])$. Now consider $d_{2^{n+1}-1}(u_{n+1})$ in degree $1+(2^{n+1}-1)\lambda$. The image of this differential lands inside a subquotient of the group $\mathrm{MU}[v_n^{-1}]^*(\mathrm{MO}[2^{n+1}])$, which we henceforth identify (using the Thom isomorphism) with the group $u_{n+1}\,\mathrm{MU}[v_n^{-1}]^*(\mathrm{BO})$.

Furthermore, classes that have survived past E_{2^n-1} must belong to the $\mathbb{Z}/2[\hat{v}_1, \ldots, \hat{v}_{n-1}, v_n^{\pm 2^n}]$-submodule of $u_{n+1}\,\mathrm{MU}[v_n^{-1}]^*(\mathrm{BO})$ generated by $u_{n+1}\hat{\mathrm{MU}}[v_n^{-1}](\mathrm{BO})$, modulo previous differentials.

This allows us to express $d_{2^{n+1}-1}(u_{n+1})$ as:

$$d_{2^{n+1}-1}(u_{n+1}) = v_n^{2^n} u_{n+1}\, w, \qquad \text{where} \qquad w \in \frac{\mathrm{MU}[v_n^{-1}]^*(\mathrm{BO})}{\langle \hat{v}_0, \ldots, \hat{v}_{n-1}\rangle},$$

for some permanent cycle w. Furthermore, we know that $d^2_{2^{n+1}-1}(u_{n+1}) = 0$. Applying this to the above expression and using the derivation property, we see that:

$$v_n^{2^{n+1}} w^2 = v_n^{2^{n+1}-2^{2n}} \hat{v}_n w.$$

Replacing w with $v_n^{-2^{2n}}\hat{w}$ for some (unique) element $\hat{w} \in \mathrm{MU}[v_n^{-1}]^{(1-2^n)(1-\lambda)}(\mathrm{BO})$, we obtain the relation $\hat{w}^2 = \hat{v}_n \hat{w}$. This implies that $\hat{w}(1 - \hat{v}_n^{-1}\hat{w}) = 0$. Since $(1 - \hat{v}_n^{-1}\hat{w})$ is a unit, we see that $\hat{w} = 0$. In other words, u_{n+1} survives the differential $d_{2^{n+1}-1}$. The proof of Theorem 6.1 is complete on observing that the spectral sequence collapses at $\mathrm{E}_{2^{n+1}}$ (by [**KW14**, Theorem 2.1(iv)]. □

The following proposition shows that we cannot expect to do better:

PROPOSITION 6.3. *There exists a vector bundle ζ such that $2^n\zeta$ is not $ER(n)$-orientable. In particular, the differential d_{2^k-1} in the Bockstein spectral sequence converging to* $\mathrm{MR}(n)^*(\mathrm{MO}[2^k])$ *is nontrivial on the generator u_k for $k \leq n$.*

PROOF. Note that if d_{2^k-1} was trivial on u_k for some $k \leq n$, the proof of the above theorem would show that $\mathrm{MO}[2^m]$ was $\mathrm{MR}(n)$-orientable for some $m \leq n$. Let us demonstrate a contradiction under that hypothesis by showing the existence of a vector bundle ζ such that $2^n\zeta$ is not $ER(n)$-orientable. Let S^α denote the one point compactification of the sign representation of $\mathbb{Z}/2$. Consider the virtual vector bundle ζ over $B\mathbb{Z}/2$ with Thom space given by:

$$Th(\zeta) = E\mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} S^{(\alpha-1)}.$$

If $2^n\zeta$ were to admit an MR(n)-orientation, then one would obtain a map representing the Thom class μ:

$$\mu : E\mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} S^{2^n(\alpha-1)} \longrightarrow MR(n).$$

Postcomposing with the inclusion of fixed points map $MR(n) \longrightarrow \mathbb{MU}[v_n^{-1}]$ (where we view $MR(n)$ as a C_2-spectrum with trivial action) and precomposing with the map to the orbits, we have the composite

$$E\mathbb{Z}/2_+ \wedge S^{2^n(\alpha-1)} \longrightarrow E\mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} S^{2^n(\alpha-1)} \longrightarrow MR(n) \longrightarrow \mathbb{MU}[v_n^{-1}],$$

which is a C_2-equivariant map (with trivial actions on the middle two terms). Taking adjoints, we obtain a map

$$S^{2^n(\alpha-1)} \longrightarrow F(E\mathbb{Z}/2_+, \mathbb{MU}[v_n^{-1}]) \simeq \mathbb{MU}[v_n^{-1}],$$

where we have used the fact that $\mathbb{MU}[v_n^{-1}]$ is cofree. Since it comes from the Thom class, this map must represent a unit in $\pi^{\mathbb{Z}/2}_{2^n(\alpha-1)}\mathbb{MU}[v_n^{-1}]$. However, the computation of the bigraded homotopy of $\mathbb{MU}[v_n^{-1}]$ given in [**HK01**] shows that there is no such class. Hence we obtain a contradiction to the existence of μ. □

REMARK 6.3. As mentioned earlier, Theorem 6.1 descends to an $ER(n)$-orientation for MO$[2^{n+1}]$. However, this is by no means optimal. For example, for $n = 1$ we know that $ER(1)$ is 2-localized real K-theory. Hence a bundle ξ is $ER(1)$-orientable if and only if it is *Spin*. This is equivalent to $w_1(\xi) = w_2(\xi) = 0$. This holds for bundles of the form $\xi = 4\zeta$, but clearly there are Spin bundles that are not divisible by 4. Similarly, for $ER(2)$, the results of [**KS04**] suggest that a bundle ξ is $ER(2)$-orientable if and only if $w_1(\xi) = w_2(\xi) = w_4(\xi) = 0$, which is clearly true for bundles of the form $\xi = 8\zeta$. It is a compelling question to find a nice answer in general for when a bundle is $ER(n)$-orientable, or even to show that an answer to this question may be given in closed form.

REMARK 6.4. The above Theorem 6.1 corrects an error given in the proof of [**KW14**, Theorem 1.4]. The induction process for the construction of u_{n+1} in [**KW14**] began with a class in the Bockstein spectral sequence converging to $ER(n)^*(\mathrm{MO}[2])$. Unfortunately, that class turns out to not be conjugation invariant as required. Our current argument starts with a manifestly invariant class in the Bockstein spectral sequence converging to $ER(n)^*(\mathrm{MO}[4])$ and use that to generate the other permanent cycles. The rest of the argument is essentially the same as in [**KW14**].

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Department of Mathematics, Johns Hopkins University, Baltimore, USA
Email address: `nitu@math.jhu.edu`

Department of Mathematics, University of Rochester, Rochester, USA
Email address: `vlorman@math.jhu.edu`

Department of Mathematics, Johns Hopkins University, Baltimore, USA
Email address: `wsw@math.jhu.edu`