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The $ER(n)$-cohomology of $BO(q)$ and real Johnson–Wilson orientations for vector bundles

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Abstract

Using the Bockstein spectral sequence developed previously by the authors, we compute the ring $ER(n)^*(BO(q))$ explicitly. We then use this calculation to show that the ring spectrum $MO[2^{n+1}]$ is $ER(n)$-orientable (but not $ER(n+1)$-orientable), where $MO[2^{n+1}]$ is defined as the Thom spectrum for the self-map of BO given by multiplication by 2^{n+1}.

1. *Introduction*

The $p = 2$ Johnson–Wilson theory, $E(n)$, is a well-known complex-oriented cohomology theory with coefficients given by:

$$E(n)^* = \mathbb{Z}_{(2)}[v_1, v_2, \ldots, v_{n-1}, v_n^{\pm 1}],$$

where $|v_i| = -2(2^i - 1)$. The orientation map $MU \longrightarrow E(n)$ can be constructed in the category of $\mathbb{Z}/2$-equivariant MU-module spectra, with the $\mathbb{Z}/2$-action on MU representing complex conjugation c. As shown by Hu and Kriz in [**5**], the spectra $E(n)$ are suitably complete so that the homotopy fixed point spectra $E(n)^{h\mathbb{Z}/2}$ agree with $E(n)^{\mathbb{Z}/2}$. We call this ring spectrum the 'real' Johnson–Wilson theory, $ER(n)$.

The main result is very straightforward and simple to state. There are Conner–Floyd Chern classes, $\hat{c}_k \in ER(n)^*(BO(q))$, for $0 < k \leqslant q$, and corresponding complex conjugate classes, $\hat{c}_k^*$, with their degrees given by $-k2^{n+2}(2^{n-1} - 1)$.

Theorem 1.1. *There is a canonical isomorphism*:

$$ER(n)^*(BO(q)) \simeq ER(n)^*[[\hat{c}_1, \ldots, \hat{c}_q]]/(\hat{c}_1 - \hat{c}_1^*, \ldots, \hat{c}_q - \hat{c}_q^*).$$

This follows, in a non-trivial way, from the similar result for $E(n)$ [**12**, **13**, **18**, **19**]. In the process, we give a nice description of $ER(n)^*$, which has some value in its own right, and its computation. We show that our Bockstein spectral sequence from $E(n)^*$ to $ER(n)^*$ takes place in a particularly nice category over operations. This allows us to use the established Landweber flatness of $E(n)^*(BO(q))$ to accomplish the above result.

Corollary 1.2. *There is a canonical isomorphism*:

$$ER(n)^*(BO) \simeq ER(n)^*[[\hat{c}_1, \hat{c}_2, \ldots]]/(\hat{c}_1 - \hat{c}_1^*, \hat{c}_2 - \hat{c}_2^*, \ldots).$$

Remark 1.3. When $n = 1$, $ER(1) = KO_{(2)}$, and this gives a description of $KO^*_{(2)}(BO)$ that is quite different from the usual one from [**1**] (and later [**2**]), where $KO^0(BO)$ is given by representations as $RO(O)\hat{}$. Our work is also clearly connected to [**3**], where $KO^*(BO(q))$ is computed.

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We take a brief excursion into vector bundles with an $ER(n)$ orientation. Although clearly not the complete answer, we prove the following theorem.

Define the bundle $2^k\xi$ over BO to be the pullback of the universal bundle ξ along the multiplication by 2^k-map $[2^k] : BO \longrightarrow BO$. Let $MO[2^k]$ denote the Thom spectrum of the bundle $2^k\xi$.

THEOREM 1.4. *The Thom spectrum $MO[2^{n+1}]$ admits a canonical $ER(n)$-orientation. In particular, given any real vector bundle $\eta : V \to B$, the bundle $2^{n+1}\eta$ admits an $ER(n)$-orientation. The Thom spectrum $MO[2^{n+1}]$ does not admit an $ER(n+1)$-orientation.*

In Section 2, we set up our Bockstein spectral sequence in a slightly different way than we did in [**8**, **9**]. We describe the behavior of the spectral sequence for $ER(n)^*(\mathrm{pt})$ in Section 3 and show all the modules and differentials are in our special category in Section 4. The computation of $ER(n)^*(BO(q))$ is completed in Section 5 and the techniques are applied to orientations in Section 6.

2. *The Bockstein spectral sequence*

In the paper [**7**], we show that the homotopy ring of $ER(n)$ is a subquotient of a ring:

$$\mathbb{Z}_{(2)}[x, \hat{v}_1, \hat{v}_2, \ldots, \hat{v}_{n-1}, v_n^{\pm 1}],$$

where x is an element of $\pi_*ER(n)$ in degree $\lambda = \lambda(n) = 2^{2n+1} - 2^{n+2} + 1 = 2(2^n - 1)^2 - 1$. The classes $\hat{v}_k$ for $k \leqslant n$ exist in $ER(n)^*$ in degree $2^{n+2}(2^{n-1} - 1)(2^k - 1)$, and map to the respective classes of the same name $\hat{v}_k = v_k v_n^{-(2^n-1)(2^k-1)}$ under the canonical map from $ER(n)^*$ to $E(n)^*$, where $v_0 = 2$.

In [**7**], we also construct a fibration of spectra:

$$\Sigma^\lambda ER(n) \to ER(n) \to E(n)$$

with the first map given by multiplication with the class x. The class x is a 2-torsion class of exponent $2^{n+1} - 1$, and consequently this fibration leads to a convergent Bockstein spectral sequence described explicitly by the following theorem.

THEOREM 2.1 (cf. [**8**, Theorem 4.2] (In [**8**], we consider the untruncated version of the spectral sequence converging to zero.)). (i) *For X a spectrum, the above fibration yields a first and fourth quadrant spectral sequence of $ER(n)^*$-modules, $E_r^{i,j}(X) \Rightarrow ER(n)^{j-i}(X)$. The differential d_r has bidegree $(r, r+1)$ for $r \geqslant 1$.*

(ii) *The E_1-term is given by: $E_1^{i,j}(X) = E(n)^{i\lambda+j-i}(X)$, with*

$$d_1(z) = v_n^{-(2^n-1)}(1-c)(z) = v_n^{-(2^n-1)}(z - c(z)) \quad \text{where } c(v_i) = -v_i.$$

The differential d_r increases cohomological degree by $1 + r\lambda$ between the appropriate subquotients of $E(n)^*(X)$.

(iii) *For $r < 2^{n+1}$, the targets of the differentials, d_r, represent the image of x^r-torsion generators of $ER(n)^*(X)$ inside $E(n)^*(X)$.*

(iv) *We have $E_{2^{n+1}}(X) = E_\infty(X)$, which is described as follows: Filter $M = ER(n)^*(X)$ by $M_i = x^i M$ so that:*

$$M = M_0 \supset M_1 \supset M_2 \supset \cdots \supset M_{2^{n+1}-1} = \{0\}.$$

Then $E_\infty^{r,*}(X)$ is canonically isomorphic to M_r/M_{r+1}.

(v) *The following are all vector spaces over* $\mathbb{Z}/2$:

$$M_i/M_j, \quad j \geqslant i > 0, \quad \text{and} \quad E_r^{i,j}(X) \quad r > 1 \text{ and } i > 0.$$

(vi) $d_r(ab) = d_r(a)b + c(a)d_r(b)$. *In particular, if* $c(z) = z \in E_r(X)$, *then* $d_r(z^2) = 0$, $r > 1$.

REMARK 2.2. When X is a space, note that there is a canonical class $y \in E_1^{1,-\lambda+1}(X)$ that corresponds to the unit element under the identification of $E_1^{1,-\lambda+1}(X) = E(n)^0(X)$. This class represents the element $x \in ER(n)^{-\lambda}$, and is therefore a permanent cycle. We may use this class to simplify the notation.

$$E_1^{*,*}(X) = E_1^{0,*}(X)[y] = E(n)^*(X)[y], \quad |y| = (1, -\lambda + 1),$$
$$d_1(z) = yv_n^{-(2^n-1)}(1-c)(z) = yv_n^{-(2^n-1)}(z - c(z)) \quad \text{where } v_n \in E_1^{0,-2(2^n-1)}.$$

REMARK 2.3. The formal structure of the differentials in the Bockstein spectral sequence appears to be identical to that of the Borel homology spectral sequence for the real spectrum $\mathbb{ER}(n)$ described in [**5**]. Since they both converge to the same object, one may be tempted to conclude that *they must be the same spectral sequence*. This assertion is presumably true in the correct context. However, these spectral sequences encode somewhat different information, for example, the Bockstein spectral sequence uses as input information about the nilpotency and cofiber of the class x on $ER(n)$, while the Borel spectral sequence does not recognize the existence of this class. Therefore, in principle, the Bockstein spectral sequence admits more structure than the Borel homology spectral sequence. Note also that we have formulated the spectral sequence in cohomology. Of course, there is a corresponding homology Bockstein spectral sequence $E^r(X)$ converging to $ER(n)_*(X)$.

3. *The spectral sequence for* $X = \mathrm{pt}$

In this section, we organize the ring $ER(n)^*$ in terms of the Bockstein spectral sequence, $E_*(X)$, for X a point. The theorem is not a new calculation of $ER(n)^*$, which is already known from [**5**], and we use their results. This approach could not give us a new calculation because the very fibration the spectral sequence depends on comes out of a calculation from [**5**]. What the theorem is about is the behavior of the Bockstein spectral sequence.

Our description of $ER(n)^*$ is much nicer than the usual descriptions and gives us access to information we need for the computation of $ER(n)^*(BO(q))$. We present the spectral sequence as a sum of very nice modules over the ring $R_n = \mathbb{Z}_{(2)}[\hat{v}_1, \ldots, \hat{v}_{n-1}]$. In our description, we need the ideals $I_j = (2, \hat{v}_1, \ldots, \hat{v}_{j-1})$ for $0 < j \leqslant n$, and $I_0 = (0)$.

Although our modules are not always cyclic, they will all be associated with particularly nice elements, namely $v_n^k y^m$. It is important to observe that the modules $I_i R_n / I_j$, which naturally come up often in the proof, are trivial whenever $0 < i \leqslant j$. We set $R_n/I_{n+1} = 0$.

Our description is as follows.

THEOREM 3.1. *In the spectral sequence* $E_r(\mathrm{pt}) \Rightarrow ER(n)^*$,

(i)

$$E_1 \simeq \mathbb{Z}_{(2)}[y, \hat{v}_1, \hat{v}_2, \ldots, \hat{v}_{n-1}, v_n^{\pm 1}].$$

That is,

$$E_1^{m,*} = \mathbb{Z}_{(2)}[\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_{n-1}, v_n^{\pm 1}] \quad \text{on } y^m.$$

(ii) *The only non-zero differentials are generated by*

$$d_{2^{k+1}-1}(v_n^{-2^k}) = \hat{v}_k y^{2^{k+1}-1} v_n^{-2^{n+k}} \quad \text{for } 0 \leqslant k \leqslant n.$$

(iii) *We have* $E^{*,*}_{2^k} = E^{*,*}_{2^{k+1}-1}$, *for* $0 \leqslant k \leqslant n$, *and* $E^{*,*}_{2^{n+1}} = E^{*,*}_\infty$.

(iv) *For* $0 \leqslant j < k \leqslant n+1$,

$$E^{m,*}_{2^k} = R_n[v_n^{\pm 2^k}]/I_j \bigoplus_{j<i<k} I_i R_n[v_n^{\pm 2^{i+1}}]v_n^{2^i}/I_j \quad \text{on } y^m,$$

when $2^j - 1 \leqslant m < 2^{j+1} - 1$.

(v) *For* $0 < k \leqslant n+1$ *and* $2^k - 1 \leqslant m$

$$E^{m,*}_{2^k} = R_n[v_n^{\pm 2^k}]/I_k \quad \text{on } y^m.$$

Remark 3.2. Note two things. First, when $m = 0$ we must have $j = 0$ and $I_j = (0)$. Second, when $k = n+1$, and $2^{n+1} - 1 \leqslant m$, $E^{m,*}_\infty = 0$.

Remark 3.3. To put this all on familiar territory, recall that $E(1) = KU_{(2)}$ and $ER(1) = KO_{(2)}$. From above, we have

$$E^{*,*}_1 \simeq \mathbb{Z}_{(2)}[y, v_1^{\pm 1}].$$

Recall that d_1 is generated by $d_1(v_1^{-1}) = 2yv_1^{-2}$. Keeping in mind that $R_1 = \mathbb{Z}_{(2)}$ and $R_1/I_1 = \mathbb{Z}/2$, we get

$$E^{0,*}_2 \simeq \mathbb{Z}_{(2)}[v_1^{\pm 2}] \quad \text{and} \quad E^{m,*}_2 \simeq \mathbb{Z}/2[v_1^{\pm 2}]y^m, \quad m > 0.$$

We have d_3 is generated by $d_3(v_1^{-2}) = \hat{v}_1 y^3 v_1^{-4}$. Recalling that $\hat{v}_1 = v_1 v_1^{-(2^1-1)(2^1-1)} = 1$, this is really $d_3(v_1^{-2}) = y^3 v_1^{-4}$.

We get, with $I_1 R_1 = (2)\mathbb{Z}_{(2)}$, $E^{m,*}_\infty$ is, for $m = 0$,

$$\mathbb{Z}_{(2)}[v_1^{\pm 4}],$$
$$(2)\mathbb{Z}_{(2)}[v_1^{\pm 4}]v_1^2$$

and, for $0 < m < 3$,

$$\mathbb{Z}/2[v_1^{\pm 4}] \quad \text{on } y^m.$$

This is our usual homotopy of $KO_{(2)}$.

Remark 3.4. A more illuminating and less familiar example, except for those familiar with [**6**–**10**] (or who think $ER(2)$ is really $TMF(3)$), is the case of $ER(2)^*$. Looking briefly at just the final answer, we have $E^{0,*}_\infty$ is

$$R_2[v_2^{\pm 8}] = \mathbb{Z}_{(2)}[\hat{v}_1, v_2^{\pm 8}],$$
$$I_1 R_2[v_2^{\pm 4}] = (2)\mathbb{Z}_{(2)}[\hat{v}_1, v_2^{\pm 4}]v_2^2,$$
$$I_2 R_2[v_2^{\pm 8}] = (2, \hat{v}_1)\mathbb{Z}_{(2)}[\hat{v}_1, v_2^{\pm 8}]v_2^4.$$

This is already where most of the new interesting stuff happens. Recall that this is periodic of order 48 on v_2^8, so we can simplify by working with degrees $\mathbb{Z}/(48)$. In this case, $\hat{v}_1 = v_1 v_2^{-(2^2-1)(2^1-1)} = v_1 v_2^{-3}$ and so is in degree 16 (or -32) and we have always called this element α.

The generators $2v_2^2$ and $2v_2^6$ are in degrees -12 and -36 and go by our names of α_1 and α_3, respectively. The more interesting elements are the $2v_2^4$ and $\hat{v}_1 v_2^4$ in degrees -24 and -8, and known to us as α_2 and w, respectively. The interesting part here is that these two elements come from the first part of $(2, \hat{v}_1)\mathbb{Z}_{(2)}[\hat{v}_1, v_2^{\pm 8}]$, so that $\alpha\alpha_2 = 2w$, a relation well known to us, but easily visible here.

The element y is our x of degree -17.

We see, from the theorem, that for $E^{m,*}_\infty$, $m > 0$, we must have $0 < j < i < n + 1 = 3$, so $j = 1$ and $i = 2$, giving us $I_2 R_2[v_2^{\pm 8}]v_2^4/I_1$, or $(\hat{v}_1)\mathbb{Z}/2[\hat{v}_1, v_2^{\pm 8}]v_2^4$ on y^m and

$R_2[v_2^{\pm 8}]/I_1 = \mathbb{Z}/2[\hat{v}_1, v_2^{\pm 8}]$ on y^m, $m = 1$ or 2. Finally, there is the $R_2[v_2^{\pm 8}]/I_2 = \mathbb{Z}/2[v_2^{\pm 8}]$ on y^m for $3 \leqslant m < 7$.

Remark 3.5. As n increases, the complexity of $ER(n)^*$ also increases, but it really does it only one step at a time. The only really new thing each n adds is

$$I_n R_n[v_n^{\pm 2^{n+1}}]/I_0 = (2, \hat{v}_1, \ldots, \hat{v}_{n-1})\mathbb{Z}_{(2)}[\hat{v}_1, \ldots, \hat{v}_{n-1}, v_n^{\pm 2^{n+1}}]v_n^{2^n}.$$

We saw above how this affected $ER(2)^*$. Looking just at this for $n = 3$, we have

$$I_3 R_3[v_3^{\pm 16}]v_3^8/I_0 = (2, \hat{v}_1, \hat{v}_2)\mathbb{Z}_{(2)}[\hat{v}_1, \hat{v}_2, v_3^{\pm 16}]v_3^8.$$

From this, we would get three new generators, generalizing those we had in $ER(2)^*$:

$$A = 2v_3^8, \quad B = \hat{v}_1 v_3^8 \quad \text{and} \quad C = \hat{v}_2 v_3^8.$$

This leads to all new, but obvious, relations:

$$\hat{v}_1 A = 2B, \quad \hat{v}_2 A = 2C, \quad \hat{v}_2 B = \hat{v}_1 C,$$

and, of course:

$$\hat{v}_1 \hat{v}_2 A = 2\hat{v}_2 B = 2\hat{v}_1 C.$$

This is just a sample, but the generators and relations are easily read off from our description of $ER(n)^*$.

Remark 3.6. Note that for the final differential we have

$$d_{2^{n+1}-1}(v_n^{-2^n}) = \hat{v}_n y^{2^{n+1}-1} v_n^{-2^{n+n}}.$$

But $\hat{v}_n = v_n v_n^{-(2^n-1)(2^n-1)}$ so

$$\hat{v}_n v_n^{-2^{n+n}} = v_n v_n^{-(2^n-1)(2^n-1)} v_n^{-2^{n+n}}.$$

Recall that $v_n^{2^{n+1}}$ is the periodicity element. The exponent, modulo 2^{n+2}, of v_n in $\hat{v}_n$ is just

$$1 - (2^n-1)(2^n-1) - 2^{n+n} = 1 - 2^{2n} + 2^{n+1} - 1 - 2^{2n} = 2^{n+1}.$$

So, $d_{2^{n+1}-1}$ takes the set of generators $v_n^{b2^{n+1}+2^n}$ to the set $v_n^{a2^{n+1}} y^{2^{n+1}-1}$.

Proof of Theorem 3.1. The E_1 term is $E(n)^*[y]$, or $\mathbb{Z}_{(2)}[y, v_1, v_2, \ldots, v_{n-1}, v_n^{\pm 1}]$. However, this representation is complicated by the fact that $d_1(v_k) \neq 0$. We can replace the v_k with the equivalent classes $\hat{v}_k = v_k v_n^{-(2^n-1)(2^k-1)}$, for $k < n$, which, by [5], are permanent cycles. Also from [5], we use the fact that $\hat{v}_k v_n^{a2^{k+1}}$ is precisely $y^{2^{k+1}-1}$ torsion, which gives us the stated differentials.

Since d_1 is determined by $d_1(v_n^{-1}) = 2yv_n^{-2^n}$ and $d_1(v_n^2) = 0$, we get $d_1(v_n^{2i-1}) = 2yv_n^{-2^n+2i}$. It is now easy to read off the above E_2 and this begins our induction on k by establishing $k = 1$.

Assume our description holds for some k, $1 \leqslant k \leqslant n$. Because y is a permanent cycle, d_r commutes with y.

We need to study $d_{2^{k+1}-1}$ on $E_{2^{k+1}-1}^{m,*}$ for each term of our description of the spectral sequence.

We first observe that the differential is trivial on all of the $I_i R_n[v_n^{\pm 2^{i+1}}]/I_j$ terms in our answer. The image of the differential must lie in the group with $m \geqslant 2^{k+1} - 1$, but this group is always $R_n[v_n^{\pm 2^k}]/I_k$. Since $i < k$, the image of $I_i R_n[v_n^{\pm 2^{i+1}}]/I_j$ must be zero because every element of the source is a multiple of an element of I_i, and all such elements are zero in I_k.

We now need to compute the differentials on the terms (with various m):

$$R_n[v_n^{\pm 2^k}]/I_j \quad \text{on } y^m \quad 0 \leqslant j \leqslant k.$$

We rewrite these as $R_n[v_n^{\pm 2^{k+1}}]/I_j$ on 1 and $v_n^{2^k}$. The differential is trivial on the first part and maps the second part to the first with a higher m so that the image is $(\hat{v}_k)R_n[v_n^{2^{k+1}}]/I_k$. Thus the cokernel for $m \geqslant 2^{k+1}-1$ is $R_n[v_n^{\pm 2^{k+1}}]/I_{k+1}$ and the kernel is $I_k R_n[v_n^{\pm 2^{k+1}}]/I_j$ on $v_n^{2^k}$ as advertised. Note that when $j = k$ this is trivial. □

4. *Landweber flatness in the bigraded setting*

Let us now identify more structure on this spectral sequence in the special case above for X a point. The spectral sequence is a spectral sequence of modules over $ER(n)^*$. Define the map $\psi : E(n)^* \to ER(n)^*$ by $\psi(v_k) = \hat{v}_k$ for $k \leqslant n$. This map multiplies degrees by $(1-\lambda)/2$. We can now view the spectral sequence as being over $E(n)^*$. We want more. Our aim in this section is to show that the Bockstein spectral sequence for a point lives in the category of $E(n)_*E(n)$-comodules that are finitely presented as $E(n)_*$-modules. Since $E(n)^*$ is a particularly nice ring, this last condition is never a problem for us.

Let $BP(n)$ denote the localization of BP given by inverting v_n, that is, $BP[v_n^{-1}]$. We begin by setting things up as $BP(n)_*BP(n)$-comodules, but by [4, Theorem C], this is equivalent to working with $E(n)_*E(n)$-comodules.

Before we begin, we need to remind the reader of the basic notation. We start with the real spectrum $\mathbb{E}$ with a $\mathbb{Z}/(2)$ action on each space, indexed over $RO(\mathbb{Z}/(2)) = \mathbb{Z} \oplus \mathbb{Z}\alpha$. In other words, the spaces corresponding to $a + b\alpha \in RO(\mathbb{Z}/(2))$ in the spectrum $\mathbb{E}$ are labeled as $\mathbb{E}_{(a,b)}$. The space $\mathbb{ER}_{(*,*)}$ represents the homotopy fixed points of $\mathbb{E}_{(*,*)}$. In this notation, the spaces $\mathbb{ER}(n)_{(*,0)}$ denote our omega spectrum $ER(n)$.

There is a subtlety with grading we encounter when working in this setting. To explain this, let $\mathbb{BP}(n)$ denote the localization $[v_n^{-1}]\mathbb{BPR}$ obtained from the real (RO($\mathbb{Z}/2$))-graded Brown–Peterson spectrum on inverting the class v_n in degree $(2^n-1)(1+\alpha)$. The (bigraded) homotopy groups of $\mathbb{BP}(n)$ and $\mathbb{BP}(n) \wedge \mathbb{BP}(n)$ can be computed using the Borel homology spectral sequence, and the Atiyah–Hirzebruch spectral sequence, respectively. In particular, one observes:

$$\begin{aligned}\pi_{(*,*)}(\mathbb{BP}(n) \wedge \mathbb{BP}(n)) &\simeq \pi_{(*,*)}(\mathbb{BP}(n))[t_1, t_2, \dots,][v_n^{-1}] \\ &\simeq [v_n^{-1}]\pi_{(*,*)}(\mathbb{BPR})[t_1, t_2, \dots,][v_n^{-1}],\end{aligned}$$

where all generators are in diagonal bidegree, and the left (respectively, right) inverse of v_n denotes the inversion of the map given by multiplication by v_n on the respective $\mathbb{BPR}$ factor. We may multiply any diagonal class z in bidegree $k(1+\alpha)$ by the kth power of the invertible class $y(n) \in \mathbb{ER}(n)^{-\lambda,-1}$ (see [7]) to represent it by a class $\hat{z}$ in bidegree $k(1-\lambda)+0\alpha$. This procedure of normalizing yields a map of rings:

$$\begin{aligned}\psi : [v_n]^{-1}BP_*[t_1, t_2, \dots][v_n^{-1}] &\longrightarrow [\hat{v}_n^{-1}]\mathbb{BPR}_{(*,0)}[\hat{t}_1, \hat{t}_2, \dots][\hat{v}_n^{-1}] \\ &= \pi_{(*,0)}(\mathbb{BP}(n) \wedge \mathbb{BP}(n)),\end{aligned} \tag{4.1}$$

where the left-hand side can be identified with the Hopf algebroid $BP(n)_*BP(n)$. Indeed, it is easy to see that ψ is a map of Hopf algebroids [5, Theorem 4.11], though we do not need it here.

The map ψ that scales the grading by $(1-\lambda)/2$ comes from a composition of maps. First, it sends the generators $v_i \in \pi_*BP(n)$ to $v_i \in \pi_{(2^i-1,2^i-1)}\mathbb{BP}(n)$. Next, we get $y(n)^{2^i-1}v_i = \hat{v}_i \in \pi_{(*,0)}\mathbb{BP}(n)$. Then there are the canonical maps $\pi_{(*,0)}\mathbb{BP}(n) \to \pi_{(*,0)}\mathbb{ER}(n) \to ER(n)_*$. This explains the unusual degrees in the map ψ and shows how the $ER(n)^*$ module spectral sequence, $E_r^{*,*}(\mathrm{pt})$, is also a spectral sequence of $E(n)^*$-modules.

Our goal is to show that each $E_r^{*,*}(\mathrm{pt})$ is also a $BP(n)_*BP(n)$-comodule (equivalently, an $E(n)_*E(n)$-comodule) that is finitely presented as a $BP(n)_*$-module (equivalently, an $E(n)_*$-module) and that each differential is a map in this category.

First, we have to describe the $E(n)^*$-module structure, via ψ, of the spectral sequence. This can be confusing because the E_1 term of the spectral sequence for a point is free over $E(n)^*$ on the y^m. At this point, it is better to adopt new notation. This $E(n)^*$-module structure via ψ is equivalent to being a module over

$$\hat{E}(n)^* = \mathbb{Z}_{(2)}[\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_n^{\pm 1}]$$

by way of $E(n)^* \cong \hat{E}(n)^* \to ER(n)^* \to E(n)^*$. Now, making $E(n)^*$ an $\hat{E}(n)^*$-module is quite different from the standard structure. The primary difference is that

$$\psi(v_n) = \hat{v}_n = v_n v_n^{-(2^n-1)^2} = v_n^{-2^{n+1}(2^{n-1}-1)}.$$

Consequently, $E(n)^*$, as a module over $\hat{E}(n)^*$, is a free module on generators

$$\{1, v_n, v_n^2, \ldots, v_n^{2^{n+1}(2^{n-1}-1)-1}\}. \tag{4.2}$$

(The $n = 1$ case is an anomaly because $\hat{v}_1 = 1$. In this case, $E(1)^*$ is a free module over $\hat{E}(1)^*$ on an infinite number of generators, $\{v_1^k\}$, $k \in \mathbb{Z}$.) Note that we have, as we should have:

$$\hat{E}(n)^* \otimes_{\hat{E}(n)^*} E(n)^* \simeq E(n)^*.$$

Along the same lines as above, the rings, $R_n[v_n^{2^k}]$, $k \leqslant n + 1$, in the description and proof of the spectral sequence for a point are also free over $\hat{E}(n)^*$ on basis elements given by

$$\{1, v_n^{2^k}, v_n^{2(2^k)}, v_n^{3(2^k)}, \ldots, v_n^{(2^{n+1-k}(2^{n-1}-1)-1)(2^k)}\}.$$

(For $n = 1$, the basis is over $v_1^{j2^k}$ for $j \in \mathbb{Z}$.) Note that by the time we are done with the spectral sequence and we have $k = n + 1$, we still have elements left for our basis:

$$\{1, v_n^{2^{n+1}}, v_n^{2(2^{n+1})}, v_n^{3(2^{n+1})}, \ldots, v_n^{(2^{n-1}-2)(2^{n+1})}\},$$

where our periodicity element is $v_n^{2^{n+1}}$, not $\hat{v}_n$, and if you raise the periodicity element to the $-(2^{n-1} - 1)$th power, you then get $\hat{v}_n$. (For $n = 1$, the periodicity element is v_1^4. In the case $n = 2$, we do have the periodicity element $v_2^8 = \hat{v}_2^{-1}$.)

We are now ready to put the $BP(n)_*BP(n)$-comodule structure on the spectral sequence for a point. We begin with the E_1 term, which is free on the y^m over $E(n)^*$. The element y is odd degree so is unaffected by (co)operations, that is, it is primitive. The only comodule structure is on $\hat{E}(n)^*$ and only goes on $E(n)^*$ through it. The ring $E(n)^*$ is free over $\hat{E}(n)^*$ on generators we have already given in equation (4.2). As these generators have no connection to the operations, they are also primitive. All of the summands in the spectral sequence for all $E_r(\mathrm{pt})$ sit on primitive elements $v_n^i y^m$. The ideals I_j are known to be invariant under the (co)operations, [**15**, **16**], and for our particular case [**4**].

We need all of our module summands to be comodules and all of our differentials to be maps of comodules. We have shown that E_1 is in our category. The first differential is simply multiplication by 2 between summands so is easily in our category. Since the map d_1 is in the category, $E_2 = E_3$ is also in our category.

Assume by induction that we have $E_{2^{k+1}-1}$ is a comodule. All we have to do is show that $d_{2^{k+1}-1}$ is a comodule map and we automatically have $E_{2^{k+1}}$ is a comodule and our induction will be complete. However, our $d_{2^{k+1}-1}$ takes a cyclic summand generator to $\hat{v}_k$ in a summand in $R_n[v_n^{2^{k+1}}]/I_k$ (a free module over $\hat{E}(n)^*/I_k$). In here, $\hat{v}_k$ is a primitive, [**4**, **15**, **16**], so our differentials are in our category of comodules.

We have now shown that the entire Bockstein spectral sequence for a point belongs to the category of $BP(n)_*BP(n)$-comodules, equivalently, $E(n)_*E(n)$-comodules, that are finitely presented as $BP(n)_*$-modules ($E(n)^*$-modules). We refer the reader to [4] for details on this category.

A Landweber flat $BP(n)_*$-module is defined to be a $BP(n)_*$-module that is flat on the category of finitely presented $BP(n)_*BP(n)$-comodules [4, 17]. Recall that by [4, Theorem C], we may as well work with $E(n)$ (as $\hat{E}(n)$) instead of $BP(n)$. This leads to the following theorem.

Theorem 4.3. *Suppose that M is Landweber flat $\hat{E}(n)^*$-module, and let (E_*, d_*) denote the Bockstein spectral sequence for $X = \mathrm{pt}$. Then $(M \otimes_{\hat{E}(n)^*} E_*, \mathrm{id}_M \otimes_{\hat{E}(n)^*} d_*)$ is a spectral sequence of $ER(n)^*$-modules that converges to $M \otimes_{\hat{E}(n)^*} ER(n)^*$.*

5. $ER(n)^*(BO(q))$

In this section, we apply the results of previous sections to compute the $ER(n)$ cohomology of $BO(q)$. The key ingredients are the fact that $E(n)^*(BO(q))$ is Landweber flat, and that $\mathbb{ER}(n)$ is a real oriented theory in the sense that the universal complex line bundle over $\mathbb{C}P^\infty$ admits an $\mathbb{ER}(n)$-theory Thom class. The standard Atiyah–Hirzebruch spectral sequence argument gives $\mathbb{ER}(n)$-theory Conner–Floyd Chern classes, which we will identify as permanent cycles in the Bockstein spectral sequence for $BO(q)$.

Let $\mathbb{ER}(n)$ denote the real oriented Johnson–Wilson theory. Let $BU(q)$ be endowed with the canonical involution given by complex conjugation. By the Atiyah–Hirzebruch spectral sequence, we know that [5]

$$\mathbb{ER}(n)^{*,*}(BU(q)) = \mathbb{ER}(n)^{*,*}[[c_1, \ldots, c_q]],$$

where c_k are the (real) Conner–Floyd Chern classes in bidegree $k(1+\alpha)$. These classes lift the usual Chern classes c_k under the forgetful map $\mathbb{ER}(n)^{a,b}(BU(q)) \longrightarrow E(n)^{a+b}(BU(q))$. Now recall from [7] that there is an invertible class $y(n) \in \mathbb{ER}(n)^{-\lambda,-1}$. The class $c_k y(n)^k$ is therefore a class in bidegree $k(1-\lambda) + 0\alpha$. Let $\hat{c}_k \in \mathbb{ER}(n)^{k(1-\lambda)}(BU(q))$ be this class. Note that the $\hat{c}_k$ are lifts of the classes $c_k v_n^{k(2^n-1)} \in E(n)^*(BU(q))$. Now consider the restriction map along homotopy fixed points:

$$\mathbb{ER}(n)^{(*,0)}(BU(q)) \longrightarrow ER(n)^*(BO(q)).$$

This yields classes by the same name: $\hat{c}_k \in ER(n)^*(BO(q))$ that lift the restriction of $c_k v_n^{k(2^n-1)}$ to $E(n)^*(BO(q))$. In particular, we know that the classes $\hat{c}_k$ are permanent cycles in the Bockstein spectral sequence converging to $ER(n)^*(BO(q))$.

We may choose the classes $\hat{c}_k$ as replacements for the Chern classes and state the following old result (see [12, 13, 18, 19]):

$$\begin{aligned} E(n)^*(BO(q)) &\simeq E(n)^*[[c_1, \ldots, c_q]]/(c_1 - c_1^*, \ldots, c_q - c_q^*) \\ &= E(n)^*[[\hat{c}_1, \ldots, \hat{c}_q]]/(\hat{c}_1 - \hat{c}_1^*, \ldots, \hat{c}_q - \hat{c}_q^*), \end{aligned}$$

where $\hat{c}_k^*$ are the conjugate Chern classes defined formally by the equality:

$$\sum_{i=0}^{\infty} \hat{c}_i^* = \prod_{i=1}^{\infty} (1 + [-_{\hat{F}} x_i]) \quad \text{with } \sigma_k(x_1, x_2, \ldots,) = \hat{c}_k$$

and with the 2-typical formal group law $\hat{F}$ defined over the ring $\mathbb{Z}_{(2)}[\hat{v}_1, \ldots]$ via

$$[2]_{\hat{F}}(u) = \sum_{i=0}^{\infty}{}^{\hat{F}} \hat{v}_i u^{2^i}.$$

Definition 5.1. Any $E(n)^*$-module, M, concentrated in even degrees, is also an $\hat{E}(n)^*$-module by the inclusion $\hat{E}(n)^* \to E(n)^*$. We define the $\hat{E}(n)^*$ module $\hat{M}$ to be all elements of M in degrees that are integral multiples of $2^{n+2}(2^{n-1}-1)$. It is pretty crucial to note that the degree of $\hat{v}_k$ is in such a degree since it is in degree $(2^k-1)2^{n+2}(2^{n-1}-1)$. As a result, it will shortly turn out that $\hat{E}(n)^*$ is $\hat{M}$ for $M = E(n)^*$. We need to account for all of the other elements in M, and we claim that we have, following equation (4.2), as an $\hat{E}(n)^*$-module,

$$M \simeq \bigoplus_{j=0}^{2^{n+1}(2^{n-1}-1)-1} \hat{M} v_n^j.$$

To see this, first observe that $v_n^{2^{n+1}(2^{n-1}-1)} = \hat{v}_n^{-1}$. We want to show that the v_n^j above hit every even degree from 0 to $2^{n+2}(2^{n-1}-1)-1$, modulo $2^{n+2}(2^{n-1}-1)$. It is enough to show that the only common divisor of the degree of v_n, that is, $-2(2^n-1)$, and $2^{n+2}(2^{n-1}-1)$, is 2. Divide both by 2 and we see that 2 does not divide 2^n-1, so this is just a question of 2^n-1 and $2^{n-1}-1$ being relatively prime. We know that $(2^n-1)-2(2^{n-1}-1)=1$, so this is true.

For notational purposes, in the case where our module is $E(n)^*(X)$, we denote this by $\hat{E}(n)^*(X)$ without implying this is a cohomology theory.

We now have

$$M \simeq \hat{M} \otimes_{\hat{E}(n)^*} E(n)^*,$$

in particular, when $M = E(n)^*(X)$ or $E(n)^*$.

Remark 5.2. The ring $\hat{E}(n)^*$ has a comodule structure on it from equation (4.1). It is identical to the usual comodule structure on $E(n)^*$. If M is Landweber flat as an $E(n)^*$-module, then we claim that $\hat{M}$ is Landweber flat as an $\hat{E}(n)^*$-module. Recall that Landweber flat just means that v_k is injective on M/I_kM for all k. In our case, because v_n is a unit, multiplication by $\hat{v}_k$ is equivalent to multiplication by v_k until we have M/I_nM. This is just free over $K(n)^* = \mathbb{Z}/2[v_n^{\pm 1}]$, which, in turn, is free over $\mathbb{Z}/2[\hat{v}_n^{\pm 1}]$. Multiplication here by either v_n or $\hat{v}_n$ is injective and $M/I_{n+1}M = 0$, so if M is Landweber flat over $E(n)^*$, then $\hat{M}$ is Landweber flat over $\hat{E}(n)^*$ just by restricting to the degrees that define $\hat{M}$.

We are now ready to prove Theorem 1.1.

Proof. Our subset $\hat{E}(n)^*(BO(q))$ of $E(n)^*(BO(q))$ consists of permanent cycles in the Bockstein spectral sequence. We may re-express $E(n)^*(BO(q))$ as:

$$E(n)^*(BO(q)) \simeq (\hat{E}(n)^*[[\hat{c}_1, \ldots, \hat{c}_q]]/(\hat{c}_1 - \hat{c}_1^*, \ldots, \hat{c}_q - \hat{c}_q^*)) \otimes_{\hat{E}(n)^*} E(n)^*. \tag{5.3}$$

Writing $E(n)^*(BO(q))$ as a tensor product in this manner has the advantage of expressing it in terms of permanent cycles in the Bockstein spectral sequence for $X = BO(q)$, extended by the coefficients $E(n)^*$. We know that $E(n)^*[[\hat{c}_1, \ldots, \hat{c}_q]]/(\hat{c}_1 - \hat{c}_1^*, \ldots, \hat{c}_q - \hat{c}_q^*)$ is Landweber flat, since it can be identified with $E(n)^*(BO(q))$. This was first stated explicitly in [**12**]. It also follows by combining [**11**, **18**]. By Remark 5.2, we see that $\hat{E}(n)^*[[\hat{c}_1, \ldots, \hat{c}_q]]/(\hat{c}_1 - \hat{c}_1^*, \ldots, \hat{c}_q - \hat{c}_q^*)$ is also Landweber flat. Invoking Theorem 4.3, one has a canonical map of spectral sequences:

$$\varphi : (\hat{E}(n)^*[[\hat{c}_1, \ldots, \hat{c}_q]]/(\hat{c}_1 - \hat{c}_1^*, \ldots, \hat{c}_q - \hat{c}_q^*)) \otimes_{\hat{E}(n)^*} (E_*, d_*) \longrightarrow (E_*(BO(q)), d_*),$$

where, as mentioned earlier, the left-hand side is the Bockstein spectral sequence for a point tensored with the ring of permanent cycles: $\hat{E}(n)^*[[\hat{c}_1, \ldots, \hat{c}_q]]/(\hat{c}_1 - \hat{c}_1^*, \ldots, \hat{c}_q - \hat{c}_q^*)$, and the right-hand side is the Bockstein spectral sequence for $X = BO(q)$.

Since the map φ is manifestly an isomorphism at the E_1-term by equation (5.3), we conclude that

$$\begin{aligned} ER(n)^*(BO(q)) &\simeq (\hat{E}(n)^*[[\hat{c}_1,\ldots,\hat{c}_q]]/(\hat{c}_1-\hat{c}_1^*,\ldots,\hat{c}_q-\hat{c}_q^*)) \otimes_{\hat{E}(n)^*} ER(n)^* \\ &\simeq ER(n)^*[[\hat{c}_1,\ldots,\hat{c}_k]]/(\hat{c}_1-\hat{c}_1^*,\ldots,\hat{c}_q-\hat{c}_q^*). \end{aligned} \qquad \square$$

6. *Orientation of bundles*

Let $\xi : V \to B$ be a real vector bundle that supports an $E(n)$-orientation. To see if ξ is orientable in $ER(n)$, then we may run the Bockstein spectral sequence on the Thom space B^ξ, and ask if there is a Thom class in the $E_1(B^\xi)$-term that survives to $E_{2^{n+1}}(B^\xi)$.

Let us work in the universal (virtual) bundle ξ over $B = BO$. This bundle is not $E(n)$-orientable. However, twice it is indeed $E(n)$-orientable since $2\xi = \xi \otimes \mathbb{C}$. Another way to say this is that given the multiplication by 2-map $[2] : BO \longrightarrow BO$, the pullback bundle $[2]^*\xi$ is $E(n)$-orientable.

Definition 6.1. Define the bundle $2^k\xi$ over BO to be the pullback of the universal bundle ξ along the multiplication by 2^k-map $[2^k] : BO \longrightarrow BO$. Let $MO[2^k]$ denote the Thom spectrum of the bundle $2^k\xi$.

Let us prove an elementary result about $MO[2]$.

Proposition 6.2. *The Thom spectrum $MO[2]$ supports an $E(n)$-orientation. Furthermore, it supports a Thom class $\mu \in E(n)^0(MO[2])$ with the property that $c(\mu) = \mu$, where c is the involution induced by complex conjugation on $E(n)$.*

Proof. Recall from earlier remarks that $MO[2]$ can be identified with Thom spectrum of the virtual (complex) vector bundle given by restricting the universal bundle over BU along BO. In particular, one has a map of Thom spectra $i : MO[2] \longrightarrow MU$. Since MU supports a canonical $E(n)$-orientation given by a Thom class $\mu \in E(n)^0(MU)$, one obtains a Thom class by restriction (and denoted by the same symbol) $\mu \in E(n)^0(MO[2])$. It remains to show that $c(\mu) = \mu$ in $E(n)^0(MO[2])$. We proceed as follows.

Consider the real orientation of $E(n)$ given by a $\mathbb{Z}/2$-equivariant map:

$$\mu_1 : MU(1) \longrightarrow \Sigma^{(1+\alpha)}E(n). \tag{6.3}$$

The action of complex conjugation on $\Sigma^{1+\alpha}E(n)$ can be identified with $-c$ (the c from $E(n)$ and the -1 from the orientation-reversing action on the two sphere).

On the other hand, complex conjugation on $MU(1)$ is induced by the (complex anti-linear) self-map of the universal line bundle γ_1 over $BU(1)$ that sends a vector to its complex conjugate. This map can be seen as a (complex linear) isomorphism from $\overline{\gamma_1}$ to γ_1, where $\overline{\gamma_1}$ is the opposite complex structure on the real bundle underlying γ_1. Since $\overline{\gamma}_1$ is isomorphic to the dual bundle γ_1^*, we see that the action of complex conjugation on $MU(1)$ sends the Thom class $\mu_1 \in E(n)^2(MU(1))$ to the class $[-_F\,\mu_1]$, where F is the formal group law for $E(n)$. The spectrum $MU(1)$ and $BU(1)$ are homotopy equivalent and the Thom isomorphism is an $E(n)^*(BU(1))$ module map so that $\mu_1^2 = \mu_1 x$, where $x = c_1(\gamma_1)$.

From this, we have $\mu_1^k = \mu_1 x^{k-1}$, so if we have any power series $\sum a_i \mu_1^{i+1}$, we can rewrite this as $\mu_1 \sum a_i x^i$. Hence $[-_F\mu_1] = \mu_1[-_F x]/x$.

Incorporating this observation into the $\mathbb{Z}/2$ equivariance of μ_1 from equation (6.3) by computing on the left and the right, this translates to the equality:

$$\mu_1 \frac{[-_F x]}{x} = -c(\mu_1).$$

Let $\mu_1^{2k} : MU(1)^{\wedge 2k} \longrightarrow \Sigma^{2k(1+\alpha)}E(n)$ denote the external smash product of $2k$-copies of μ_1. It follows that:

$$c(\mu_1^{2k}) = \mu_1^{2k} \prod_{i=1}^{2k} \frac{[-_F x_i]}{x_i} = \mu_1^{2k} \frac{\sigma_{2k}(-_F x_1, -_F x_2, \ldots, -_F x_{2k})}{\sigma_{2k}(x_1, x_2, \ldots, x_{2k})},$$

where σ_{2k} denotes the $2k$th elementary symmetric function in $2k$-variables. It follows from this that the canonical orientation $\mu_{2k} : MU(2k) \longrightarrow \Sigma^{2k(1+\alpha)}E(n)$ has the property:

$$c(\mu_{2k}) = \mu_{2k} \frac{c_{2k}^*}{c_{2k}},$$

where c_i are the Conner–Floyd Chern classes with conjugates denoted by c_i^*. Now let $MO(2k)[2]$ denote the Thom spectrum of the bundle over $BO(2k)$ given by restriction along $BO(2k) \longrightarrow BU(2k)$. Let $\mu_{2k} \in E(n)^{4k}(MO(2k)[2])$ also denote the restriction of $\mu_{2k} \in E(n)^{4k}(MU(2k))$. Since $c_i = c_i^*$ in $E(n)^*(BO(2k))$, we observe that $c(\mu_{2k}) = \mu_{2k}$ in $E(n)^{4k}(MO(2k)[2])$. Taking inverse limits over k, we deduce that $c(\mu) = \mu$ in $E(n)^0(MO[2])$. □

We can now prove the first part of Theorem 1.4.

Proof. The spectral sequence for a Thom space is a module over the spectral sequence for BO. The only new concern is what happens with the Thom class. If $d_r = 0$ on the Thom class, then E_{r+1} for the Thom space is a rank 1 free module over $E_{r+1}(BO)$ on the Thom class.

Consider the Bockstein spectral sequence $E_*(MO[2])$ converging to $ER(n)^*(MO[2])$. By the previous proposition, we see that $E_1(MO[2])$ is a rank one module over $E_1(BO)$ generated by the complex orientation μ for $MO[2]$. Furthermore, we know that $c(\mu) = \mu$. It follows that $d_1(\mu) = y\, v_n^{-(2^n-1)}(1-c)(\mu) = 0$. This gives us E_2, but to begin our induction, we need to have $d_2 = 0$ on the Thom class. $d_2(\mu)$ has to hit some element zy^2 with $z \in E_2^{0,p}(BO)$ where $p = 2\lambda + 1 = 2^{2n+2} - 2^{n+2} + 5$, but there are no odd degree elements. We have $E_3(MO[2])$ is a rank 1 module over $E_3(BO)$ generated by the complex orientation μ.

Assume by induction that for $k > 1$, $E_{2^k-1}(MO[2^{k-1}])$ is a rank 1 module over $E_{2^k-1}(BO)$ on a distinguished generator u_{k-1}. Writing $2^k\xi$ as $2(2^{k-1}\xi)$, we have a factorization:

$$[2^k] = ([2^{k-1}] + [2^{k-1}]) \circ \Delta : BO \longrightarrow BO \times BO \longrightarrow BO,$$

which induces a map of Thom spectra:

$$\tau = \mu \circ \Delta : MO[2^k] \longrightarrow MO[2^{k-1}] \wedge MO[2^{k-1}] \longrightarrow MO[2^{k-1}].$$

This induces a map of spectral sequences $\tau^* : E_{2^k-1}(MO[2^{k-1}]) \longrightarrow E_{2^k-1}(MO[2^k])$. Define u_k to be $\tau^*(u_{k-1})$. Observe that $c(u_k) = u_k$ by naturality and induction from u_{k-1}. This is needed for the following equation using Theorem 2.1(vi):

$$\begin{aligned} d_{2^k-1}(u_k) &= \Delta^*\mu^* d_{2^k-1}(u_k) = \Delta^* d_{2^k-1}\mu^*(u_k) \\ &= \Delta^*(d_{2^k-1}(u_{k-1}) \wedge u_{k-1} + u_{k-1} \wedge d_{2^k-1}(u_{k-1})). \end{aligned}$$

But note that $\Delta : MO[2^k] \longrightarrow MO[2^{k-1}] \wedge MO[2^{k-1}]$ is invariant under the swap map on $MO[2^{k-1}] \wedge MO[2^{k-1}]$. Therefore, $\Delta^*(d_{2^k-1}(u_{k-1}) \wedge u_{k-1}) = \Delta^*(u_{k-1} \wedge d_{2^k-1}(u_{k-1}))$. It follows that $d_{2^k-1}(u_k)$ is a multiple of 2 and must consequently be zero since $E_{2^k-1}(MO[2^k])$ is a $\mathbb{Z}/2$-module for external degrees greater than zero. (The next claim shows that $d_{2^k-1}(u_{k-1})$ is in fact non-trivial.) It follows that u_k survives to $E_{2^k}(MO[2^k])$.

We need to show that $d_{2^k+r}(u_k) = 0$ by induction on r, for $0 \leqslant r < 2^k - 1$ to complete the induction on k. We have $d_{2^k+r}(u_k) = zy^{2^k+r}$ by induction where $|z| = (2^k + r)\lambda + 1$, $z \in E_{2^k+r}^{0,|z|}$ (from Theorem 2.1(ii)). Now by the flatness of $E(n)^*(BO)$, we have the Thom isomorphism:

$$E_{2^k+r}(MO[2^k]) = E_{2^k+r}(\mathrm{pt}) \otimes_{\hat{E}(n)^*} \hat{E}(n)^*(BO).$$

From Corollary 1.2, Theorems 3.1 and 4.3, we know the only elements to survive to sit on y^{2^k+r} are elements from BO and R_n/I_k sitting on $v_n^{a2^k}$. The degrees of the c_k are divisible by 2^{n+2} and the degree of elements in R_n/I_k sitting on $v_n^{a2^k}$ are divisible by 2^{k+1} because $v_n^{a2^k}$ is and every element in R_n/I_k is made up of multiples of elements $\hat{v}_j$ with $j \geqslant k$, and all of them have degree divisible by 2^{n+2}. Modulo 2^{k+1}, $|z| = 2^k + r + 1$, which is less than 2^{k+1}, so z must be zero as it is not in a degree with any elements.

The proof of the first part of Theorem 1.4 is complete on observing that the spectral sequence collapses at $E_{2^{n+1}}$. □

The last part of Theorem 1.4 is complete with the following.

Claim 6.4. *There exists a vector bundle ζ such that $2^n\zeta$ is not $ER(n)$-orientable. In particular, the last possible differential $d_{2^{n+1}-1}$ in the Bockstein spectral sequence converging to $ER(n)^*(MO[2^n])$ is non-trivial on the generator u_n.*

Proof. Let S^α denote the one-point compactification of the sign representation of $\mathbb{Z}/2$. Consider the virtual vector bundle ζ over $B\mathbb{Z}/2$ with Thom space given by:

$$\mathrm{Th}(\zeta) = E\mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} S^{(\alpha-1)}.$$

If $2^n\zeta$ were to admit an $ER(n)$-orientation, then one obtains a $\mathbb{Z}/2$-equivariant map representing the Thom class μ:

$$\mu : E\mathbb{Z}/2_+ \wedge S^{2^n(\alpha-1)} \longrightarrow ER(n) \longrightarrow E(n).$$

By adjointness, one obtains a class $\mu \in \pi_{2^n(\alpha-1)}\mathbb{ER}(n)$, where $\mathbb{ER}(n)$ denotes the RO($\mathbb{Z}/2$)-graded real spectrum representing Johnson–Wilson theory [7]. This class has the property that it restricts to a unit on forgetting the $\mathbb{Z}/2$-equivariant structure. However, the computation of the bigraded homotopy of $\mathbb{ER}(n)$ given in [5] shows that there is no such class. Hence we obtain a contradiction to the existence of μ. □

Remark 6.5. Theorem 1.4 is by no means optimal. For example, for $n = 1$ we know that $ER(1)$ is 2-localized real K-theory. Hence a bundle ζ is $ER(1)$-orientable if and only if it is *spin*. This is equivalent to $w_1(\zeta) = w_2(\zeta) = 0$. This holds for bundles of the form $\zeta = 4\xi$, but clearly there are spin bundles that are not divisible by 4. Similarly, for $ER(2)$, the results of [14] suggest that a bundle ζ is $ER(2)$-orientable if and only if $w_1(\zeta) = w_2(\zeta) = w_4(\zeta) = 0$, which is clearly true for bundles of the form $\zeta = 8\xi$. It is a compelling question to find a nice answer in general for when a bundle is $ER(n)$-orientable, or even to show that an answer to this question may be given in closed form.

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