

BIEQUIVARIANT MAPS ON SPHERES AND TOPOLOGICAL COMPLEXITY OF LENS SPACES

JESÚS GONZÁLEZ

*Departamento de Matemáticas
Centro de Investigación y de Estudios Avanzados del IPN
Apartado Postal 14-740, Mexico City, 07000, Mexico
jesus@math.cinvestav.mx*

MAURILIO VELASCO

*Colegio de Ciencia y Tecnología
Universidad Autónoma de la Ciudad de México
San Isidro 15, Lomas de San Lorenzo, Iztapalapa
Mexico City, 09790, Mexico
maurilio.velasco.fuentes@uacm.edu.mx*

W. STEPHEN WILSON

*Department of Mathematics and the School of Education
Johns Hopkins University
421 Krieger Hall, Baltimore, MD 21218, USA
ws@math.jhu.edu*

Received 21 November 2011

Revised 5 June 2012

Accepted 17 June 2012

Published 11 January 2013

Weighted cup-length calculations in singular cohomology led Farber and Grant in 2008 to general lower bounds for the topological complexity of lens spaces. We replace singular cohomology by connective complex K -theory, and weighted cup-length arguments by considerations with biequivariant maps on spheres to improve on Farber–Grant’s bounds by arbitrarily large amounts. Our calculations are based on the identification of key elements conjectured to generate the annihilator ideal of the toral bottom class in the ku -homology of the classifying space for a rank-2 abelian 2-group.

Keywords: Connective complex K -theory; $\mathbb{Z}_2^e$ -biequivariant maps; topological complexity.

Mathematics Subject Classification 2010: 55M30, 55N15, 55N22, 68T40

1. Topological Complexity of 2-Torsion Lens Spaces

The concept of topological complexity was introduced by Farber in [4] motivated by one of the most basic problems in robotics: given a mechanical system $\mathcal{S}$, one wants

to determine an algorithm or a program capable of taking the system from any given initial state A to any given final state B under certain given constraints. Thus, a motion planning program for $\mathcal{S}$ is a set of rules that specify a movement of the system from any given initial state to any other given final state. The problem can be formalized mathematically in the following way. Let X denote the configuration space of $\mathcal{S}$, and let PX be the function space of all continuous paths $\gamma : [0, 1] \rightarrow X$ in X . There is a fibration $\pi_X : PX \rightarrow X \times X$ that associates to every $\gamma \in PX$ the ordered pair formed by the initial and final points of γ , i.e. $\pi_X(\gamma) = (\gamma(0), \gamma(1))$. In its most basic form, the motion planning problem in X asks to construct a function $s : X \times X \rightarrow PX$ such that the composition $\pi_X \circ s$ is the identity; that is, s must be a cross-section of π_X . The natural restriction that the section s be continuous can only hold when X is contractible; in general, continuity will hold on neighborhoods U_i covering $X \times X$, called *local domains*, on each of which π_X should admit a continuous local cross-section s_i , called the *i*th *local rule*. Such a set of local domains and local rules is called a *motion planner* in X .

The (normalized) topological complexity of X , denoted here by $\overline{\text{TC}}(X)$, is one less than the lowest possible number of local rules among motion planners in X . In other words, $\overline{\text{TC}}(X)$ is the normalized Schwarz genus of π_X (so, the standard convention for the Lusternik–Schnirelmann category is now imposed on Farber’s original definition).

The concept of topological complexity captures a number of interesting phenomena. To begin with, as shown by Farber, $\overline{\text{TC}}(X)$ gives a sharp measure of the intrinsic discontinuities in the motion planning problem in X . On the other hand, $\overline{\text{TC}}(X)$ depends only on the homotopy type of X and, since the diagonal $\Delta : X \rightarrow X \times X$ is homotopy equivalent to π_X , the weighted cup-length of the zero-divisors in any multiplicative cohomology theory of X gives lower bounds for $\overline{\text{TC}}(X)$. The latter is the key observation leading (with singular cohomology) to Farber–Grant’s general lower bounds for the topological complexity of lens spaces (Theorem 6.1 in the final section of this paper). But most striking is the connection with a classical problem in differential topology: For the n -dimensional real projective space P^n , Farber, Tabachnikov, and Yuzvinsky show in [6] that

$$\overline{\text{TC}}(P^n) = \text{Imm}(P^n) \tag{1.1}$$

when $n \neq 1, 3, 7$. Here $\text{Imm}(P^n)$ stands for the Euclidean immersion dimension of P^n , i.e. the smallest positive integer d such that P^n can be immersed in $\mathbb{R}^d$.

The above property can be extrapolated to the case of lens spaces by recalling from [1] that, for $n \neq 1, 3, 7$, $\text{Imm}(P^n)$ is the smallest positive integer m for which there is a $\mathbb{Z}_2$ -biequivariant map $S^n \times S^n \rightarrow S^m$, where $\mathbb{Z}_2$ acts antipodally on each sphere.

Definition 1.1. For positive integers n and t consider the standard $\mathbb{Z}_t$ -action on the $(2n + 1)$ -dimensional sphere S^{2n+1} , and let $b_{n,t}$ denote the smallest positive integer m for which there is a $\mathbb{Z}_t$ -biequivariant map $S^{2n+1} \times S^{2n+1} \rightarrow S^{2m+1}$.

Let $L^{2n+1}(t)$ denote the standard $(2n+1)$ -dimensional t -torsion lens space, the orbit space of the action in Definition 1.1 above. The main result in [10] partially extends (1.1) by showing

$$\overline{\text{TC}}(L^{2n+1}(t)) = 2b_{n,t} + \varepsilon_{n,t}, \quad \varepsilon_{n,t} \geq 0, \quad (1.2)$$

where in fact $\varepsilon_{n,t} \in \{0, 1\}$ if t is even.^a

We focus on $b_{n,2^e}$, which will also be denoted by $b(n, e)$. While $b(n, 1)$ captures up to parity the (still undetermined) immersion dimension of P^{2n+1} , the function $b(n, e)$ has an easy description for e large enough: [10, Proposition 2.2] claims

$$b(n, e) = \begin{cases} 2n, & e > \alpha(n); \\ 2n - 1, & e = \alpha(n), \end{cases} \quad (1.3)$$

where $\alpha(n)$ denotes the number of ones that appear in the binary expansion of n . This paper's goal (Theorem 1.1 below) is to analyze the “first” unsolved case in (1.3): $e = \alpha(n) - 1$.

The main result in [3] claims that $P^{2(m+\alpha(m)-1)}$ does not admit an immersion in $\mathbb{R}^{4m-2\alpha(m)}$. Therefore (1.1) and (1.2) imply

$$b(m + \alpha(m) - 1, 1) \geq 2m - \alpha(m). \quad (1.4)$$

More generally, considerations of the immersion dimension of lens spaces led to the following conjecture in [12].

Conjecture 1.1. *For $1 \leq e \leq \alpha(m)$, $b(m + \alpha(m) - e, e) \geq 2m - \alpha(m) + e - 1$.*

This should be considered as an alternative point of view toward an eventual understanding of the intricacies in the “TC-approach” to the immersion dimension of *odd-dimensional* projective spaces (cf. [10]). Namely, although half a century of experience suggests that the numerical value of (1.1) might look like $2n - k(n)\alpha(n) + o(\alpha(n))$ with $1 \leq k(n) \leq 6$, determining the nature of $k(n)$ and the form of $o(\alpha(n))$ is currently a major open task. However, if the “correction term” $\alpha(m) - 1$ in the first entry of the b -function in (1.4) were to be ignored, $k(n) = 2$ and $o(\alpha(n)) \geq 0$ would provide a rather satisfying general lower bound for (1.1), at least in the case of odd-dimensional projective spaces. Conjecture 1.1 extends such an idealistic bound to (the topological complexity of) 2^e -torsion lens spaces in such a way that the required correction term $\alpha(m) - e$ gets smaller as the lens-space torsion increases. For instance, after ignoring the correction term “+1”, Theorem 1.1 below should be thought of as giving the *expected* answer for the next case in (1.3).

In view of (1.3) and (1.4), Conjecture 1.1 is true for $e = 1$, and sharp for $e = \alpha(m)$. It is also known to hold in many other cases with $e = 2$ (cf. [12, Theorem 2.4]). One of the main achievements of this paper is a proof of Conjecture 1.1

^aThere is a gap in the proof given in [10] of the estimates for $\varepsilon_{n,t}$. The first author thanks José García-Calines and Lucile Vandembroucq for noticing the problem and for discussions leading to a fixing of the gap. Details are discussed in the final section of this paper.

for $e = \alpha(m) - 1$, with its corresponding application to the topological complexity of lens spaces.

Theorem 1.1. $b(m + 1, \alpha(m) - 1) \geq 2m - 2$ provided $\alpha(m) \geq 2$.

Corollary 1.1. $\overline{\text{TC}}(L^{2m+3}(2^{\alpha(m)-1})) \geq 4m - 4$ provided $\alpha(m) \geq 2$.

The proof of Theorem 1.1 is given in Sec. 5. Corollary 1.1 improves by arbitrarily large amounts on Farber–Grant’s general lower bound for the topological complexity of lens spaces. Indeed, if $\nu(m)$ stands for the exponent in the largest 2-power dividing m then, as indicated in Proposition 6.1 in the final section of the paper, [5, Theorem 11] asserts in the case of $L^{2m+3}(2^{\alpha(m)-1})$ that

$$\overline{\text{TC}}(L^{2m+3}(2^{\alpha(m)-1})) \geq 4m - 2^{\nu(m)+2} - 1, \quad (1.5)$$

provided $\alpha(m + 1) \geq \alpha(m)$ — i.e. when (1.3) does not apply. But Corollary 1.1 improves the lower bound in (1.5) by $2^{\nu(m)+2} - 3$ units.

2. On the Annihilator of the Toral Class in $ku_*(\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^e})$

Let ku stand for the connective cover of complex K -theory, and write ku_*X for the reduced ku -homology of a space (or spectrum) X . The obvious projection $\mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}_{2^k} \times \mathbb{Z}_{2^e}$ determines a “toral” map $T^2 = S^1 \times S^1 \rightarrow B(\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^e})$. Since the top cell of T^2 splits off after a suspension, this yields a stable map $S^2 \rightarrow B(\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^e})$ and a corresponding “toral” class $\tau = \tau_{k,e} \in ku_2B(\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^e})$. Note that τ actually lies in the ku_* -direct summand $ku_*B(\mathbb{Z}_{2^k}) \otimes_{ku_*} ku_*B(\mathbb{Z}_{2^e})$.

The first result in this section (proved in Sec. 3) identifies key elements in $\text{Ann}_{ku_*}(\tau)$, the ku_* -annihilator ideal of τ .

Theorem 2.1. *Let $v_1 \in ku_2$ correspond to Bott periodicity. For $k \geq e$ and $1 \leq j \leq e$ consider the elements*

$$\varepsilon_j = 2^{e-j} v_1^{2^{j-1}(k-e+3)-2} \in ku_*.$$

Then $\varepsilon_j \cdot \tau_{k,e} = 0$, for $j = 0, \dots, e$, where we set $\varepsilon_0 = 2^e$.

Plenty of evidence (some of which is discussed in the following paragraphs) points toward the possibility that the elements ε_j in Theorem 2.1 generate $\text{Ann}_{ku_*}(\tau)$. For instance, Theorem 2.2 below — a crucial ingredient in our proof of Theorem 1.1 — settles the initial steps in such a task. The following potential picture, which implies that the ε_j would indeed generate $\text{Ann}_{ku_*}(\tau)$, arose from extensive computations mimicking those in the classical Conner–Floyd conjecture.

Conjecture 2.1. *Let $I = I_{k,e}$ be the ideal of ku_* generated by the elements ε_j for $j = 0, \dots, e$. Then there is a ku_* -filtration of $ku_*B(\mathbb{Z}_{2^k}) \otimes_{ku_*} ku_*B(\mathbb{Z}_{2^e})$ whose associated graded object is ku_*/I -free, and has (the class of) $\tau_{k,e}$ as a basis element.*

Remark 2.1. An important motivation for Theorem 2.1 and specially Conjecture 2.1 comes from a desire to prove the corresponding statements with ku replaced by the 2-primary Brown–Peterson spectrum (or, for that matter, by any $BP\langle n \rangle$ with $n \geq 2$). This would yield a $(\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^e})$ -analogue of the classical Conner–Floyd conjecture. An immediate consequence of such a potential result is that the BP-projective dimension of $B(\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^e})$ would be 2 — the $(\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^e})$ -case of an old conjecture of Landweber. (Some of these goals — for $k = e = 2$, as well as for $e = 1$ — have been accomplished in Nakos’ Ph.D. work [16].) Furthermore, on the applications side, and even more interesting, is the fact that a proof of the BP-version of Conjecture 2.1 would complete an important step toward proving the general case of Conjecture 1.1 (cf. [12]).

The starting point for the second result in this section comes from the observation (Proposition 4.1) that, since ku_* is polynomial on a *single* variable, the verification that the elements ε_j in Theorem 2.1 generate $\text{Ann}_{ku_*}(\tau)$ follows easily from the next conjecture.

Conjecture 2.2. *None of the elements $2^{e-j}v_1^{2^{j-1}(k-e+3)-3}$ ($1 \leq j \leq e$) annihilates $\tau_{k,e}$.*

Conjecture 2.2 is true for $k \geq e = 1$ and $k \geq e = 2$. More generally, the following result is proved in Sec. 4.

Theorem 2.2. *Neither of the elements $2^{e-1}v_1^{k-e}$ and $2^{e-2}v_1^{2(k-e)+3}$ annihilates the toral class $\tau_{k,e}$ (the latter element makes sense only for $e \geq 2$). Consequently, the elements ε_j in Theorem 2.1 generate $\text{Ann}_{ku_*}(\tau_{k,e})$ provided $e \leq 2$ and $k \geq e$.*

Remark 2.2. Conjecture 2.2 for $j = 3$ and $k = e$ and, therefore, the last sentence in Theorem 2.2 for $k = e \leq 3$ are proved in the Ph.D. thesis of the second author.^b As a consequence, it is deduced that, for $k = e \geq 4$, none of the elements

$$2^{e-j}v_1^{11 \cdot 2^{j-3}-2} \quad (4 \leq j \leq e), \quad (2.1)$$

annihilates $\tau_{e,e}$. Note that, for $j = 4$, this says that $2^{e-4}v_1^{20}\tau_{e,e} \neq 0$, involving a power of v_1 which is only one less than that described by Conjecture 2.2. Although the *difference* between the exponents in v_1 in (2.1) and Conjecture 2.2 (for $k = e$) grows exponentially in j , it is its *quotient* which displays a good asymptotic behavior, as it tends to $\frac{11}{12}$ as $t \rightarrow \infty$. The proofs of these facts are given in [17, Chap. 3], and are based on a much more elaborate argument than that given in Sec. 4 to prove Theorem 2.2. Since we have no (say, TC-)application for these extended results, we leave the interested reader to look in [17] for proof details of the facts mentioned in this paragraph.

^bThe thesis extends in fact the range $k = e \leq 3$ to $k = e \leq 4$, for which Conjecture 2.1 is also verified. This depends on extensive computer-based calculations.

The rest of the section is devoted to preparing the grounds for the proof of Theorem 2.1 — the actual proof is done in Sec. 3.

We use the shorthand $ku_*(e)$ and $ku_*(k, e)$, respectively, for the reduced ku_* -homology of $\Sigma^{-1}B\mathbb{Z}_{2^e}$ and $\Sigma^{-2}B\mathbb{Z}_{2^k} \wedge B\mathbb{Z}_{2^e}$. The suspended spectra are taken just for notational convenience as bottom classes become zero-dimensional. It is well known (see, for instance, [8, Sec. 3]) that $ku_*(e)$ has a length-1 ku_* -resolution

$$0 \longrightarrow L_1 \xrightarrow{\partial_e} L_0 \longrightarrow ku_*(e) \longrightarrow 0,$$

where $L_1 = L_0$ is the ku_* -free module on generators z_i for $i \geq 0$. Each z_i has dimension $2i$ and will also be denoted by (i) . The map ∂_e comes from the 2^e -series for the multiplicative formal group law $[2^e](x) = \frac{(1+v_1x)^{2^e}-1}{v_1} = \sum_{i=1}^{2^e} \binom{2^e}{i} v_1^{i-1} x^i$. Explicitly,

$$\partial_e(i) = \sum_{s \geq 0}^{2^e-1} a_s(i-s), \quad (2.2)$$

where $(\ell) = 0$ for $\ell < 0$, and $a_s = \binom{2^e}{s+1} v_1^s$. In particular, the ku_* -Künneth spectral sequence for $\Sigma^{-2}B\mathbb{Z}_{2^k} \wedge B\mathbb{Z}_{2^e}$ collapses to the usual Landweber short exact sequence

$$0 \rightarrow ku_*(k) \otimes_{ku_*} ku_*(e) \rightarrow ku_*(k, e) \rightarrow \Sigma \mathrm{Tor}_1^{ku_*}(ku_*(k), ku_*(e)) \rightarrow 0.$$

The attention in this and the next two sections focuses on the tensor group $ku_*(k) \otimes_{ku_*} ku_*(e)$, in which the toral class lies. An important computational tool will be given by the Smith ku_* -morphism $\delta : ku_*(e) \rightarrow ku_*(e)$ determined by

$$\delta(i) = (i-1).$$

We have the two endomorphisms $\delta_1 = \delta \otimes 1$ and $\delta_2 = 1 \otimes \delta$ of $ku_*(k) \otimes_{ku_*} ku_*(e)$ through which we can define a T -module structure on $ku_*(k) \otimes_{ku_*} ku_*(e)$, where $T = ku_*[[\delta_1, \delta_2]]$ is the power series ring on two variables δ_1 and δ_2 with coefficients in ku_* . An element $(a) \otimes (b) \in ku_*(k) \otimes_{ku_*} ku_*(e)$ will simply be denoted by (a, b) . For instance, the toral class $\tau \in ku_0(k, e)$ corresponds to $(0, 0)$. We will generically denote by $((s))$ any linear combination of elements of the form (a, b) with $a + b = s$.

Proposition 2.1. *For $j = 1, \dots, e$ any element $2^{k-j}v_1^{2^j+2^{j-1}-2}(a, b)$ lies in the T -module generated by terms of the form*

$$2^{k-s}v_1^{2^s+2^{j-1}-2}((a+b+2^j-2^s)), \quad \text{with } j < s \leq k.$$

The proof of Proposition 2.1 and the deduction of Theorem 2.1 from Proposition 2.1 are given in the next section. It will be convenient to write

$$2^{k-j}v_1^{2^j+2^{j-1}-2}(a, b) \equiv \sum_{s=j+1}^k 2^{k-s}v_1^{2^s+2^{j-1}-2}((a+b+2^j-2^s)) \quad (2.3)$$

for expressing the conclusion in Proposition 2.1. Here, the congruence symbol is to be read as “modulo T -multiples of the elements on the right-hand side”. This notation will be in force throughout the following sections.

Remark 2.3. Although the proof of Proposition 2.1 is relatively straightforward, and has the $k = e$ case of Theorem 2.1 as an obvious consequence, in Sec. 3 we need to appeal to a rather involved process in order to derive the general case of Theorem 2.1 from Proposition 2.1. In a sense, our approach to Theorem 2.1 fills in the gap for the case $k > e$. The other major feature of our proof for Theorem 2.1 has already been discussed in Remark 2.1, namely, the possibility of extending this result to the BP-case. In this respect, it is to be observed that Proposition 2.1 claims, in particular, that not only the toral class, but the whole tensor product $ku_*(k) \otimes_{ku_*} ku_*(e)$ is killed by $v_1^{2^{e-1}(k-e+3)-2}$ when $k = e$. This situation might also hold for any $k \geq e$, but perhaps not with BP. Thus, in an eventual BP-generalization of these results, the present direct form of Proposition 2.1 might need to be replaced by the type of (inductive) methods in Sec. 3.

3. Proof of Theorem 2.1

The arithmetical manipulations in this section are based on the standard fact that the highest power of 2 dividing the binomial coefficient $\binom{2^e}{s+1}$ is

- (i) equal to $e - \ell$, if $s + 1 = 2^\ell$, for some $\ell = 0, \dots, e$;
- (ii) greater than $e - \ell$, if $2^\ell < s + 1 < 2^{\ell+1}$, for some $\ell = 0, \dots, e - 1$.

For instance, in terms of the notation set up in (2.3), the relation imposed by (2.2) on the second tensor factor of $ku_*(k) \otimes ku_*(e)$ yields

$$2^e(a, b) \equiv \sum_{s=1}^e 2^{e-s} v_1^{2^s-1}(a, b - 2^s + 1),$$

a ($j = 0$)-version of (2.3) when $k = e$ as long as we think of 2^{-1} as being 1. The proof of Proposition 2.1, which is given next, makes a systematic use of analogous considerations based on a suitable combination of the relations coming from both tensor factors.

Proof of Proposition 2.1. Set $g(\ell) = 2^\ell - 1$ and, for a generator (a, b) of $ku_*(k) \otimes ku_*(e)$, write $2^{k-j+1} v_1^{g(j-1)}(a) = 2^{k-j+2} A + A'$ and $2^{e-j} v_1^{g(j)}(b) = 2^{e-j+1} B + B'$ with

$$\begin{aligned} A &\equiv \sum_{s=0}^{j-2} 2^{j-s-2} v_1^{g(s)}(a + g(j-1) - g(s)), & B &\equiv \sum_{s=0}^{j-1} 2^{j-s-1} v_1^{g(s)}(b + g(j) - g(s)), \\ A' &\equiv \sum_{s=j}^k 2^{k-s} v_1^{g(s)}(a + g(j-1) - g(s)), & B' &\equiv \sum_{s=j+1}^e 2^{e-s} v_1^{g(s)}(b + g(j) - g(s)). \end{aligned}$$

Note that the terms coming from (ii) above are meant to be taken care of by the suitable use of the congruence symbols (of course, $B' = 0$ when $j = e$, and $A = 0$ when $j = 1$). Then, with $k = e + d$, $d \geq 0$ we get

$$\begin{aligned} & 2^{k-j} v_1^{g(j)+g(j-1)}(a, b) \\ &= 2^d v_1^{g(j-1)}(a, 2^{e-j} v_1^{g(j)}(b)) \\ &\equiv 2^d v_1^{g(j-1)} \left(a, 2^{e-j+1} B + \sum_{s=j+1}^e 2^{e-s} v_1^{g(s)}(b + g(j) - g(s)) \right) \\ &= 2^{k-j+1} v_1^{g(j-1)}(a, B) + \sum_{s=j+1}^e 2^{k-s} v_1^{g(s)+g(j-1)}(a, b + g(j) - g(s)). \end{aligned}$$

The summation in the previous line has the form required in (2.3). Thus we only need to deal with the term $2^{k-j+1} v_1^{g(j-1)}(a, B)$ which is congruent to

$$2^{k-j+2}(A, B) + \sum_{s=j}^k 2^{k-s} v_1^{g(s)}(a + g(j-1) - g(s), B). \quad (3.1)$$

The last summation is congruent to

$$\sum_{s=j}^k 2^{k-s} v_1^{g(s)} \left(a + g(j-1) - g(s), \sum_{\sigma=0}^{j-1} 2^{j-\sigma-1} v_1^{g(\sigma)}(b + g(j) - g(\sigma)) \right)$$

or, in the proposed shortened form, to

$$\sum_{s=j}^k \sum_{\sigma=0}^{j-1} 2^{k+j-s-\sigma-1} v_1^{g(s)+g(\sigma)}((a + b + g(j) + g(j-1) - g(s) - g(\sigma))) \quad (3.2)$$

for each of whose summands one has $1 \leq s + \sigma + 1 - j \leq k$. Those with $s + \sigma + 1 - j > j$ are (T -multiples of terms) of the form required in (2.3) since, in that case, $g(s) + g(\sigma) \geq g(s + \sigma + 1 - j) + g(j-1)$, as can easily be verified. Those with $s + \sigma + 1 - j = j$ take in fact the form $2^{k-j} v_1^{g(s)+g(\sigma)}(\alpha, \beta)$, where now $g(s) + g(\sigma) \geq g(j) + g(j-1)$, $\alpha < a$, and $\beta > b$. Finally, those with $s + \sigma + 1 - j = \mu$ for a fixed $1 \leq \mu < j$ and necessarily with $\mu \geq \sigma + 1$ are taken into account within

$$\sum_{\sigma=0}^{\mu-1} 2^{k-\mu} v_1^{g(\mu+j-\sigma-1)+g(\sigma)}((\dots)),$$

where numbers inside double parenthesis are forced by dimensional reasons. But since $\mu < j$, all terms in the last summation are T -multiples of that with $\sigma = \mu - 1$. We deduce that the summation in (3.1) is congruent to

$$\begin{aligned} & \sum_{s>j}^k 2^{k-s} v_1^{g(s)+g(j-1)}((\dots)) \\ &+ \sum_{\mu=1}^{j-1} 2^{k-\mu} v_1^{g(j)+g(\mu-1)}((\dots)) + \sum_{(\alpha, \beta) \in \Lambda} 2^{k-j} v_1^{g(j)+g(j-1)}(\alpha, \beta), \end{aligned}$$

where $\Lambda \subseteq \{(\alpha, \beta) | \alpha + \beta = a + b, \alpha < a, \beta > b\}$. On the other hand, the first term in (3.1) is

$$\begin{aligned}
 & 2^{k-j+2}(A, B) \\
 &= 2^{d+1}(A, 2^{e-j+1}B) \\
 &\equiv 2^{d+1} \left(A, \sum_{s=j}^e 2^{e-s} v_1^{g(s)} (b + g(j) - g(s)) \right) \\
 &= \sum_{s=j}^e 2^{k-s+1} v_1^{g(s)} (A, b + g(j) - g(s)) \\
 &\equiv \sum_{s=j}^e 2^{k-s+1} v_1^{g(s)} \left(\sum_{\sigma=0}^{j-2} 2^{j-\sigma-2} v_1^{g(\sigma)} (a + g(j-1) - g(\sigma)), b + g(j) - g(s) \right) \\
 &= \sum_{s=j}^e \sum_{\sigma=0}^{j-2} 2^{k-1+j-s-\sigma} v_1^{g(s)+g(\sigma)} ((a + b + g(j) + g(j-1) - g(s) - g(\sigma))),
 \end{aligned}$$

which is similar to (3.2) and, therefore, the corresponding analysis applies, with the exception that, in the case $s + \sigma + 1 - j = j$, the relations $\alpha < a$ and $\beta > b$ might not necessarily hold but, instead, one gets the strict inequality $g(s) + g(\sigma) > g(j) + g(j-1)$ from the new restriction $\sigma \leq j-2$.

Putting everything together, there results that $2^{k-j} v_1^{g(j)+g(j-1)}(a, b)$ is congruent to

$$\begin{aligned}
 & \sum_{s>j}^k 2^{k-s} v_1^{g(s)+g(j-1)}((\dots)) + \sum_{\mu=1}^{j-1} 2^{k-\mu} v_1^{g(j)+g(\mu-1)}((\dots)) \\
 & + \sum_{R>0} 2^{k-j} v_1^{g(j)+g(j-1)+R}((\dots)) + \sum_{(\alpha, \beta) \in \Lambda} 2^{k-j} v_1^{g(j)+g(j-1)}(\alpha, \beta). \quad (3.3)
 \end{aligned}$$

Now, for $1 \leq \mu \leq j-1$ (holding only for $j > 1$) one has $g(j) + g(\mu-1) > g(\mu) + g(\mu-1)$ so that, by induction, each term on the second summation in (3.3) becomes

$$2^{k-\mu} v_1^{g(j)+g(\mu-1)}((\dots)) \equiv \sum_{\lambda>\mu} 2^{k-\lambda} v_1^{g(\lambda)+g(j)+g(\mu-1)-g(\mu)}((\dots)).$$

Here, terms with $\lambda > j$ are easily seen to be contained (up to congruence) in the first summation of (3.3), whereas those with $\lambda = j$ are contained in the third summation of (3.3). But the terms $2^{k-\lambda} v_1^{g(\lambda)+g(j)+g(\mu-1)-g(\mu)}((\dots))$ with $1 \leq \mu < \lambda \leq j-1$ are easily seen to be T -multiples of the corresponding λ th term in the second summation of (3.3). Therefore, an auxiliary inductive process on $\mu = 1, \dots, j-1$ allows us to get rid of the second summation in the expression (3.3) for $2^{k-j} v_1^{g(j)+g(j-1)}(a, b)$. Then, by iterating the resulting formula, we can also get rid, first, of the last summations in (3.3) and, then, of the third summation in (3.3). $\square$

One further formula is needed before proving Theorem 2.1. We use the relation imposed by (2.2) on the second tensor factor to write

$$2^{e-j}v_1^{g(j)}(a, b) \equiv \sum_{s \in I_j} 2^{e-s}v_1^{g(s)}((a + b + g(j) - g(s))) \quad (3.4)$$

for $j = 1, \dots, e$, where $I_j = \{s | 0 \leq s \leq e, s \neq j\}$. Then, for $j < \ell \leq d + j$ (recall $d = k - e \geq 0$) we have

$$\begin{aligned} & 2^{k-\ell}v_1^{2^{j-1}(\ell+3-j)-2}(a, b) \\ &= 2^{d+j-\ell}v_1^{2^{j-1}(\ell+1-j)-1} \cdot 2^{e-j}v_1^{g(j)}(a, b) \\ &\equiv 2^{d+j-\ell}v_1^{2^{j-1}(\ell+1-j)-1} \left(\sum_{s \in I_j} 2^{e-s}v_1^{g(s)}((a + b + g(j) - g(s))) \right) \\ &= \sum_{s \in I_j} 2^{k+j-\ell-s}v_1^{2^{j-1}(\ell-j)+g(j-1)+g(s)}((a + b + g(j) - g(s))). \end{aligned} \quad (3.5)$$

Note that, in the last summation, $1 \leq \ell + s - j \leq k$. In particular, we could use Proposition 2.1 in order to get rid of some such summands (those with a high enough power of v_1). However the resulting expression seems to become unnecessarily complicated. Instead, all summands in (3.5) will be taken care of (in the arguments below) by means of a suitable inductive process.

We now start working toward the proof of Theorem 2.1. The relation $2^e(0, 0) = 0$ is obvious as it comes directly from the second tensor factor. In fact, in $ku_*(e)$ one can easily prove (see, for instance, [9, Corollaries 2.6 and 2.7])

$$2^{e+i}(i) = 0 \quad \text{and} \quad 2^{e+i-1}v_1(i) = 2^{e+i}(i+1), \quad \text{for } i \geq 0. \quad (3.6)$$

This implies the $(j = 1)$ -case in the conclusion of Theorem 2.1:

$$\begin{aligned} 2^{e-1}v_1^{k-e+1}(0, 0) &= 2^e v_1^{k-e}(0, 1) \\ &= 2^{e+1}v_1^{k-e-1}(0, 2) = \dots = 2^k(0, d+1) = 0, \end{aligned} \quad (3.7)$$

where the last equality comes directly from the bottom relation in the first tensor factor. However, the $(j > 1)$ -cases in the conclusion of Theorem 2.1 are much more cumbersome to derive. The auxiliary constructions below are intended to organize an elaborate process whose main idea is to use (3.5) as a generalization of (3.7) in order to fill in the gap left by (2.3) when $k > e$ (this is a slightly more detailed description of the first two sentences in Remark 2.3). Thus, throughout the rest of this section we assume $k = e + d$ with $d > 0$.

Definition 3.1. Set $p(0) = -1$ and $p(\sigma) = (d+1)g(\sigma-1)$ for $\sigma \geq 1$. Consider the set J consisting of all pairs of non-negative integers (i, j) satisfying $1 \leq j \leq d + \sigma$ whenever $p(\sigma-1) < i \leq p(\sigma)$ with $1 \leq \sigma \leq e$. The number σ above is determined

by i ; yet at times we will denote it as $\sigma(i, j)$. Then, for $(i, j) \in J$, set

$$s(i, j) = \begin{cases} d + 1 + i - 2^{j-1} & \text{for } j \leq \sigma(i, j), \\ d + 1 + i - 2^{\sigma(i, j)-1}(j - \sigma(i, j) + 1) & \text{for } j \geq \sigma(i, j), \end{cases}$$

$$u(i, j) = \begin{cases} g(j) + g(\sigma(i, j) - 1) & \text{for } j \leq \sigma(i, j), \\ 2^{\sigma(i, j)-1}(j + 3 - \sigma(i, j)) - 2 & \text{for } j \geq \sigma(i, j), \end{cases}$$

and

$$E(i, j) = 2^{k-j} v_1^{u(i, j)}((s(i, j))).$$

Note that, for $1 \leq \sigma \leq e$, $s(p(\sigma), d + \sigma) = 0$ and $u(p(\sigma), d + \sigma) = 2^{\sigma-1}(d + 3) - 2$, so that $E(p(\sigma), d + \sigma)$ reduces to the term $\varepsilon_\sigma \cdot (0, 0)$ in Theorem 2.1. Thus, Theorem 2.1 is a particular case of the following more complete result.

Theorem 3.1. $E(i, j) = 0$ for $(i, j) \in J$.

The main computational task in this section is to show that, for any $(i, j) \in J$ and in terms of the convention set up in (2.3),

$$E(i, j) \equiv \sum E(i', j'), \quad (3.8)$$

where the summation runs over (some of) those $(i', j') \in J$ which, in the lexicographic order of J , satisfy $(i', j') < (i, j)$. Theorem 3.1 will then be a consequence of the following.

Lemma 3.1. $E(0, 1) = 0$.

Proof. $E(0, 1) = 2^{k-1} v_1((d)) \equiv \sum_{s \geq 2}^k 2^{k-s} v_1^{g(s)}((d + 2 - 2^s))$ in view of (2.3). But, in view of the first relation in (3.6), each of these summands is trivial as $d - 2^s + 2 + e \leq k - s$ for $s \geq 2$. $\square$

The remainder of this section is devoted to establishing (3.8). By the way of example, we first complete the analysis for the case $i = 0$ (that is $\sigma = 1$) started in the proof of Lemma 3.1. Thus, for $1 < j \leq d + 1$, (3.5) yields

$$E(0, j) = 2^{k-j} v_1^j((d + 1 - j)) = \sum_{s \in I_1} 2^{k+1-j-s} v_1^{j-1+g(s)}((d + 3 - j - 2^s)),$$

where I_1 is defined in (3.4). The term with $s = 0$ corresponds to $E(0, j - 1)$ whereas, as in the proof of Lemma 3.1, the terms with $s \geq 2$ are trivial in view of (3.6). The proof of (3.8) in the general case is as follows.

Proof of (3.8). Suppose $p(\sigma - 1) < i \leq p(\sigma)$, with $\sigma \geq 2$, and consider $E(i, j)$ as given in Definition 3.1 (so that $1 \leq j \leq d + \sigma$).

Case $1 \leq j \leq \sigma$. From (2.3) we have

$$\begin{aligned} E(i, j) &= 2^{k-j} v_1^{g(j)+g(\sigma-1)} ((d+1+i-2^{j-1})) \\ &= \sum_{s>j}^k 2^{k-s} v_1^{g(s)+g(\sigma-1)} ((d+1+i+2^{j-1}-2^s)). \end{aligned}$$

The summation really runs for $j < s < d + \sigma$ since, when $s \geq d + \sigma$ (and given that $i \leq p(\sigma)$ and $j \leq \sigma$), one actually has $d+1+i+2^{j-1}-2^s < 0$. But in this restricted range, one easily checks that the s th summand in the last summation is a T -multiple of $E(i-1, s)$. Of course such a fact has to be verified by dividing into cases, depending on whether $s \geq \sigma$ or $s \leq \sigma$, and whether $i-1 > p(\sigma-1)$ or $i-1 = p(\sigma-1)$. The actual verifications are left to the reader and, for the sake of illustration, we only sketch a representative situation. When $\sigma \leq s$ and $i-1 > p(\sigma-1)$ one has

$$E(i-1, s) = 2^{k-s} v_1^{2^{\sigma-1}(s+3-\sigma)-2} ((d+i-2^{\sigma-1}(s-\sigma+1)))$$

so that we only need to verify that $d+1+i+2^{j-1}-2^s \leq d+i-2^{\sigma-1}(s-\sigma+1)$ (for which it is convenient to keep in mind that, in the present situation, $j \leq \sigma \leq s$ with $j < s$) and that $2^{\sigma-1}(s+3-\sigma)-2 \leq g(s)+g(\sigma-1)$.

Case $\sigma < j \leq d + \sigma$. Using (3.5) we now have

$$\begin{aligned} E(i, j) &= 2^{k-j} v_1^{2^{\sigma-1}(j+3-\sigma)-2} ((d+1+i-2^{\sigma-1}(j-\sigma+1))) \\ &= \sum_{s \in I_\sigma} 2^{k+\sigma-j-s} v_1^{2^{\sigma-1}(j-\sigma)+g(\sigma-1)+g(s)} \\ &\quad \times ((d+1+i-2^{\sigma-1}(j-\sigma+1)+2^\sigma-2^s)). \end{aligned}$$

This time the summation is relevant only for $j+s-\sigma \leq d+\sigma-1$ because, otherwise, the summands have a ((*negative*))-part. One then verifies (again taking into consideration suitable cases, so that Definition 3.1 is applied correctly) that summands with $s < \sigma$ (so that $1 \leq j+s-\sigma < j$) are T -multiples of $E(i, j+s-\sigma)$, whereas summands with $s > \sigma$ are T -multiples of $E(i-1, j+s-\sigma)$. $\square$

4. Proof of Theorem 2.2

In this section $k \geq e$ are fixed, and attention is focused on the ku_* -annihilator ideal of the toral class $\tau = \tau_{k,e} \in ku_2 B(\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^e})$.

Proposition 4.1. *Let $2^{p_i} v_1^{q_i}$, $i = 1, \dots, T$, be elements annihilating τ such that no $2^{p_i} v_1^{q_i-1}$ annihilates τ , $i = 2, 3, \dots, T$. If $p_{i+1} = p_i - 1$, $p_T = 0$, and $0 = q_1 < q_2 < \dots < q_T$, then in fact $\text{Ann}(\tau) = (2^{p_i} v_1^{q_i})_{i=1, \dots, T}$.*

Proof. Let $f \in \text{Ann}(\tau)$ be a homogeneous element. Since $2^{p_1} \tau = 0$ (and odd integers represent units in $\mathbb{Z}_{2^{p_1}}$), we can assume $f = 2^m v_1^n$. If $m \geq p_1$, then evidently

$2^m v_1^n \in (2^{p_i} v_1^{q_i})_{i=1, \dots, T}$. Assume $m = p_\ell$, $\ell = 2, \dots, T$. If $n < q_\ell$, then $2^{p_\ell} v_1^{q_\ell-1}$ is a multiple of $2^m v_1^n \in \text{Ann}(\tau)$ and therefore we would have $2^{p_\ell} v_1^{q_\ell-1} \in \text{Ann}(\tau)$, which contradicts the hypothesis. Thus we must have $n \geq q_\ell$, so that $f = 2^m v_1^n$ is a multiple of $2^{p_\ell} v_1^{q_\ell}$, and evidently $f \in (2^{p_i} v_1^{q_i})_{i=1, \dots, T}$. $\square$

This section's strategy is to work directly in the free ku_* -module F with basis the pairs (i, j) , $i, j \geq 0$, and observe that Conjecture 2.2 can be proved by checking that it is not possible to have in F relations of the form

$$2^{e-j} v_1^{f(j)-1}(0, 0) = \sum_{\alpha, \beta} c_{\alpha, \beta} g_{\alpha, \beta} + \sum_{\alpha, \beta} d_{\alpha, \beta} h_{\alpha, \beta}, \quad c_{\alpha, \beta}, d_{\alpha, \beta} \in \mathbb{Z}, \quad (4.1)$$

where $f(j) = 2^{j-1}(k - e + 3) - 2$,

$$g_{\alpha, \beta} = \binom{2^e}{1} v_1^{f(j)-\alpha-\beta-1}(\alpha, \beta) + \binom{2^e}{2} v_1^{f(j)-\alpha-\beta}(\alpha, \beta - 1) + \dots, \quad (4.2)$$

$$h_{\alpha, \beta} = \binom{2^k}{1} v_1^{f(j)-\alpha-\beta-1}(\alpha, \beta) + \binom{2^k}{2} v_1^{f(j)-\alpha-\beta}(\alpha - 1, \beta) + \dots, \quad (4.3)$$

and where the summations in (4.1) run over indexes α and β subject to

$$\alpha, \beta \geq 0, \quad \alpha + \beta < f(j). \quad (4.4)$$

The 2-divisibility properties of binomial coefficients will play a crucial role in our arguments, and we record for future reference a well-known result stating the form in which we will make use of these 2-divisibility properties.

Lemma 4.1. *The functions α and ν defined respectively in (1.3) and (1.5) satisfy $\nu\binom{a}{b} = \alpha(b) + \alpha(a - b) - \alpha(a)$. In particular, if a, b, c are non-negative integers with $b \leq a$ and $1 \leq c < 2^{a-b+1}$, the binomial coefficient $\binom{2^a}{c}$ is divisible by 2^b (a more precise form of this assertion — not needed in the sequel — has already been noted at the beginning of Sec. 3).*

Throughout the rest of the section we will assume, to reach a contradiction, that an equation of the form (4.1) has been given with $j \leq 2$. The next two propositions will be central.

Proposition 4.2.

$$\nu(c_{\alpha, \beta}) \geq \begin{cases} \alpha + \beta & \text{if } j = 1, \\ \alpha + \beta - d - 3 & \text{if } j = 2. \end{cases}$$

Proof. Case $j = 1$. We proceed by inverse induction on $\alpha + \beta$. Notice that, in agreement with (4.4), the range of interest is $0 \leq \alpha + \beta \leq d$ (we keep the notation $d = k - e$ introduced in the previous section). Let us focus attention on a basis element (α, β) in F with $\alpha + \beta = d$. In these conditions the terms $g_{\alpha, \beta}$ and $h_{\alpha, \beta}$ are the only ones contributing in (4.1) with multiples of the basis element (α, β) .

The contributions are $2^e c_{\alpha,\beta}$ and $2^k d_{\alpha,\beta}$, respectively. Therefore, we should have $\nu(c_{\alpha,\beta}) \geq k - e$.

We now assume $\nu(c_{\alpha',\beta'}) \geq \alpha' + \beta'$ whenever $\alpha' + \beta' > p$ for some $p < d$, and demonstrate that $\nu(c_{\alpha,\beta}) \geq p$ given α and β with $\alpha + \beta = p$. Again, we focus attention on the basis element (α, β) in F , and the way it can arise on the right-hand side of (4.1). By virtue of (4.3), the contributions from the second summation in (4.1) must arise from summands having an index of the form (r, β) , $r \geq \alpha$. For these summands we have

$$\nu\left(\binom{2^k}{r - \alpha + 1}\right) \geq p + e,$$

due to Lemma 4.1, as $r - \alpha + 1 = r + \beta - p + 1 \leq d - p + 1 < 2^{d-p+1}$ and $p + e \leq k$ — the latter inequality holds because $p = \alpha + \beta \leq d = k - e$. On the other hand, the contributions that arise from the first summation in (4.1) come from indexes of the form (α, s) , with $s \geq \beta$, and by induction satisfy the inequality $\nu(c_{\alpha,s}) + \nu\left(\binom{2^e}{s - \beta + 1}\right) \geq p + e$ for $s > \beta$. Indeed,

$$\nu\left(\binom{2^e}{s - \beta + 1}\right) \geq p + e - \alpha - s$$

holds because the conditions of Lemma 4.1 are satisfied: $p + e - \alpha - s \leq e$ since $p = \alpha + \beta < \alpha + s$, while $s - \beta + 1 < 2^{\alpha+s-p+1}$ since $s - \beta + 1 = s + \alpha - p + 1 < 2^{\alpha+s-p+1}$. Therefore, when $s = \beta$ we must also have $\nu(c_{\alpha,\beta}) + \nu\left(\binom{2^e}{1}\right) \geq p + e$; that is

$$\nu(c_{\alpha,\beta}) \geq p = \alpha + \beta.$$

Case $j = 2$. We proceed by inverse induction on $\alpha + \beta$. This time the range under consideration is $0 \leq \alpha + \beta < f(2) = 2d + 4$. So, to ground the induction, assume $\alpha + \beta = 2d + 3$. As in previous cases, we focus attention on the basis element (α, β) , and the way it can arise on the right-hand side of (4.1). Only $g_{\alpha,\beta}$ and $h_{\alpha,\beta}$ are relevant, and the corresponding contributions to the summand (α, β) are $2^e c_{\alpha,\beta}$ and $2^k d_{\alpha,\beta}$. Therefore, we must have $\nu(c_{\alpha,\beta}) \geq k - e$.

Now, we assume that $\nu(c_{\alpha',\beta'}) \geq \alpha' + \beta' - d - 3$ whenever $\alpha' + \beta' > p$ for some $p < 2d + 3$, and demonstrate that $\nu(c_{\alpha,\beta}) \geq p - d - 3$ when $\alpha + \beta = p$. We focus attention on the basis element (α, β) in F . In view of (4.3), the contributions of the second summation in (4.1) must arise from summands with index of the form (r, β) , $r \geq \alpha$. For these we have

$$\nu\left(\binom{2^k}{r - \alpha + 1}\right) \geq p + e - d - 3,$$

because of Lemma 4.1 in view of the fact that $p + e - d - 3 \leq k$ since $p = \alpha + \beta \leq 2d + 3 = 2(k - e) + 3$ and $r - \alpha + 1 = r + \beta - p + 1 \leq 2d + 3 - p + 1 < 2^{2d-p+4}$. On the other hand, the contributions from the first summation of (4.1) arise from indexes of the form (α, s) , with $s \geq \beta$, and by induction satisfy the inequality

$\nu(c_{\alpha,s}) + \nu\binom{2^e}{s-\beta+1} \geq p + e - d - 3$ for $s > \beta$. Indeed,

$$\nu\binom{2^e}{s-\beta+1} \geq p + e - \alpha - s$$

holds because the conditions of Lemma 4.1 are satisfied: $p + e - \alpha - s \leq e$ since $p = \alpha + \beta \leq \alpha + s$, while $s - \beta + 1 < 2^{\alpha+s-p+1}$ since $s - \beta + 1 = \alpha + s - p + 1 < 2^{\alpha+s-p+1}$. Therefore, when $s = \beta$ we must also have $\nu(c_{\alpha,\beta}) + \nu\binom{2^e}{1} \geq p + e - d - 3$, that is $\nu(c_{\alpha,\beta}) \geq p - d - 3 = \alpha + \beta - d - 3$. $\square$

Proposition 4.3. *In the case $j = 2$, the inequality of Proposition 4.2 can be improved to $\nu(c_{\alpha,\beta}) \geq \alpha + \beta + i - d - 3$ when the following conditions are fulfilled:*

- (a) $d + 4 - i \leq \alpha + \beta \leq 2d - 2i + 4$;
- (b) $\alpha \leq d + 3 - i$;
- (c) $1 \leq i \leq d$.

Proof. We proceed by double induction, first on i in a range limited by (c), and then on $\alpha + \beta$ in the range specified by (a). Let us consider a triple (i, α, β) satisfying (a), (b), and (c), and assume inductively that the proposition has been verified for any other triple (i', α', β') that, in addition to fulfilling (a), (b), and (c), satisfies one of the following conditions:

- (d) $i' < i$;
- (e) $i' = i$ and $\alpha' + \beta' < \alpha + \beta$.

Of course, the induction hypotheses associated with (d) and (e) are empty at the start of the corresponding induction. We note that the three conditions (a), (b), and (c) imply $\beta \geq 1$ and $\alpha + \beta \geq 4$. In particular, $(\alpha, \beta - 1)$ and $(0, 0)$ are different basis elements and, consequently, the total coefficient T with which $(\alpha, \beta - 1)$ appears on the right-hand side of (4.1) must be null. Then, in view of (4.2), the conclusion we need in order to complete the induction translates into verifying that the contribution to T from the summand with index (α, β) in the first summation of (4.1) is divisible by

$$2^{\alpha+\beta+i+e-d-4} \tag{4.5}$$

(notice that in view of (a), the exponent in (4.5) is at least e). But since $T = 0$, it suffices to verify that the rest of the contributions to T from the right-hand side of (4.1) are also divisible by (4.5).

By virtue of (4.3), the contributions of the second summation in (4.1) must arise from summands with index of the form $(r, \beta - 1)$, $r \geq \alpha$. For these it will be enough to verify that $\binom{2^k}{r-\alpha+1}$ is divisible by (4.5). But this will be a consequence of Lemma 4.1, as soon as we establish the inequalities

- (f) $\alpha + \beta + i + e - d - 4 \leq k$;
- (g) $r - \alpha + 1 < 2^{k+d-\alpha-\beta-i-e+5}$.

The first condition is equivalent to $\alpha + \beta \leq 2d - i + 4$ and, therefore, it is guaranteed by (a). On the other hand, in case of the index $(r, \beta - 1)$, the second inequality in (4.4) asserts that $r - \alpha + 1 \leq 2d - \alpha - \beta + 5$, and since $2^m \geq 2m$ for every $m \in \mathbb{Z}$, (g) will follow as soon as the inequality

$$2d - \alpha - \beta + 5 < 2(k - \alpha - \beta - i - e + d + 5) \quad (4.6)$$

is justified. But an elementary arithmetic manipulation shows that the latter inequality is equivalent to the second inequality in (a).

It remains to consider the contributions coming from summands in the first summation of (4.1) having index of the form (α, s) , with $s \geq \beta - 1$, $s \neq \beta$. For these we have to verify

$$\nu(c_{\alpha,s}) + \nu\left(\binom{2^e}{s - \beta + 2}\right) \geq \alpha + \beta + i + e - d - 4. \quad (4.7)$$

The case $s = \beta - 1$ follows from the case (e) of the induction, except when $\alpha + \beta = d + 4 - i$ — corresponding to the beginning of the induction — in which case (4.7) is obvious. Hence, throughout the rest of the proof we will assume

$$s > \beta. \quad (4.8)$$

Likewise, when $i = 1$, Proposition 4.2 implies that (4.7) is a consequence of the inequality $\nu\left(\binom{2^e}{s - \beta + 2}\right) \geq \beta + e - s$, which in turn follows easily from Lemma 4.1. Thus we will also assume

$$i \geq 2. \quad (4.9)$$

Furthermore, when $\alpha + s = 2d + 3$ (recall that the upper limit for $\alpha + s$ is set by (4.4)), Proposition 4.2 asserts that (4.7) is a consequence of the inequality $\nu\left(\binom{2^e}{s - \beta + 2}\right) \geq \alpha + \beta + i + e - 2d - 4$, which in turn follows from Lemma 4.1. Indeed, the conditions

- (j) $\alpha + \beta + i \leq 2d + 4$
- (k) $s - \beta + 2 < 2^{2d - \alpha - \beta - i + 5}$

required by the above-mentioned lemma are deduced directly from (a) — as with (4.6), for (k) it is convenient to keep in mind the simple inequality $2^m \geq 2m$ for every $m \in \mathbb{Z}$. Thus, we will assume in addition

$$\alpha + s < 2d + 3. \quad (4.10)$$

Now, when $\alpha + s \leq 2d - 2i + 6$, the triple $(i - 1, \alpha, s)$ satisfies the conditions (a), (b), and (c) of Proposition 4.3. Then, the case (d) of the induction assures that (4.7) is a consequence of the inequality $\nu\left(\binom{2^e}{s - \beta + 2}\right) \geq \beta + e - s$ which, just as in the considerations previous to (4.9), follows directly from Lemma 4.1. Therefore, we can now assume

$$2d - 2i + 6 < \alpha + s. \quad (4.11)$$

Note that (4.10) and (4.11) imply that (4.9) specializes to $i \geq 3$, while (4.11) and (a) imply that (4.8) specializes to

$$s > \beta + 2. \quad (4.12)$$

At this point we are in a position to repeat the analysis performed after (4.10): When $\alpha + s \leq 2d - 2i + 8$, and by virtue of (4.11), the triple $(i - 2, \alpha, s)$ satisfies the conditions (a), (b), and (c) of Proposition 4.3. So, the clause (d) of the induction assures that (4.7) is a consequence of the relation $\nu\binom{2^e}{s-\beta+2} \geq \beta + e - s + 1$. In turn, the latter inequality is guaranteed by Lemma 4.1 — the verification of the second hypothesis in that lemma uses (4.12) together with the fact that $2^m > m + 2$, for $m > 2$. As a result, we can assume that (4.11) specializes to

$$2d - 2i + 8 < \alpha + s.$$

Recursively, assume that, for some $j \geq 3$, (4.7) has been proved except for

$$2(d - i + j + 1) < \alpha + s. \quad (4.13)$$

Note that (4.10) and (4.13) imply that $i \geq j + 1$; whereas (4.13) and (a) imply

$$s > \beta + 2(j - 1). \quad (4.14)$$

Then, if (4.13) is satisfied together with the inequality $\alpha + s \leq 2(d - i + j + 2)$, the triple $(i - j, \alpha, s)$ fulfills the conditions (a), (b), and (c) in Proposition 4.3. Therefore, the case (d) of the induction assures that (4.7) is a consequence of the inequality $\nu\binom{2^e}{s-\beta+2} \geq \beta + e - s + j - 1$, which in turn is guaranteed by Lemma 4.1 (the verification of the second hypothesis in the aforementioned lemma uses (4.14) together with the fact that $2^m > m + j$, for $m > j$). This completes the recursive process (which is finite, in view of (4.10)), concluding the verification of (4.7) and, thus, the proof of Proposition 4.3. $\square$

The main result of this section, Conjecture 2.2 for $j \leq 2$, will be used in the proof of Proposition 5.2 which is a key step in the proof of Theorem 1.1 and Corollary 1.1 — our main applications to the topological complexity of lens spaces. Explicitly, the proof of Theorem 2.2 is as follows:

Proof of Theorem 2.2. The easy part is for $j = 1$, for which we will see that all contributions to the term $v_1^d(0, 0)$ on the right-hand side of (4.1) have integer coefficients divisible by 2^e . The contributions that come from the second summation arise from indexes of the form $(r, 0)$ with $r \geq 0$ and the corresponding coefficient is divisible by $\binom{2^k}{r+1}$. But $\nu\binom{2^k}{r+1} \geq e$ in view of Lemma 4.1. Indeed, the relation $r + 1 < 2^{k-e+1}$ is assured by (4.4), which in our case translates into $r \leq d = k - e$. On the other hand, the contributions that come from the first summation in (4.1) arise from indexes of the form $(0, s)$ with $s \geq 0$ and they have coefficient $\binom{2^e}{s+1}c_{0,s}$. In such cases, making use of Proposition 4.2, it suffices to verify the inequality $\nu\binom{2^e}{s+1} \geq e - s$. But the latter relation is an easy consequence of Lemma 4.1.

The crux of the matter is dealing with the case $j = 2$. For organizational purposes we start by settling a few preliminary steps.

Step 1. Let us focus attention on the multiples of a basis element of the form (α, β) with $\alpha + \beta = 2d + 3$. As we have noticed before, such a basis element arises on the right-hand side of (4.1) only from the summand with index (α, β) , in both summations of (4.1). Thus, we obtain the relation

$$0 = 2^e c_{\alpha, \beta} + 2^k d_{\alpha, \beta}. \quad (4.15)$$

But in view of Proposition 4.2, the coefficient $c_{\alpha, \beta}$ in (4.15) takes the form $c_{\alpha, \beta} = 2^d c'_{\alpha, \beta}$ for some integer $c'_{\alpha, \beta}$. In these terms, the only information we need from (4.15) is given by the mod-2 congruences

$$\begin{aligned} c'_{1, 2d+2} + d_{1, 2d+2} &\equiv 0, \\ c'_{2, 2d+1} + d_{2, 2d+1} &\equiv 0. \end{aligned} \quad (4.16)$$

Step 2. Let us now focus attention on the way a basis element (α, β) with $\alpha + \beta = 2d + 2$ arises on the right-hand side of (4.1). The total coefficient of (the v_1 -multiple of) (α, β) is

$$0 = 2^e c_{\alpha, \beta} + \binom{2^e}{2} c_{\alpha, \beta+1} + 2^k d_{\alpha, \beta} + \binom{2^k}{2} d_{\alpha+1, \beta}.$$

The first summand on the right-hand side of this expression vanishes modulo 2^k provided $0 \leq \alpha \leq 2$ because, if $e < k$, Proposition 4.3 (with the triple $(1, \alpha, \beta)$) produces $\nu(c_{\alpha, \beta}) \geq d$. We thus have the congruences modulo 2^k

$$0 \equiv \binom{2^e}{2} c_{\alpha, \beta+1} + \binom{2^k}{2} d_{\alpha+1, \beta}, \quad \text{for } 0 \leq \alpha \leq 2. \quad (4.17)$$

Making use of the fact that $\nu\left(\binom{2^q}{2}\right) = q - 1$, together with the expression $c_{\alpha, \beta+1} = 2^d c'_{\alpha, \beta+1}$ described above, we see that (4.17) is equivalent to the following three congruences modulo 2:

$$\begin{aligned} 0 &\equiv c'_{0, 2d+3} + d_{1, 2d+2}, \\ 0 &\equiv c'_{1, 2d+2} + d_{2, 2d+1}, \\ 0 &\equiv c'_{2, 2d+1} + d_{3, 2d}. \end{aligned}$$

Together with the congruences in (4.16), this leads to the mod-2 congruence

$$0 \equiv c'_{0, 2d+3} + d_{3, 2d}. \quad (4.18)$$

Step 3 (conclusion of the proof). Let us consider the way in which the basis element $(0, 2d)$ arises on the right-hand side of (4.1). Notice that this basis element appears on the left-hand side of (4.1) only when $e = k$. So, the total coefficient with which (the v_1^3 -multiple of) $(0, 2d)$ appears in (4.1) is

$$\begin{aligned} 2^{e-2} \delta(e, k) &= 2^e c_{0, 2d} + \binom{2^e}{2} c_{0, 2d+1} + \binom{2^e}{3} c_{0, 2d+2} + \binom{2^e}{4} c_{0, 2d+3} \\ &\quad + 2^k d_{0, 2d} + \binom{2^k}{2} d_{1, 2d} + \binom{2^k}{3} d_{2, 2d} + \binom{2^k}{4} d_{3, 2d}, \end{aligned} \quad (4.19)$$

where $\delta(e, k)$ is Kronecker's delta. We next show that each of the terms on the right-hand side of (4.19) vanishes modulo 2^{k-1} , except perhaps for the second, fourth, and last terms.

Case of $2^e c_{0,2d}$. The claim is obvious for $e+1 \geq k$. When $k > e+1$, Proposition 4.3 (with the triple $(i, \alpha, \beta) = (2, 0, 2d)$) produces $\nu(c_{0,2d}) \geq d-1$, so that $\nu(2^e c_{0,2d}) \geq e+d-1 = k-1$.

Case of $\binom{2^e}{3} c_{0,2d+2}$. On the one hand, we have $\nu\left(\binom{2^e}{3}\right) = e$, and on the other hand, Proposition 4.2 implies that $\nu(c_{0,2d+2}) \geq d-1$, so that $\nu\left(\binom{2^e}{3} c_{0,2d+2}\right) \geq e+d-1 = k-1$.

Case of the fifth, sixth, and seventh terms on the right-hand side of (4.19). The affirmation is obvious since $\nu\left(\binom{2^k}{2}\right) = k-1$ and $\nu\left(\binom{2^k}{3}\right) = k$.

Therefore, modulo 2^{k-1} , (4.19) reduces to

$$2^{e-2} \delta(e, k) \equiv \binom{2^e}{2} c_{0,2d+1} + \binom{2^e}{4} c_{0,2d+3} + \binom{2^k}{4} d_{3,2d}. \quad (4.20)$$

Now, when $e = k$, the first term on the right-hand side of (4.20) is trivial, whereas the rest of the binomial coefficients are exactly divisible by 2^{k-2} , so that (4.20) simplifies to the congruence

$$1 \equiv c_{0,2d+3} + d_{3,2d}$$

modulo 2, which contradicts (4.18) and, thus, completes the proof of Theorem 2.2 for $e = k$. On the other hand, when $k > e$, Eq. (4.20) takes the form

$$0 \equiv \binom{2^e}{2} c_{0,2d+1} + \binom{2^e}{4} c_{0,2d+3} + \binom{2^k}{4} d_{3,2d} \quad (4.21)$$

modulo 2^{k-1} . But in terms of the notation $c_{0,2d+3} = 2^d c'_{0,2d+3}$ and the corresponding fact that $c_{0,2d+1} = 2^{d-1} c'_{0,2d+1}$ with $c'_{0,2d+1}$ an odd integer (justified by Lemma 4.2 below), (4.21) translates into $0 \equiv c'_{0,2d+1} + c'_{0,2d+3} + d_{3,2d} \equiv 1 + c'_{0,2d+3} + d_{3,2d}$ modulo 2 which, again, contradicts (4.18). $\square$

It only remains to prove that $c_{0,2d+1} = 2^{d-1} c'_{0,2d+1}$ with $c'_{0,2d+1}$ an odd integer. Note that the inequality $\nu(c_{0,2d+1}) \geq d-1$ is guaranteed for $d \geq 2$ by Proposition 4.3 with $(i, \alpha, \beta) = (1, 0, 2d+1)$. Hence, our aim is to prove that such inequality can be refined to an equality for $d \geq 1$. We prove in fact the following.

Lemma 4.2. *In the case $j = 2$ of (4.1), we have $\nu(c_{0,2\ell+1}) = \ell - 1$ for $\ell = 1, 2, \dots, d$.*

Proof. We proceed by induction on ℓ . For $\ell = 1$, we start by analyzing the coefficients of $v_1^{2d+3}(0, 0)$ on both sides of (4.1). On the left-hand side, the coefficient is 2^{e-2} . On the right-hand side, the coefficients coming from the second summation

arise from summands having index $(\alpha, 0)$ with

$$\alpha + 1 \leq 2^k \quad \text{and} \quad \alpha \leq 2d + 3 \quad \text{— the latter in view of (4.4).} \quad (4.22)$$

The relevant coefficient $\binom{2^k}{\alpha+1}$ is divisible by 2^{e-1} in view of Lemma 4.1 since the required hypotheses $e - 1 \leq k$ and $\alpha + 1 < 2^{k-e+2}$ are satisfied — the latter follows from the second inequality in (4.22) and the facts that $d \geq 1$ and

$$2m < 2^m \quad \text{for all } m \geq 3. \quad (4.23)$$

On the other hand, the coefficients coming from the first summation of (4.1) arise from summands having an index $(0, \beta)$ with

$$\beta \leq 2d + 3, \quad \text{and} \quad \beta + 1 \leq 2^e. \quad (4.24)$$

For such summands, the relevant coefficient is $c_{0,\beta} \binom{2^e}{\beta+1}$. We show in the next paragraph that

$$\nu(c_{0,\beta}) + \nu \binom{2^e}{\beta+1} \geq e - 1 \quad (4.25)$$

for $\beta \neq 3$. Once this is done, the remaining coefficient to analyze — the one with $\beta = 3$ — will be forced to be divisible by 2^{e-2} but not by 2^{e-1} , so that $\nu(c_{0,3}) + \nu \binom{2^e}{4} = e - 2$, that is $\nu(c_{0,3}) = 0$, completing the proof for $\ell = 1$.

When $0 \leq \beta \leq 2$ we have

$$\nu \binom{2^e}{\beta+1} = \begin{cases} \nu \binom{2^e}{1} = e & \text{if } \beta = 0, \\ \nu \binom{2^e}{2} = e - 1 & \text{if } \beta = 1, \\ \nu \binom{2^e}{3} = e & \text{if } \beta = 2, \end{cases}$$

implying (4.25). So, we can safely assume $\beta \geq 4$. If β is even, (4.25) is evident as $\nu \binom{2^e}{\beta+1} = e$. If $\beta = 2m + 1$, then $m \geq 2$ and (4.24) reads

$$2m + 1 \leq 2d + 3, \quad \text{i.e. } m \leq d + 1,$$

$$2m + 2 \leq 2^e, \quad \text{i.e. } m < 2^{e-1}.$$

We now consider the following two cases.

Case $m = d + 1$. In view of Proposition 4.2, it suffices to verify the inequality

$$2m + 1 - d - 3 + \nu \binom{2^e}{\beta+1} \geq e - 1,$$

that is

$$\nu \binom{2^e}{\beta+1} \geq e - d - 1.$$

But this is a consequence of Lemma 4.1 since the two required hypotheses $e - d - 1 \leq e$ and $\beta + 1 < 2^{d+2}$ hold — the latter follows from (4.23) and the fact that $\beta + 1 = 2d + 4$.

Case $m \leq d$. The three hypotheses in Proposition 4.3 for $i = d + 1 - m$ and $c_{0,\beta}$ are

$$d + 4 - d - 1 + m \leq \beta \leq 2d - 2d - 2 + 2m + 4,$$

$$0 \leq d + 3 - d - 1 + m,$$

$$1 \leq d + 1 - m \leq d,$$

or, equivalently

$$m + 3 \leq \beta \leq 2m + 2,$$

$$0 \leq 2 + m,$$

$$1 \leq m \leq d,$$

all of which are obvious. Therefore it suffices to check that

$$\beta + d + 1 - m - d - 3 + \nu\left(\frac{2^e}{\beta + 1}\right) \geq e - 1.$$

But since $\beta = 2m + 1$, this simplifies to $\nu\left(\frac{2^e}{\beta + 1}\right) \geq e - m$ which, in turn, follows from Lemma 4.1 as the two required hypotheses $e - m \leq e$ and $\beta + 1 < 2^{m+1}$ hold — the latter follows from (4.23) and $\beta = 2m + 1$. This completes the verification of (4.25) and, thus, of the case $\ell = 1$ of the lemma.

For the inductive step of the proof ($\ell \geq 2$), we analyze the coefficients for $v_1^{2d-2\ell+5}(0, 2\ell - 2)$ on both sides of (4.1). The coefficient is zero on the left-hand side. On the right-hand side, the coefficients coming from the second summation arise from summands having index $(\alpha, 2\ell - 2)$ with

$$\alpha + 1 \leq 2^k \quad \text{and} \quad 2\ell + \alpha - 2 \leq 2d + 3. \quad (4.26)$$

In view of Lemma 4.1, the coefficient for such a summand $\binom{2^k}{\alpha+1}$ is divisible by $2^{e+\ell-2}$ as $k \geq k - 2 = e + d - 2 \geq e + \ell - 2$, $\alpha + 1 \leq 2d + 6 - 2\ell$ by (4.26), and $2d + 6 - 2\ell < 2^{d-\ell+3} = 2^{k-e-\ell+3}$ by (4.23). On the other hand, the coefficients coming from the first summation in (4.1) arise from summands having an index $(0, \beta)$ with

$$2\ell - 2 \leq \beta, \quad \beta \leq 2d + 3, \quad \text{and} \quad \beta - 2\ell + 3 \leq 2^e.$$

We claim that the coefficient $c_{0,\beta}\binom{2^e}{\beta-2\ell+3}$ of such a summand is also divisible by $2^{e+\ell-2}$ provided $\beta \neq 2\ell + 1, 2\ell - 1$. Indeed, we have the following cases.

Case $\beta = 2d + 3$. In view of Proposition 4.2, it suffices to verify

$$\beta - d - 3 + \nu\left(\frac{2^e}{\beta - 2\ell + 3}\right) \geq e + \ell - 2,$$

that is

$$\nu\left(\frac{2^e}{\beta - 2\ell + 3}\right) \geq e + \ell - d - 2,$$

which follows from Lemma 4.1 as the two required hypotheses $\ell \leq d + 2$ and $\beta - 2\ell + 3 = 2d - 2\ell + 6 < 2^{d+3-\ell}$ hold — the latter is by virtue of (4.23) and since $d + 3 - \ell \geq 3$.

Case $\beta = 2\ell - 2$. We need to verify

$$\nu(c_{0,2\ell-2}) + \nu\binom{2^e}{1} \geq e + \ell - 2,$$

that is

$$\nu(c_{0,2\ell-2}) \geq \ell - 2, \tag{4.27}$$

and we can safely assume $\ell \geq 3$. The hypotheses of Proposition 4.3 for $c_{0,2\ell-2}$ with $i = d - \ell + 3$ read as

$$\begin{aligned} d + 4 - (d + 3 - \ell) &\leq 2\ell - 2 \leq 2d - 2(d + 3 - \ell) + 4, \\ 0 &\leq d + 3 - d + \ell - 3, \\ 1 &\leq d - \ell + 3 \leq d, \end{aligned}$$

which are equivalent to

$$\begin{aligned} 1 + \ell &\leq 2\ell - 2 \leq 2\ell - 2, \\ 0 &\leq \ell, \\ 3 &\leq \ell \leq d + 2, \end{aligned}$$

respectively, all of which are clearly satisfied. Therefore (4.27) follows from Proposition 4.3 in this case.

Case $\beta = 2\ell$. We now need to verify $\nu(c_{0,2\ell}) + \nu\binom{2^e}{3} \geq e + \ell - 2$, or alternatively

$$\nu(c_{0,2\ell}) \geq \ell - 2, \tag{4.28}$$

which follows from Proposition 4.3 for $c_{0,2\ell}$ with $i = d + 1 - \ell$. This time the required hypotheses are

$$\begin{aligned} d + 4 - (d + 1 - \ell) &\leq 2\ell \leq 2d - 2(d + 1 - \ell) + 4, \\ 0 &\leq d + 3 - (d + 1 - \ell), \\ 1 &\leq d + 1 - \ell \leq d, \end{aligned}$$

or equivalently

$$\begin{aligned} 3 &\leq \ell \quad \text{and} \quad 2\ell \leq 2\ell + 2, \\ 0 &\leq \ell + 2, \\ 1 &\leq \ell \leq d, \end{aligned}$$

respectively, which clearly hold (except for $\ell = 2$, in which case (4.28) is obvious).

Case where β satisfies

$$2\ell + 2 \leq \beta \leq 2d + 2 \quad \text{and} \quad \beta - 2\ell + 3 \leq 2^e. \quad (4.29)$$

We have to verify the inequality

$$\nu(c_{0,\beta}) + \nu\left(\frac{2^e}{\beta - 2\ell + 3}\right) \geq e + \ell - 2. \quad (4.30)$$

Let $[r]$ denote the integral part of a real number r . We claim that the three hypotheses of Proposition 4.3 for $c_{0,\beta}$ with $i = d + 2 - [\frac{\beta+1}{2}]$ hold, namely:

$$d + 4 - d - 2 + \left[\frac{\beta + 1}{2}\right] \leq \beta \leq 2d - 2d - 4 + 2 \left[\frac{\beta + 1}{2}\right] + 4, \quad (4.31)$$

$$0 \leq d + 3 - d - 2 + \left[\frac{\beta + 1}{2}\right], \quad (4.32)$$

$$1 \leq d + 2 - \left[\frac{\beta + 1}{2}\right] \leq d. \quad (4.33)$$

The inequality (4.32) is clear. The inequalities in (4.31) are verified by a straightforward argument: If $\beta = 2m$, then (4.31) is equivalent to $2 + m \leq 2m \leq 2m$, which is true since, by (4.29), $m \geq 2$. If $\beta = 2m + 1$, then (4.31) is equivalent to $2 + m + 1 \leq 2m + 1 \leq 2m + 2$, which again is valid because $m \geq 2$. Lastly, the second inequality in (4.33) is evident, while the first one follows from (4.29):

$$\left[\frac{\beta + 1}{2}\right] \leq \left[\frac{2d + 3}{2}\right] = d + 1.$$

Thus, the conclusion in Proposition 4.3 allows us to deduce (4.30) from

$$\beta + d + 2 - \left[\frac{\beta + 1}{2}\right] - d - 3 + \nu\left(\frac{2^e}{\beta - 2\ell + 3}\right) \geq e + \ell - 2,$$

that is

$$\nu\left(\frac{2^e}{\beta - 2\ell + 3}\right) \geq e + \ell - \beta + \left[\frac{\beta + 1}{2}\right] - 1. \quad (4.34)$$

To prove (4.34), we check that the hypotheses in Lemma 4.1 are satisfied. The first one is

$$e + \ell - \beta + \left[\frac{\beta + 1}{2}\right] - 1 \leq e,$$

or equivalently

$$\ell + \left[\frac{\beta + 1}{2}\right] \leq \beta + 1,$$

which is fulfilled since, by (4.29), we have $\ell + 1 \leq \lfloor \frac{\beta}{2} \rfloor$, so that

$$\ell + \left\lfloor \frac{\beta + 1}{2} \right\rfloor \leq \left\lfloor \frac{\beta}{2} \right\rfloor - 1 + \left\lfloor \frac{\beta + 1}{2} \right\rfloor \leq \beta + 1;$$

the last inequality can be verified in a simple direct way. The second hypothesis in Lemma 4.1 in order to verify (4.34) is that

$$\beta - 2\ell + 3 < 2^{\beta+2-\ell-\lfloor \frac{\beta+1}{2} \rfloor}. \quad (4.35)$$

If $\beta = 2m + 1$, (4.35) is equivalent to $2m + 1 - 2\ell + 3 < 2^{2m+1-\ell-m-1+2} = 2^{m-\ell+2}$, which is true by virtue of (4.23), since $m - \ell + 2 \geq 3$ (by (4.29)). If $\beta = 2m$, then (4.35) is equivalent to $2m - 2\ell + 3 < 2^{2m-\ell-m+2} = 2^{m-\ell+2}$, which is true for the same reasons as in the case of an odd β .

Therefore, except for the two cases not yet analyzed, namely the terms in the first summation in (4.1) with index $(0, 2\ell - 1)$ and $(0, 2\ell + 1)$, all the coefficients of contributions to the term $v_1^{2d-2\ell+5}(0, 2\ell - 2)$ have been verified to be divisible by $2^{e+\ell-2}$. We consider now the case $\beta = 2\ell - 1$. The greatest power of 2 that divides the corresponding contribution is given by

$$\nu(c_{0,2\ell-1}) + \nu\binom{2^e}{2} = \nu(c_{0,2(\ell-1)+1}) + e - 1 = \ell - 2 + e - 1 = e + \ell - 3,$$

where the penultimate equality holds by induction. Consequently, the coefficient coming from the case $\beta = 2\ell + 1$ must be divisible by $2^{e+\ell-3}$, but not by $2^{e+\ell-2}$; that is

$$e + \ell - 3 = \nu(c_{0,2\ell+1}) + \nu\binom{2^e}{4} = \nu(c_{0,2\ell+1}) + e - 2.$$

Thus $\nu(c_{0,2\ell+1}) = \ell - 1$, completing the induction. $\square$

5. Proof of Theorem 1.1

The proof of Theorem 1.1 is based on Propositions 5.1 and 5.2 below, the first of which follows from [10, Proposition 2.7 and Note 2.6]. The proof of Proposition 5.2 will be given at the end of this section.

Proposition 5.1. *There is a $\mathbb{Z}_{2^e}$ -biequivariant map $S^{2m+1} \times S^{2m+1} \rightarrow S^{2\ell+1}$ if and only if there is a map $\beta : L^{2m+1}(2^e) \times L^{2m+1}(2^e) \rightarrow L^{2\ell+1}(2^e)$ such that the diagram*

$$\begin{array}{ccc} L^\infty(2^e) \times L^\infty(2^e) & \xrightarrow{\mu} & L^\infty(2^e) \\ \uparrow & & \uparrow \\ L^{2m+1}(2^e) \times L^{2m+1}(2^e) & \xrightarrow{\beta} & L^{2\ell+1}(2^e) \end{array}$$

is homotopy commutative, where μ is the H -space product on $L^\infty(2^e) = K(\mathbb{Z}/2^e, 1)$, and both vertical maps are inclusions.

Proposition 5.2. *If $\alpha(n) \geq 3$, the element $2^{\alpha(n)-3}v_1^3x^{n+1}y^{n+1}$ is non-zero in the ring $R_n = ku_*[x, y]/([2^{\alpha(n)-1}](x), [2^{\alpha(n)-1}](y), x^{n+2}, y^{n+2})$.*

Here is how we use these two results in order to argue Theorem 1.1: We assume $b(n+1, \alpha(n)-1) < 2n-2$ and derive a contradiction. Since the case $\alpha(n) = 2$ follows from (1.4), we will assume throughout the section that $\alpha(n) \geq 3$. By Proposition 5.1, the restriction to $L^{2n+3}(2^{\alpha(n)-1}) \times L^{2n+3}(2^{\alpha(n)-1})$ of the H -product in $L^\infty(2^{\alpha(n)-1})$ is homotopic to a map $\beta : L^{2n+3}(2^{\alpha(n)-1}) \times L^{2n+3}(2^{\alpha(n)-1}) \rightarrow L^{4n-5}(2^{\alpha(n)-1})$. Apply ku -cohomology to the homotopy commutative diagram

$$\begin{array}{ccccc}
 & & & L^{2n+2}(2^{\alpha(n)-1}) \times L^{2n+2}(2^{\alpha(n)-1}) & \\
 & & \swarrow & \downarrow & \\
 & L^\infty(2^{\alpha(n)-1}) \times L^\infty(2^{\alpha(n)-1}) & \longleftarrow & L^{2n+3}(2^{\alpha(n)-1}) \times L^{2n+3}(2^{\alpha(n)-1}) & \\
 \swarrow \pi \times \pi & \downarrow \mu & & \downarrow \beta & \\
 \mathbb{C}P^\infty \times \mathbb{C}P^\infty & L^\infty(2^{\alpha(n)-1}) & \longleftarrow & L^{4n-5}(2^{\alpha(n)-1}) & \\
 \downarrow \mu' & \swarrow \pi & & \swarrow \pi & \\
 \mathbb{C}P^\infty & \mathbb{C}P^{2n-3} & \longleftarrow & \mathbb{C}P^{2n-3} &
 \end{array}$$

where unlabeled maps are inclusions, μ' is the H -product of $\mathbb{C}P^\infty$, and $\pi : L^{2k+1}(2^{\alpha(n)-1}) \rightarrow \mathbb{C}P^k$ ($0 \leq k \leq \infty$) stands for the canonical projection (a morphism of H -spaces for $k = \infty$).

It is well known that the maps π^* and $(\pi \times \pi)^*$ induced in ku -theory take respectively the forms $ku_*[[x]] \rightarrow ku_*[[x]]/([2^{\alpha(n)-1}](x))$ and $ku_*[[x, y]] \rightarrow ku_*[[x, y]]/([2^{\alpha(n)-1}](x), [2^{\alpha(n)-1}](y))$, where x and y stand for complex orientation elements of cohomological dimension 2 and, as usual in this context, ku_* is graded over the non-positive integers. Further, in even dimensions the ring $ku^*(L^{2n+2}(2^{\alpha(n)-1}) \times L^{2n+2}(2^{\alpha(n)-1}))$ agrees with R_n , and the “restriction” map $ku^{\text{even}}(L^\infty(2^{\alpha(n)-1}) \times L^\infty(2^{\alpha(n)-1})) \rightarrow R_n$ is surjective with kernel generated by x^{n+2} and y^{n+2} (cf. [8, Proposition 3.1]). Chase the element $x \in ku^*(\mathbb{C}P^\infty)$ across the diagram above, from the lower left corner to the upper right corner, recalling that $(\mu')^*(x) = x + y + v_1xy$ is the multiplicative formal group. Since $x \in ku^*(\mathbb{C}P^{2n-3})$ satisfies $x^{2n-2} = 0$, we get the relation $(x + y + v_1xy)^{2n-2} = 0$ in R_n . The proof of Theorem 1.1 will be complete once we show that, up to units in this ring, $x(x + y + v_1xy)^{2n-2} = 2^{\alpha(n)-3}v_1^3x^{n+1}y^{n+1}$, the element in Proposition 5.2.

In view of the relations $x^{n+2} = y^{n+2} = 0$, we have

$$\begin{aligned}
 x(x + y + v_1xy)^{2n-2} &= x(x + y)^{2n-2} + \binom{2n-2}{1}(x + y)^{2n-3}v_1x^2y \\
 &\quad + \binom{2n-2}{2}(x + y)^{2n-4}v_1^2x^3y^2 \\
 &\quad + \binom{2n-2}{3}(x + y)^{2n-5}v_1^3x^4y^3.
 \end{aligned}$$

These four summands reduce, respectively, to

$$\begin{aligned} & \binom{2n-2}{n} x^{n+1} y^{n-2} + \binom{2n-2}{n-1} x^n y^{n-1} \\ & + \binom{2n-2}{n-2} x^{n-1} y^n + \binom{2n-2}{n-3} x^{n-2} y^{n+1}, \end{aligned} \quad (5.1)$$

$$\begin{aligned} & \binom{2n-2}{1} \left[\binom{2n-3}{n-1} x^{n-1} y^{n-2} + \binom{2n-3}{n-2} x^{n-2} y^{n-1} \right. \\ & \left. + \binom{2n-3}{n-3} x^{n-3} y^n \right] v_1 x^2 y, \end{aligned} \quad (5.2)$$

$$\binom{2n-2}{2} \left[\binom{2n-4}{n-2} x^{n-2} y^{n-2} + \binom{2n-4}{n-3} x^{n-3} y^{n-1} \right] v_1^2 x^3 y^2, \quad (5.3)$$

$$\binom{2n-2}{3} \left[\binom{2n-5}{n-3} x^{n-3} y^{n-2} \right] v_1^3 x^4 y^3. \quad (5.4)$$

Next we analyze the divisibility by 2 of these binomial coefficients assuming

$$\alpha(n+1) \geq \alpha(n) \quad (5.5)$$

(Theorem 1.1 follows from (1.3) for $\alpha(n) > \alpha(n+1)$). In the following arguments we make free use of the relations $\nu\binom{a}{b} = \alpha(b) + \alpha(a-b) - \alpha(a)$, $\alpha(k-1) = \alpha(k) - 1 + \nu(k)$, and $2^i z^j = 0$ for $i+j \geq n + \alpha(n)$ and $z \in \{x, y\}$ (the latter was pointed out in (3.6) to come from [9, Corollary 2.6]).

Lemma 5.1. *The first and last terms in (5.1) vanish.*

Proof.

$$\begin{aligned} \nu\binom{2n-2}{n} &= \alpha(n) + \alpha(n-2) - \alpha(n-1) \\ &= \alpha(n) + \alpha(n-1) - 1 + \nu(n-1) - \alpha(n-1) \\ &= \alpha(n) - 1 + \nu(n-1) \geq \alpha(n) - 1, \end{aligned}$$

so that in fact $\binom{2n-2}{n} x^{n+1} = 0$, which takes care of the first summand in (5.1). Similarly, the last summand in (5.1) is taken care of by observing that $\nu\binom{2n-2}{n-3} \geq \alpha(n+1) - 2 + \nu(n-1) + \nu(n-2) \geq \alpha(n) - 1$, where the last inequality makes use of (5.5). $\square$

The rest of the coefficients in (5.1) are analyzed in the following result.

Lemma 5.2. *Up to units in R_n we have*

- $\binom{2n-2}{n-1} x^n y^{n-1} = 2^{\alpha(n)-3} v_1^3 x^{n+1} y^{n+1}$ and $\binom{2n-2}{n-2} x^{n-1} y^n = 0$, for n odd;
- $\binom{2n-2}{n-1} x^n y^{n-1} = 0$ and $\binom{2n-2}{n-2} x^{n-1} y^n = 2^{\alpha(n)-3} v_1^3 x^{n+1} y^{n+1}$, for n even.

Proof. We only consider the case with n odd; the situation for n even is similar and, thus, left to the reader. With $n = 2k + 1$ we have

$$\begin{aligned}\nu\binom{2n-2}{n-1} &= \alpha(n-1) + \alpha(n-1) - \alpha(n-1) \\ &= \alpha(n) - 1 + \nu(2k+1) = \alpha(n) - 1.\end{aligned}$$

Therefore, up to a 2-local unit, the second summand in (5.1) is $2^{\alpha(n)-1}x^ny^{n-1}$. Now, by using the $2^{\alpha(n)-1}$ -series twice, we get

$$\begin{aligned}2^{\alpha(n)-1}x^ny^{n-1} &= 2^{\alpha(n)-1}xx^{n-1}y^{n-1} \\ &= x^{n-1}y^{n-1}\left(-\binom{2^{\alpha(n)-1}}{2}v_1x^2 - \binom{2^{\alpha(n)-1}}{3}v_1^2x^3 - \binom{2^{\alpha(n)-1}}{4}v_1^3x^4 - \dots\right) \\ &= -\binom{2^{\alpha(n)-1}}{2}v_1x^{n+1}y^{n-1} = -\binom{2^{\alpha(n)-1}}{2}v_1y^2x^{n+1}y^{n-3} \\ &= x^{n+1}y^{n-3}\left(2^{\alpha(n)-1}y + \binom{2^{\alpha(n)-1}}{3}v_1^2y^3 + \binom{2^{\alpha(n)-1}}{4}v_1^3y^4 + \dots\right) \\ &= 2^{\alpha(n)-1}x^{n+1}y^{n-2} + \binom{2^{\alpha(n)-1}}{3}v_1^2x^{n+1}y^n + \binom{2^{\alpha(n)-1}}{4}v_1^3x^{n+1}y^{n+1}.\end{aligned}$$

The first two terms in the last expression vanish as $2^{\alpha(n)-1}x^{n+1} = 0$, while the fact that $\nu\binom{2^{\alpha(n)-1}}{4} = \alpha(n) - 3$ means that, up to a 2-local unit, the third term takes the form $2^{\alpha(n)-3}v_1^3x^{n+1}y^{n+1}$. On the other hand

$$\begin{aligned}\nu\binom{2n-2}{n-2} &= \alpha(n-2) + \alpha(n) - \alpha(n-1) \\ &= \alpha(n-1) - 1 + \nu(n-1) + \alpha(n) - \alpha(n-1) \\ &= \alpha(n) - 1 + \nu(2k) \geq \alpha(n),\end{aligned}$$

and the third summand in (5.1) vanishes as $2^{\alpha(n)}y^n = 0$. $\square$

The proof of Theorem 1.1 is finally completed by noticing that every summand in (5.2)–(5.4) vanishes (such a verification is similar to the previous arithmetical manipulations and, therefore, left to the reader).

The section closes with the proof of Proposition 5.2.

Proof of Proposition 5.2. As a ku_* -module, the ring R_n decomposes as the direct sum of a free ku_* -module generated by the unit 1, and a ku_* -module M_n generated by the elements x^ry^s with $0 \leq r, s \leq n+1, r+s > 0$ subject only to relations of the form

- $a_0x^{r+1}y^s + a_1x^{r+2}y^s + a_2x^{r+3}y^s + \dots$
- $a_0x^ry^{s+1} + a_1x^ry^{s+2} + a_2x^ry^{s+3} + \dots$

(for r and s as above) where $a_i = \binom{2^{\alpha(n)-1}}{i+1} v_1^i$. In turn, M_n is isomorphic to the ku_* -module M'_n generated by elements (i, j) with $0 \leq i, j \leq n+1$ and $i+j < 2n+2$ subject only to the relations

- $a_0(i-1, j) + a_1(i-2, j) + \cdots$
- $a_0(i, j-1) + a_1(i, j-2) + \cdots$

(for r and s as above). An isomorphism $M_n \cong M'_n$ identifies $x^r y^s$ with $(n+1-r, n+1-s)$ — note that ku_* recovers its non-negative grading in the ku_* -module structure of M'_n . Now, we have an obvious map $M'_n \rightarrow M''_n$ where M''_n is generated by elements (i, j) with $0 \leq i, j \leq n+1$ subject only to relations of the form

- $a_0(i, j) + a_1(i-1, j) + a_2(i-2, j) + \cdots$
- $a_0(i, j) + a_1(i, j-1) + a_2(i, j-2) + \cdots$

(for r and s as above). Note that, in M''_n , the generator $(n+1, n+1)$ has been included, and a slightly larger set of relations has been imposed, namely, besides the relations in M'_n , M''_n also has the relations starting as $a_0(n+1, n+1) + a_1(n+1, n) + \cdots$ and $a_0(n+1, n+1) + a_1(n, n+1) + \cdots$. Finally, M''_n maps canonically to $ku_*(L^\infty(2^{\alpha(n)-1})) \otimes ku_*(L^\infty(2^{\alpha(n)-1}))$ — where i and j vary over all non-negative integers. Since the composition $M_n \cong M'_n \rightarrow M''_n \rightarrow ku_*(L^\infty(2^{\alpha(n)-1})) \otimes ku_*(L^\infty(2^{\alpha(n)-1}))$ sends $x^{n+1}y^{n+1}$ to $(0, 0)$, Proposition 5.2 follows from Theorem 2.2 since, as an element of $ku_*(L^\infty(2^{\alpha(n)-1})) \otimes ku_*(L^\infty(2^{\alpha(n)-1}))$, $2^{\alpha(n)-3}v_1^3(0, 0) \neq 0$. $\square$

6. Comparison with Previous TC-Results

The following general lower bound for the topological complexity of lens spaces was proved by Farber and Grant in [5].

Theorem 6.1 ([5, Theorem 11]). *Let k and ℓ be integers with $0 \leq k, \ell \leq m$. If t does not divide $\binom{k+\ell}{k}$, then $\overline{\text{TC}}(L^{2m+1}(t)) \geq 2(k+\ell) + 1$.*

The main task in this section is to complete the details of (1.5) and, therefore, of the fact that, for lens spaces of the form $L^{2n+3}(2^{\alpha(n)-1})$, Corollary 1.1 improves on Theorem 6.1 by arbitrarily large amounts.

Proposition 6.1. *For a positive integer n with $\alpha(n) \geq 2$ and $\alpha(n+1) \geq \alpha(n)$, $\max\{k+\ell : 0 \leq k, \ell \leq n+1 \text{ and } 2^{\alpha(n)-1} \nmid \binom{k+\ell}{k}\} = 2n - 2^{\nu(n)+1} - 1$.*

Remark 6.1. The relevance of the hypothesis $\alpha(n+1) \geq \alpha(n)$ comes from [11, Example 5.9]: the topological complexity of $L^{2n+3}(2^{\alpha(n)-1})$ is completely understood for $\alpha(n+1) < \alpha(n)$ — in which case Theorem 6.1 is optimal. On the other hand, note that the condition $\alpha(n+1) \geq \alpha(n)$ is equivalent to $\nu(n+1) \leq 1$, which holds except for $n \equiv 3 \pmod{4}$.

The proof of Proposition 6.1 for $\nu(n) = 0$ reduces to a simple verification: using the formulas $\nu\binom{a}{b} = \alpha(a-b) + \alpha(b) - \alpha(a)$ and $\alpha(k-1) = \alpha(k) - 1 + \nu(k)$ it is easy

to check that $\nu\binom{2n+2}{n+1}$, $\nu\binom{2n+1}{n+1}$, $\nu\binom{2n}{n+1}$, $\nu\binom{2n}{n}$, $\nu\binom{2n-1}{n+1}$, $\nu\binom{2n-1}{n}$, $\nu\binom{2n-2}{n+1}$, $\nu\binom{2n-2}{n}$, and $\nu\binom{2n-2}{n-1}$ are all greater than or equal to $\alpha(n) - 1$, while

$$\begin{aligned}\nu\binom{2n-3}{n} &= \alpha(n) + \alpha(n-3) - \alpha(2(n-1) - 1) \\ &= \alpha(n) + \alpha(n-2) - 1 + \nu(n-2) - \alpha(n-1) - \nu(n-1) \\ &= \alpha(n) - 2 + \nu(n-2) = \alpha(n) - 2.\end{aligned}$$

However, the argument for $\nu(n) > 0$ is arithmetically cumbersome and, for the reader's benefit, we split the required verifications into a few preliminary steps. Note that, for $\nu(n) > 0$, the inequality

$$\max \left\{ k + \ell : 0 \leq k, \ell \leq n+1 \text{ and } 2^{\alpha(n)-1} \nmid \binom{k+\ell}{k} \right\} \leq 2n - 2^{\nu(n)+1} - 1$$

in Proposition 6.1 is a consequence of the following result.

Lemma 6.1. *Let k, n , and t be non-negative integers with $\alpha(n) \geq 2$. The inequality $\nu\binom{t}{k} \geq \alpha(n) - 1$ holds provided*

$$\nu(n) > 0, \tag{6.1}$$

$$2n - 2^{\nu(n)+1} \leq t \leq 2n + 2, \quad \text{and} \tag{6.2}$$

$$\left\lceil \frac{t+1}{2} \right\rceil \leq k \leq n+1. \tag{6.3}$$

In preparation for the proof of Lemma 6.1, we introduce some supplementary notation which will be in force through the rest of the section. Set $\nu = \nu(n)$ and let $n = 2^{\nu_0} + 2^{\nu_1} + \dots$ with $\nu = \nu_0 < \nu_1 < \dots$ be the binary expansion of n . From (6.2) we have

$$2n - 2^{\nu+1} = 2^{\nu_1+1} + 2^{\nu_2+1} + \dots \leq t \leq 2 + 2^{\nu_0+1} + 2^{\nu_1+1} + \dots$$

and, if we set $t_0 = t - (2^{\nu_1+1} + 2^{\nu_2+1} + \dots)$, we have

$$0 \leq t_0 \leq 2 + 2^{\nu_0+1}. \tag{6.4}$$

Further, since $\nu_0 \geq 1$, it is clear that, in the binary expansion of t_0 , only powers 2^j with $j \leq \nu_0 + 1$ are involved. In particular, from $t = t_0 + 2^{\nu_1+1} + 2^{\nu_2+1} + \dots$ we have

$$\alpha(t) = \alpha(t_0) + \alpha(2^{\nu_1+1} + 2^{\nu_2+1} + \dots) = \alpha(t_0) + \alpha(n) - 1. \tag{6.5}$$

Likewise, if we set $k_0 = k - (2^{\nu_1} + 2^{\nu_2} + \dots)$, from (6.3) one has

$$\begin{aligned}\left\lceil \frac{t_0+1}{2} \right\rceil + 2^{\nu_1} + 2^{\nu_2} + \dots &= \left\lceil \frac{t_0+1+2^{\nu_1+1}+\dots}{2} \right\rceil \\ &= \left\lceil \frac{t+1}{2} \right\rceil \leq k \leq 1 + 2^{\nu_0} + 2^{\nu_1} + \dots,\end{aligned}$$

that is

$$\left\lfloor \frac{t_0 + 1}{2} \right\rfloor \leq k_0 \leq 1 + 2^{\nu_0}. \quad (6.6)$$

Recalling again the assumption $\nu_0 \geq 1$, it is now clear that, in the binary expansion of k_0 only powers 2^j with $j \leq \nu_0$ are involved. In particular, from $k = k_0 + 2^{\nu_1} + 2^{\nu_2} + \dots$ we obtain

$$\alpha(k) = \alpha(k_0) + \alpha(2^{\nu_1} + 2^{\nu_2} + \dots) = \alpha(k_0) + \alpha(n) - 1. \quad (6.7)$$

Lastly, we claim that

$$t_0 - k_0 < 2^{\nu_0+1}. \quad (6.8)$$

For, otherwise, from (6.4) and (6.6) we would have

$$2 + 2^{\nu_0+1} \geq t_0 \geq k_0 + 2^{\nu_0+1} \geq \left\lfloor \frac{t_0 + 1}{2} \right\rfloor + 2^{\nu_0+1}, \quad (6.9)$$

so that $2 \geq \left\lfloor \frac{t_0+1}{2} \right\rfloor$, i.e. $t_0 \leq 4$. But (6.9) also gives

$$t_0 \geq k_0 + 2^{\nu_0+1} \geq 2^{\nu_0+1} \geq 2^2 = 4,$$

implying in fact $t_0 = 4$. However, the inequality $t_0 \geq \left\lfloor \frac{t_0+1}{2} \right\rfloor + 2^{\nu_0+1}$ in (6.9) would now say $4 \geq 2 + 2^{\nu_0+1} \geq 2 + 4 = 6$, which is a contradiction.

We are now ready to prove Lemma 6.1.

Proof of Lemma 6.1. From (6.5) and (6.7) we obtain

$$\begin{aligned} \nu \binom{t}{k} &= \alpha(k) + \alpha(t - k) - \alpha(t) \\ &= \alpha(k_0) + \alpha(n) - 1 + \alpha(t - k) - \alpha(t_0) - \alpha(n) + 1 \\ &= \alpha(k_0) + \alpha(t - k) - \alpha(t_0). \end{aligned}$$

Hence, taking into account the definitions of k_0 and t_0 , we have

$$\nu \binom{t}{k} = \alpha(k_0) + \alpha(t_0 - k_0 + 2^{\nu_1} + 2^{\nu_2} + \dots) - \alpha(t_0). \quad (6.10)$$

At this point we consider two possibilities.

Case $k_0 \leq t_0$. $t_0 - k_0$ is a non-negative integer whose binary expansion involves exclusively powers 2^j with $j \leq \nu_0$, by virtue of (6.8). Thus (6.10) transforms into

$$\begin{aligned} \nu \binom{t}{k} &= \alpha(k_0) + \alpha(t_0 - k_0) + \alpha(2^{\nu_1} + 2^{\nu_2} + \dots) - \alpha(t_0) \\ &= \alpha(k_0) + \alpha(t_0 - k_0) - \alpha(t_0) + \alpha(n) - 1 \\ &= \nu \binom{t_0}{k_0} + \alpha(n) - 1 \geq \alpha(n) - 1, \end{aligned}$$

as asserted by Lemma 6.1.

Case $k_0 > t_0$. $k_0 - t_0$ is a positive integer whose binary expansion involves exclusively powers 2^j with $j \leq \nu_0$, by virtue of (6.6). Thus, the central term on the right-hand side of (6.10) becomes

$$\begin{aligned} \alpha(t_0 - k_0 + 2^{\nu_1} + 2^{\nu_2} \cdots) &= \alpha(2^{\nu_2} + \cdots) + \alpha(2^{\nu_1} - (k_0 - t_0)) \\ &= \alpha(n) - 2 + \alpha(2^{\nu_1} - 1 - (k_0 - t_0 - 1)) \\ &= \alpha(n) - 2 + \nu_1 - \alpha(k_0 - t_0 - 1) \\ &= \alpha(n) - 2 + \nu_1 - (\alpha(k_0 - t_0) - 1 + \nu(k_0 - t_0)), \end{aligned}$$

so that

$$\begin{aligned} \nu \binom{t}{k} &= \alpha(k_0) - \alpha(t_0) + \alpha(n) - 2 + \nu_1 - \alpha(k_0 - t_0) + 1 - \nu(k_0 - t_0) \\ &= \alpha(k_0) - \alpha(t_0) - \alpha(k_0 - t_0) + \alpha(n) - 1 + \nu_1 - \nu(k_0 - t_0) \\ &= -\nu \binom{k_0}{t_0} + \alpha(n) - 1 + \nu_1 - \nu(k_0 - t_0). \end{aligned}$$

Thus, under the present hypothesis (summarized as $0 \leq t_0 < k_0 \leq 1 + 2^{\nu_0}$ with $0 < \nu_0 < \nu_1$), the conclusion of Lemma 6.1 is equivalent to $\nu_1 \geq \nu(k_0 - t_0) + \nu \binom{k_0}{t_0}$. The proof is then complete in view of Lemma 6.2 below. $\square$

Lemma 6.2. *For integers i, j , and μ with $\mu > 0$ and $0 \leq j < i \leq 2^{\mu+1} - 1$, we have $\mu + 1 \geq \nu(i - j) + \nu \binom{i}{j}$.*

Proof. The hypothesis means that the binary expansions of i, j , and $i - j$ involve exclusively powers 2^ℓ with $\ell \leq \mu$, say

$$\begin{aligned} i - j &= 2^{i_0} + 2^{i_1} + \cdots + 2^{i_a}, \\ j &= 2^{j_0} + 2^{j_1} + \cdots + 2^{j_b}, \end{aligned}$$

with $i_0 < i_1 < \cdots < i_a \leq \mu$ and $j_0 < j_1 < \cdots < j_b \leq \mu$, where $i_0 = \nu(i - j)$. By Kummer's theorem [14], the number C of binary carries in the sum of $i - j$ and j is $C = \nu \binom{i}{j}$. But the condition $i_a \leq \mu$ implies that the maximum possible of such carries is $\mu - (i_0 - 1)$. Thus, $\nu \binom{i}{j} = C \leq \mu - i_0 + 1$, as asserted. $\square$

We are now ready to prove Proposition 6.1.

Proof of Proposition 6.1. It suffices to check that $\nu \binom{2n - 2^{\nu(n)+1} - 1}{n} < \alpha(n) - 1$ for $\nu(n) > 0$. Setting as above $\nu = \nu(n)$, we have

$$\begin{aligned} \nu \binom{2n - 2^{\nu+1} - 1}{n} &= \alpha(n) + \alpha(n - 2^{\nu+1} - 1) - \alpha(2(n - 2^\nu) - 1) \\ &= \alpha(n) + \alpha(n - 2^{\nu+1}) - 1 + \nu(n - 2^{\nu+1}) \\ &\quad - (\alpha(n - 2^\nu) - 1 + \nu(2(n - 2^\nu))) \\ &= \alpha(n) + \alpha(n - 2^{\nu+1}) - 1 + \nu(n - 2^{\nu+1}) \\ &\quad - \alpha(n - 2^\nu) - \nu(n - 2^\nu). \end{aligned}$$

Since $n = 2^\nu a$, where a is an odd integer greater than 1 (recall that $\alpha(n) \geq 2$), we now have

$$\begin{aligned} \nu \binom{2n - 2^{\nu+1} - 1}{n} &= \alpha(2^\nu \cdot a) + \alpha(2^\nu(a - 2)) - 1 + \nu(2^\nu(a - 2)) - \alpha(2^\nu(a - 1)) \\ &\quad - \nu(2^\nu(a - 1)) \\ &= \alpha(a) + \alpha(a - 2) - 1 + \nu - \alpha(a - 1) - \nu - \nu(a - 1) \\ &= \alpha(a) + \alpha(a - 1) - 1 + \nu(a - 1) - 1 - \alpha(a - 1) - \nu(a - 1) \\ &= \alpha(a) - 2 = \alpha(n) - 2 < \alpha(n) - 1, \end{aligned}$$

as asserted. □

We close the paper with indications on how to solve a gap in [10] — so to justify our use of (1.2). In Lemma 4.1 of that paper it is claimed that there is no $\mathbb{Z}_t$ -axial map

$$\beta : L^{2m+1}(t) \times L^{2m+1}(t) \rightarrow L^{2m+1}(t) \tag{6.11}$$

when $t > 2$ — that is, a map β for which the diagram in Proposition 5.1 (with $\ell = m$, and where 2^e is replaced by t) is homotopy commutative. Such an assertion (which is well known to fail for $t = 2$) is not well argued for a general t in the proof of [10, Lemma 4.1]: although the argument correctly shows that a $\mathbb{Z}_t$ -axial map (6.11) can exist only when both t and $m + 1$ are prime powers,^c the incompatibility with the relation “ $x^2 = 0$ ” asserted at the end of that argument (and which really meant to use cohomology with $\mathbb{Z}_t$ -coefficients) overlooked the graded commutativity in the cohomology ring

$$H^*(L^{2m+1}(t) \times L^{2m+1}(t); \mathbb{Z}_t) \cong H^*(L^{2m+1}(t); \mathbb{Z}_t) \otimes H^*(L^{2m+1}(t); \mathbb{Z}_t).$$

In any case, what is important for our purposes is that the above problem hurts no other result in [10]. Indeed, Lemma 4.1 in that paper is used only in the proof of its main Theorem 2.9 — stated here as the estimates for $\varepsilon_{n,t}$ in (1.2). Now, the easy proof of the inequality $\varepsilon_{n,t} \geq 0$ is given in [10, Lemma 3.1 and Proposition 3.2] independently of the problematic Lemma 4.1. It is the inequality $\varepsilon_{n,t} \leq 1$, asserted for even t , which makes use of the potentially faulty Lemma 4.1. However, as noted in [7, Proposition 24], the argument in the proof of [7, Lemma 17] shows that [10, Lemma 4.1] is true even when t is a 2-power.

Acknowledgments

The authors thank Peter Landweber and the anonymous referee for many useful comments on preliminary versions of this work. The first author was supported by Conacyt Research Grant 102783 during the time this research was conducted.

^cAlthough strictly a different property, the converse of this assertion would seem to be related to the stable parallelizability of lens spaces (a well-understood property by [15]), and to the existence of H -space multiplications on localized lens spaces (a well-understood property by [2, 13]).

References

- [1] J. Adem, S. Gitler and I. M. James, On axial maps of a certain type, *Bol. Soc. Mat. Mexicana* **17** (1972) 59–62.
- [2] W. Browder, Higher torsion in H -spaces, *Trans. Amer. Math. Soc.* **108** (1963) 353–375.
- [3] D. M. Davis, A strong nonimmersion theorem for real projective spaces, *Ann. of Math.* **120** (1984) 517–528.
- [4] M. Farber, Topological complexity of motion planning, *Discrete Comput. Geom.* **29** (2003) 211–221.
- [5] M. Farber and M. Grant, Robot motion planning, weights of cohomology classes, and cohomology operations, *Proc. Amer. Math. Soc.* **136** (2008) 3339–3349.
- [6] M. Farber, S. Tabachnikov and S. Yuzvinsky, Topological robotics: Motion planning in projective spaces, *Int. Math. Res. Not.* **34** (2003) 1853–1870.
- [7] J. M. García-Calines and L. Vandembroucq, On the topological complexity and the homotopy cofibre of the diagonal map, preprint (2011); arXiv:math.AT/1106.4943v1.
- [8] J. González, Connective K -theoretic Euler classes and nonimmersions of 2^k lens spaces, *J. London Math. Soc. (2)* **63** (2001) 247–256.
- [9] ———, A generalized Conner–Floyd conjecture and the immersion problem for low 2-torsion lens spaces, *Topology* **42** (2003) 907–927.
- [10] ———, Topological robotics in lens spaces, *Math. Proc. Cambridge Philos. Soc.* **139** (2005) 469–485.
- [11] J. González and P. Landweber, Symmetric topological complexity of projective and lens spaces, *Algebr. Geom. Topol.* **9** (2009) 473–494.
- [12] J. González and L. Zárate, BP-theoretic instabilities to the motion planning problem in 4-torsion lens spaces, *Osaka J. Math.* **43** (2006) 581–596.
- [13] J. Harper, Regularity of finite H -spaces, *Illinois J. Math.* **23** (1979) 330–333.
- [14] E. E. Kummer, Über die Ergänzungssätze zu den allgemeinen Reciprocitätssätzen, *J. Reine Angew. Math.* **44** (1852) 93–146.
- [15] B. H. Li and Z. Z. Tang, Codimension 1 and 2 immersions of lens spaces, in *Differential Geometry and Topology*, Lecture Notes in Mathematics, Vol. 1369 (Springer, 1989), pp. 152–163.
- [16] G. Nakos, On the Brown–Peterson homology of certain classifying spaces, Ph.D. thesis, The Johns Hopkins University (1985).
- [17] M. Velasco, On the ku_* -homology of $\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^e}$ and its application to the topological complexity of lens spaces, Ph.D. thesis, CINVESTAV-IPN (2010).