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Stable splitting of the complex connective K-theory of $BO(n)$

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ABSTRACT

We give the complete stable splitting of the complex connective K-theory of the classifying space of the n-th orthogonal group for all $n \geqslant 1$. Only three types of spectra show up: bu, mod 2 Eilenberg–Mac Lane spectra, and $bu \wedge RP^{\infty}$.

1. Introduction

Before we state the main result of this paper, let's recall what we need. Let bu be the complex connective K-theory, $RP^{\infty} = BO(1)$ be the infinite real projective space, $H\mathbb{Z}/2$ be the $\mathbb{Z}/2$ Eilenberg–Mac Lane spectrum, $BO(n)$ be the classifying space of the n-th orthogonal group, and $H^*(X)$, $\widetilde{H}^*(X)$ be the unreduced mod 2 cohomology and the reduced mod 2 cohomology of X respectively. Throughout the paper the spaces, the spectra, or the homotopy equivalences are localized at prime 2.

Let A be the mod 2 Steenrod algebra and $E = \mathbb{Z}/2\langle Q_0, Q_1\rangle$ ($Q_0 = Sq^1$ and $Q_1 = Sq^3 + Sq^2Sq^1$) be the exterior algebra on Q_0 and Q_1, which is a subalgebra of A. Since E is a subalgebra of A, $\widetilde{H}^*(X)$ is an E-module for any space or spectrum X. By the Cartan formula $Sq^i(xy) = \sum_{j=0}^{i} Sq^j(x)Sq^{i-j}(y)$, we know Q_0 and Q_1 act as derivations, that is, $Q_k(xy) = Q_k(x)y + xQ_k(y)$.

Recall that $H^*(BO(n)) = \mathbb{Z}/2[w_1, w_2, \dots, w_n]$, the polynomial algebra on the Stiefel–Whitney classes.

The technical theorem we need is an analysis of the E-module structure of $\widetilde{H}^*(BO(n))$.

Theorem 1.1. *As an E-module, $\widetilde{H}^*(BO(n))$ is isomorphic to*

$$D_1^* \oplus D_2^* \oplus M,$$

where D_1^* is a trivial E-module with E-generators

$$w_2^{2m_1} w_4^{2m_2} \cdots w_{2k}^{2m_k} \quad \text{such that} \sum_{i=1}^{k} m_i > 0,\ 2k \leqslant n,$$

D_2^* is an E-module, free over the exterior algebra on Q_0, with E-generators

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$$w_1^{2j-1} w_2^{2m_1} w_4^{2m_2} \cdots w_{2t}^{2m_t} \quad \textit{such that} \sum_{i=1}^{t} m_i \geqslant 0,\ j \geqslant 1,\ 2t \leqslant n-1,$$

and

$$Q_1\big(w_1^{2j-1} w_2^{2m_1} w_4^{2m_2} \cdots w_{2t}^{2m_t}\big) = Q_0\big(w_1^{2j+1} w_2^{2m_1} w_4^{2m_2} \cdots w_{2t}^{2m_t}\big).$$

M is a free E-module with a basis $T = \{d_j \mid j \in \Lambda\}$.

Now we state the main result of this paper.

Theorem 1.2. *For each $n \geqslant 1$, there is a stable splitting*

$$bu \wedge BO(n) \simeq \Big[\bigvee_{\alpha} \Sigma^{\alpha} H\mathbb{Z}/2\Big] \vee \Big[\bigvee_{\beta} \Sigma^{\beta} bu\Big] \vee \Big[\bigvee_{\gamma} \Sigma^{\gamma} bu \wedge RP^{\infty}\Big],$$

where $\alpha = \deg d_j$, $d_j \in T$, i.e. the generators of M. The β, and their degrees, correspond to generators of D_1^*. The γ, and their degrees, correspond to the monomials $w_2^{2m_1} w_4^{2m_2} \cdots w_{2t}^{2m_t}$, $\sum_{i=1}^{t} m_i \geqslant 0$, $2t \leqslant n-1$.

Remark 1.3. For $n = 1$ we have $BO(1) = RP^{\infty}$ and the first two parts of the theorem are trivial and there is only one term in the third part corresponding to $t = 0$, so the result is trivial, but correct.

Remark 1.4. Note that the generators of D_1^* are the mod 2 reductions of the monomials of the Pontrjagin classes of $BO(n)$. Consequently, they also index the stable copies of bu in the theorem. Likewise, the copies of $bu \wedge RP^{\infty}$ are indexed over the mod 2 reductions of the monomials of the Pontrjagin classes of $BO(n-1)$.

The E-module structure of $\widetilde{H}^*(BO(n))$ was known already [3], but we need the special multiplicative form of the structure for D_1^* and D_2^* in order to construct a map:

$$g : bu \wedge BO(n) \to \Big[\bigvee_{\alpha} \Sigma^{\alpha} H\mathbb{Z}/2\Big] \vee \Big[\bigvee_{\beta} \Sigma^{\beta} bu\Big] \vee \Big[\bigvee_{\gamma} \Sigma^{\gamma} bu \wedge RP^{\infty}\Big]. \tag{1.5}$$

Once g is constructed we prove it induces an isomorphism on the mod 2 cohomology, hence g is a homotopy equivalence.

The rest of this paper is organized as follows: In Section 2, we study the E-module structure of $\widetilde{H}^*(BO(n))$ and prove Theorem 1.1. In Section 3, we construct the map g of Eq. (1.5). In Section 4, we prove Theorem 1.2.

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2. The E-module structure of $\widetilde{H}^*(BO(n))$

In this section we will study the E-module structure of $\widetilde{H}^*(BO(n))$ and prove Theorem 1.1.

Recall that $H^*(RP^{\infty}) = H^*(BO(1)) = \mathbb{Z}/2[w_1]$. Moreover, we have

$$H^*\Big(\bigtimes_{i=1}^{n} RP^{\infty}\Big) = \mathbb{Z}/2[x_1, x_2, \ldots, x_n], \quad \deg(x_i) = 1.$$

Via the classifying map for the sum of the canonical line bundles $f : \bigtimes_{i=1}^{n} RP^{\infty} \to BO(n)$, we have

$$H^*\Big(\bigtimes_{i=1}^{n} RP^{\infty}\Big) \supset H^*\big(BO(n)\big) = \mathbb{Z}/2[\sigma_1, \sigma_2, \ldots, \sigma_n],$$

where σ_i is the elementary symmetric function $\sum x_1 \cdots x_i$, $1 \leqslant i \leqslant n$. Finally, let E_i be the exterior algebra $\mathbb{Z}/2\langle Q_0, Q_1, \ldots, Q_{i-1}\rangle$. We have $E = E_2 = \mathbb{Z}/2\langle Q_0, Q_1\rangle$ and $E_\infty = \mathbb{Z}/2\langle Q_0, Q_1, Q_2, \ldots\rangle$. Note that E_k, $k > 1$, is free over E.

We start with what is known about the E_∞-module structure of $\widetilde{H}^*(BO(n))$.

Theorem 2.1. *([3, Theorem 2.1]) As vector spaces over $\mathbb{Z}/2$,*

$$H^*\big(BO(n)\big) \cong \sum_{i=0}^{n} E_i G_i,$$

where the E_∞-module generators are given by $\sum_{i=0}^n G_i$ and a basis for G_k is given by all symmetric functions

$$\sum x_1^{2i_1+1}x_2^{2i_2+1}\cdots x_k^{2i_k+1}x_{k+1}^{2j_1}\cdots x_{k+q}^{2j_q},\quad k+q\leqslant n,$$

with $0\leqslant i_1\leqslant i_2\leqslant\cdots\leqslant i_k$, and $0<j_1\leqslant j_2\leqslant\cdots\leqslant j_q$; and if the number of j equal to j_u is odd, then there is some $s\leqslant k$ such that

$$2i_s+2^s<2j_u<2i_s+2^{s+1}.$$

Remark 2.2. This doesn't really give the complete E_∞-module structure of $\widetilde{H}^*(BO(n))$, but there is enough information to get the complete E-module structure. Since E_k, $k>1$, is free over E, this establishes the E-free part, $M=\sum_{i=2}^n E_iG_i$, of $\widetilde{H}^*(BO(n))$ and we capture our E-free generators, $d_j\in T$, for Theorem 1.1.

Taking a close look at G_0, we have:

$$\sum x_1^{2j_1}\cdots x_q^{2j_q},\quad q\leqslant n,$$

and $0<j_1\leqslant j_2\leqslant\cdots\leqslant j_q$; and the number of j equal to j_u is even. The E action on G_0 is trivial.

Taking a close look at G_1, we have:

$$\sum x_1^{2i_1+1}x_2^{2j_1}\cdots x_{1+q}^{2j_q},\quad 1+q\leqslant n,$$

with $0\leqslant i_1$, and $0<j_1\leqslant j_2\leqslant\cdots\leqslant j_q$; and the number of j equal to j_u is even, because the final condition in the theorem is vacuous. It is important to observe that:

$$Q_0\Big(\sum x_1^{2i_1+3}x_2^{2j_1}\cdots x_{1+q}^{2j_q}\Big)=Q_1\Big(\sum x_1^{2i_1+1}x_2^{2j_1}\cdots x_{1+q}^{2j_q}\Big).$$

Thus we have the complete E-module structure of $\widetilde{H}^*(BO(n))$.

Remark 2.3. Similarly, we have:

$$Q_0\big(w_1^{2j+3}w_2^{2m_1}w_4^{2m_2}\cdots w_{2t}^{2m_t}\big)=Q_1\big(w_1^{2j+1}w_2^{2m_1}w_4^{2m_2}\cdots w_{2t}^{2m_t}\big).$$

Definition 2.4. Let N be a sub-E-module of $H^*(BO(n))$, with basis y_i, $i>2m\geqslant 0$, and with the degree of y_i equal to i. Let $Q_0y_{2k+1}=y_{2k+2}$ and $Q_1y_{2k+1}=y_{2k+4}$.

Remark 2.5. The sub-E-module, E_1G_1 of $\widetilde{H}^*(BO(n))$, is the direct sum of modules of the form N above. So is D_2^*.

Lemma 2.6. *Let N be as in the above definition, then either N is entirely in the free part, $N\subset M$, or entirely out of the free part, i.e. $N\cap M=0$.*

Proof. Assume that for some k, y_{2k+1} is in the free part, that is, in $\sum_{i=2}^n E_iG_i=M$. Since the splitting is as E-modules, this means both y_{2k+2} and y_{2k+4} must also be in the free part. We have $Q_0y_{2k+3}=y_{2k+4}$. There are no odd elements in E_0G_0, and all of the odd elements in E_1G_1 have Q_0 non-zero on them and this is also in E_1G_1. So if y_{2k+3} is not in the free part, neither is y_{2k+4}, a contradiction. Induction then shows that all y_i, $i>2k$, are in the free part.

If $k>m$, we have $Q_1y_{2k-1}=y_{2k+2}$. A similar argument to that just given shows that y_{2k-1} is in the free part. We can now use downward induction to finish the result starting with y_{2k+1}.

If we have some y_{2k} in the free part, we can use the above argument to show that y_{2k-1} is also, and that will complete the proof by the first part. □

Since $Q_1Q_1=0$, we can take the homology of $\widetilde{H}^*(BO(n))$ with respect to Q_1.

Lemma 2.7. *The Q_1 homology of $\widetilde{H}^*(BO(n))$ is given by*:

$$w_2^{2m_1}w_4^{2m_2}\cdots w_{2k}^{2m_k},\quad \sum_{i=1}^k m_i>0,\ 2k\leqslant n$$

and

$$w_1^2w_2^{2m_1}w_4^{2m_2}\cdots w_{2t}^{2m_t},\quad \sum_{i=1}^t m_i\geqslant 0,\ 2t\leqslant n-1.$$

Proof. If we were happy with the answer in terms of elements from Theorem 2.1, it would be easy. The splitting $\widetilde{H}^*(BO(n)) \simeq \sum E_i G_i$ is as E-modules. All we need to know about it is the structure of $\widetilde{H}^*(BO(n))$ as a module over the exterior algebra on Q_1, $\mathbb{Z}/2\langle Q_1\rangle$. We already know that the free part, M, is free over this, so it contributes nothing to the Q_1 homology. The E_1G_1 part is just a direct sum of E-modules of the sort N. The Q_1 homology of N is just given by the single element y_{2+2m}, second lowest degree element in N. Note that E_1G_1 is the same size as the second part of our answer in the theorem, but they are not the same elements. E_0G_0 is trivial and is all Q_1 homology. This is the same size as the first part of our theorem's answer, but, again, they are not the same elements.

This change of bases for the Q_1 homology is the crucial step in the proof of the main theorem. We need the multiplicative nature of the new basis, so we proceed to show that it gives the Q_1 homology.

The elements in question are all Q_1 cycles. If when written as symmetric functions, they are seen to not be Q_1 boundaries, we have the correct Q_1 homology. For our present purposes, we write the symmetric product as:

$$\sum x_1^{j_1} x_2^{j_2} \cdots x_k^{j_k}, \quad k \leqslant n,$$

with

$$j_1 \geqslant j_2 \geqslant \cdots \geqslant j_k.$$

Order the symmetric functions lexicographically so j_1 is biggest. The Stiefel–Whitney class w_k is well known to be represented by

$$\sum x_1 x_2 \cdots x_k.$$

We see that our element

$$w_2^{2m_1} w_4^{2m_2} \cdots w_{2k}^{2m_k}$$

has, when expanded as a sum of symmetric functions, highest term given by

$$\sum x_1^{j_1} x_2^{j_2} \cdots x_k^{j_k},$$

where $j_1 = j_2 = \sum_{i=1}^k 2m_i,\ j_3 = j_4 = \sum_{i=2}^k 2m_i,\ \ldots,\ j_{k-1} = j_k = 2m_k$. This lead term is just a different way of writing the basis for G_0. If the product has terms that are not in G_0, then there must be some j_u with an odd number of $j = j_u$, and $j_u \geqslant 4$. (Recall that all the Stiefel–Whitney classes are even and squared, so all of the terms in the symmetric products are even.) Replacing one j_u with $j_u - 3$, we see that the original element is in the image of Q_1 from this new element, and, as such, is a boundary. This shows that these Stiefel–Whitney classes are Q_1 homologous to those given by G_0.

For the other part, we just have $k < n$ and a x_{k+1}^2 on the end of our lead term. The lead term of the Stiefel–Whitney classes with the w_1^2 maps to the appropriate representative of the Q_1 homology of E_1G_1. Similar to the above, any terms that do not lie here are in the image of Q_1. This shows that the elements in the Stiefel–Whitney version are Q_1 homologous to the generators we found in E_1G_1. $\square$

Proof of Theorem 1.1. It is easy to see that, as vector spaces, D_1^* is the same size as E_0G_0 and D_2^* is the same size as E_1G_1. Both also have the same E-module structures. We need to show that the bases for D_1^* and D_2^* as given can be used to replace the bases G_0 and G_1. To do this, all we have to do is show that $D_1^* \oplus D_2^*$ has no elements in the free part, which has already been determined. D_1^* is all non-trivial in Q_1 homology, so none of these elements can be in the free part. D_2^* is the direct sum of E-modules of the sort N. Such an N is either entirely in the free part or entirely out of it, from Lemma 2.6. Each N has non-trivial Q_1 homology, which is not in the free part, so the intersection of D_2^* and M, the free part, is trivial. $\square$

3. The stable map g

The map g, Eq. (1.5), comes in three parts. The part that maps to the Eilenberg–Mac Lane spectra is easy. We have the E-free generators of $\widetilde{H}^*(BO(n))$, and each one gives a stable map

$$BO(n) \longrightarrow \Sigma^{\deg d_i} H\mathbb{Z}/2.$$

$H\mathbb{Z}/2$ is a module spectrum over bu, so this gives us maps:

$$bu \wedge BO(n) \to bu \wedge \Sigma^{\deg d_i} H\mathbb{Z}/2 \to \Sigma^{\deg d_i} H\mathbb{Z}/2.$$

The next part we deal with is to find a map

$$BO(n) \longrightarrow \Sigma^{\beta} bu$$

that takes $1 \in \Sigma^{\beta} H^*(bu)$ to $w_2^{2m_1} w_4^{2m_2} \cdots w_{2t}^{2m_t} \in \widetilde{H}^*(BO(n))$, where $2t \leqslant n$.

This is standard fare. The Atiyah–Hirzebruch spectral sequence for $bu^*(BU(n))$ collapses because $H^*(BU(n))$ is even degree and free. This gives maps $BU(n) \to \Sigma^\beta bu$ that take 1 to any monomial in the Chern classes. By the splitting principle and the fact that $c_1 \in H^2(CP^\infty)$ maps to $w_1^2 \in H^2(RP^\infty)$, we know that c_k maps to w_k^2 by the map $BO(n) \to BU(n)$. Just take the map that goes to $c_2^{m_1} c_4^{m_2} \cdots c_{2t}^{m_t}$. The map we want is the composite

$$BO(n) \to BU(n) \to \Sigma^\beta bu.$$

From this, and the fact that bu is a ring spectrum, we get:

$$bu \wedge BO(n) \to bu \wedge BU(n) \to bu \wedge \Sigma^\beta bu \to \Sigma^\beta bu.$$

For the third part of the map g, we need the multiplication on bu, the standard map $\det : BO(n) \to RP^\infty$, and the map

$$BO(n) \longrightarrow \Sigma^\gamma bu$$

obtained from $w_2^{2m_1} w_4^{2m_2} \cdots w_{2t}^{2m_t}$ with $2t < n$ to construct:

$$\begin{aligned} bu \wedge BO(n) &\xrightarrow{bu \wedge \Delta} bu \wedge \big(BO(n) \times BO(n)\big) \to bu \wedge \big(BO(n) \wedge BO(n)\big) \\ &\to bu \wedge \big(\Sigma^\gamma bu \wedge BO(n)\big) \to bu \wedge \big(\Sigma^\gamma bu \wedge RP^\infty\big) \to \Sigma^\gamma bu \wedge RP^\infty. \end{aligned}$$

Combine all of these maps together to get g.

4. Proof of Theorem 1.2

In this final section we will prove Theorem 1.2. First we recall what we need following [2]. Suppose M and N are left A-modules with the actions μ_M and μ_N, then $M \otimes N$ is also a left A-module with the action defined by the composite map

$$A \otimes M \otimes N \xrightarrow{\psi \otimes M \otimes N} A \otimes A \otimes M \otimes N \xrightarrow{A \otimes T \otimes N} A \otimes M \otimes A \otimes N \xrightarrow{\mu_M \otimes \mu_N} M \otimes N,$$

where ψ is the diagonal map of A and $T(a \otimes b) = (b \otimes a)$ is the twist map (recall we are working over $\mathbb{Z}/2$). We write ${}_D(M \otimes N)$ to indicate $M \otimes N$ with this left action. Moreover, ${}_L(M \otimes N)$ indicates the extended A action over M with A acting only on M. Note that it is simple to show that ${}_D(M \otimes N) \simeq {}_D(N \otimes M)$.

If B is a Hopf subalgebra of A, we know that M and N are left B-modules, hence ${}_D(M \otimes N)$ is a left B-module. Also we know that A is both a right B-module and a left A-module, hence $A \otimes_B N$ is a left A-module with the extended action over A.

Proposition 4.1. *([2, Proposition 1.7]) If B is a Hopf subalgebra of A, M a left A-module, N a left B-module, then*

$${}_D\big[M \otimes (A \otimes_B N)\big] \cong {}_L\big[A \otimes_B {}_D(M \otimes N)\big]$$

as left A-modules.

Remark 4.2. Let N be $\mathbb{Z}/2$ and B be E in Proposition 4.1. Since

$${}_D\big[M \otimes (A \otimes_E \mathbb{Z}/2)\big] \cong {}_D\big[(A \otimes_E \mathbb{Z}/2) \otimes M\big]$$

and

$${}_D(M \otimes \mathbb{Z}/2) \cong M,$$

this isomorphism becomes

$$\theta : {}_L[A \otimes_E M] \xrightarrow{\cong} {}_D\big[(A \otimes_E \mathbb{Z}/2) \otimes M\big]$$

and is given by $\theta(a \otimes x) = \sum a' \otimes 1 \otimes a''x$, with inverse $\theta^{-1}(a \otimes 1 \otimes x) = \sum a' \otimes \chi(a'')x$, where $\psi(a) = \sum a' \otimes a''$ and χ is the conjugation map. For details on the maps, see [1] and Proposition 1.1 of [2].

Proof of Theorem 1.2. Now we show that the g of Section 3 induces an isomorphism on the mod 2 cohomology. Recall that $H^*(bu) \cong A/A(Q_0, Q_1) \cong A \otimes_E \mathbb{Z}/2$ and we have

$$H^*(bu \wedge X) \cong H^*(bu) \otimes \widetilde{H}^*(X) \cong (A \otimes_E \mathbb{Z}/2) \otimes \widetilde{H}^*(X) \overset{\theta^{-1}}{\cong} A \otimes_E \widetilde{H}^*(X)$$

for any space X, where θ^{-1} is the isomorphism described in Remark 4.2. By Theorem 1.1, we have $\widetilde{H}^*(BO(n)) \cong D_1^* \oplus D_2^* \oplus M$ as E-modules. So,

$$H^*\big(bu \wedge BO(n)\big) \cong A \otimes_E \widetilde{H}^*\big(BO(n)\big) \cong A \otimes_E \big(D_1^* \oplus D_2^* \oplus M\big). \tag{4.3}$$

We will look at each piece of the E-module

$$D_1^* \oplus D_2^* \oplus M.$$

For each basis element, b, of D_1^*, we have a map $BO(n) \to \Sigma^\alpha bu$ that takes $1 \in H^*(\Sigma^\alpha bu)$ to b. For the map g this corresponds to:

$$bu \wedge BO(n) \to bu \wedge \Sigma^\alpha bu \to \Sigma^\alpha bu$$

with the last map coming from multiplication on bu. This takes $1 \in H^*(\Sigma^\alpha bu)$ to $1 \otimes \Sigma^\alpha 1$ and then onto $1 \otimes b$. Recalling that D_1^* is a trivial E-module, the A-module summand in $H^*(bu \wedge BO(n))$ associated with b is just, from Eq. (4.3),

$$A \otimes_E \Sigma^\alpha \mathbb{Z}/2 \simeq H^*\big(\Sigma^\alpha bu\big).$$

Since this is cyclic and the generator goes to the generator, we get an isomorphism. This takes care of all elements from D_1^* in the map g.

We next move to consider D_2^*. This is free over the exterior algebra on Q_0, but as E-modules it is the sum of copies of N from Definition 2.4. The start of the definition of g for this part comes from the map $BO(n) \to RP^\infty$. This map takes w_1 to w_1, and so takes w_1^k to w_1^k. For this bottom case, the map g is defined in Section 3 by

$$bu \wedge BO(n) \to bu \wedge \big(BO(n) \times BO(n)\big) \to bu \wedge \big(BO(n) \wedge BO(n)\big) \to bu \wedge \big(bu \wedge RP^\infty\big) \to bu \wedge RP^\infty.$$

The map $BO(n) \to bu$ is just the map associated with $1 \in bu^*(BO(n))$. Algebraically, $1 \otimes w_1^k$ maps to $1 \otimes 1 \otimes w_1^k$ to $1 \otimes 1 \otimes w_1^k$ to $1 \otimes w_1^k$. For the maps that come from $w_2^{2m_1} w_4^{2m_2} \cdots w_{2t}^{2m_t}$ of degree γ with $2t < n$, we have:

$$bu \wedge BO(n) \to bu \wedge \big(BO(n) \times BO(n)\big) \to bu \wedge \big(BO(n) \wedge BO(n)\big) \to bu \wedge \big(\Sigma^\gamma bu \wedge RP^\infty\big) \to \Sigma^\gamma bu \wedge RP^\infty.$$

Now $1 \otimes w_1^k$ goes to $1 \otimes 1 \otimes w_1^k$ and then onto

$$1 \otimes w_2^{2m_1} w_4^{2m_2} \cdots w_{2t}^{2m_t} \otimes w_1^k$$

and finally to

$$1 \otimes w_2^{2m_1} w_4^{2m_2} \cdots w_{2t}^{2m_t} w_1^k.$$

This is a map from $H^*(\Sigma^\gamma bu) \otimes H^*(RP^\infty)$ to $H^*(bu) \otimes H^*(BO(n))$ that has as its isomorphic image $H^*(bu)$ tensored with an N summand of D_2^*. This completes all of the D_2^* part of the isomorphism.

All that is left to consider is the free part, M. For every free generator, we have a map $BO(n) \to \Sigma^{\deg d_i} H\mathbb{Z}/2$. In g this becomes:

$$bu \wedge BO(n) \to bu \wedge \Sigma^{\deg d_i} H\mathbb{Z}/2 \to \Sigma^{\deg d_i} H\mathbb{Z}/2.$$

In cohomology, the generator of the free A-module (on one generator) goes to $1 \otimes 1$ and then onto $1 \otimes d_i$. The element d_i is a free E-generator of M, so this map is really a map $A \to A \otimes_E E \simeq A$ (from Eq. (4.3)) that takes the generator to the generator. This completes the isomorphism on part M.

Hence g induces an isomorphism on the mod 2 cohomology and it follows that g is a homotopy equivalence. So this completes the proof of Theorem 1.2. □

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