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$k(n)$-Torsion-Free H-Spaces and $P(n)$-Cohomology

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Abstract. The H-space that represents Brown–Peterson cohomology $\mathrm{BP}^k(-)$ was split by the second author into indecomposable factors, which all have torsion-free homotopy and homology. Here, we do the same for the related spectrum $P(n)$, by constructing idempotent operations in $P(n)$-cohomology $P(n)^k(-)$ in the style of Boardman–Johnson–Wilson; this relies heavily on the Ravenel–Wilson determination of the relevant Hopf ring. The resulting $(i-1)$-connected H-spaces Y_i have free connective Morava K-homology $k(n)_*(Y_i)$, and may be built from the spaces in the Ω-spectrum for $k(n)$ using only v_n-torsion invariants.

We also extend Quillen's theorem on complex cobordism to show that for any space X, the $P(n)_*$-module $P(n)^*(X)$ is generated by elements of $P(n)^i(X)$ for $i \geq 0$. This result is essential for the work of Ravenel–Wilson–Yagita, which in many cases allows one to compute BP-cohomology from Morava K-theory.

Introduction

We exploit the close relationship between the connective Morava K-theory spectrum $k(n)$, whose coefficient ring is $k(n)_* = \mathbb{F}_p[v_n]$, and the spectrum $P(n)$ with $P(n)_* = \mathbb{F}_p[v_n, v_{n+1}, v_{n+2}, \dots]$, where $\mathbb{F}_p$ denotes the field with p elements. These ring spectra are defined for each prime p (suppressed from almost all the notation) and integer $n \geq 0$. Most of our work generalizes the case $n = 0$ (see [Wi75]), where $k(0) = H(\mathbb{Z}_{(p)})$, the Eilenberg–Mac Lane spectrum for $\mathbb{Z}_{(p)}$ (the integers localized at p), and $P(0) = \mathrm{BP}$, the Brown–Peterson spectrum, with $\mathrm{BP}_* = \mathbb{Z}_{(p)}[v_1, v_2, v_3, \dots]$.

In Section 1 we present three groups of results. First, we give a structure theorem for a class of H-spaces that may be defined entirely in terms of $k(n)$. Second, starting from $P(n)$, we construct examples of such H-spaces, which we use to prove our structure theorem. Third, there are consequences for the structure of $P(n)$-(co)homology: we find (i) a Quillen-type result, that $P(n)^*(X)$ is generated as a module by elements of $P(n)^i(X)$ for $i \geq 0$, (ii) a Landweber-type filtration theorem, and (iii) a bound on the homological dimension of $P(n)$-homology.

All these results depend on the Ravenel–Wilson calculation [RW96] of the Hopf ring for $P(n)$, which encodes the unstable operations in $P(n)$-cohomology. All the machinery of [Bo95, BJW95] becomes available, making $P(n)^*(-)$ our sixth example of a cohomology theory whose operations we can handle in a uniform manner.

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Notation

We fix throughout a prime p and an integer $n > 0$. Because it occurs so frequently, we find it convenient to write $N = p^n - 1$.

[For completeness, we include the *results* for $p = 2$. Modifications are required because (i) our ring spectra are no longer commutative, and (ii) one of our test spaces, real projective space, has different cohomology. Shorter comments, like this one, are enclosed within square brackets. Longer comments form a subsection. A few proofs are substantial enough to be deferred to a forthcoming paper [Bo].**]**

All spaces are assumed to be homotopy-equivalent to CW-complexes. Identity maps and homomorphisms are denoted by id.

We use much notation and terminology from [BJW95]. A ring spectrum E defines a homology theory $E_*(-)$ and a cohomology theory $E^*(-)$, both multiplicative with coefficient ring $E_* = \pi_*^S(E)$. Then $E^i(-)$ is represented (on the homotopy category Ho of *unbased* spaces) by the i-th space $\underline{E}_i$ of the Ω-spectrum for E.

Because we deal mainly with homology and homotopy groups rather than cohomology, we use *homology* degrees throughout (unlike [BJW95]), assigning the degree i to elements of $E_i(X)$ and $\pi_i(X)$. This forces elements of $E^i(X)$ to have degree $-i$. We thus write E_* for the coefficient ring, even when working with cohomology; in particular, $E^i(\text{point}) = E_{-i}$. So the Hazewinkel generator v_i has degree $2(p^i - 1)$.

The *algebraic suspension* ΣM of a graded group M is a copy of M with all degrees raised by one: an element $x \in M_i$ gives rise to $\Sigma x \in (\Sigma M)_{i+1}$.

As in [RW96], $E(x, \dots)$ denotes the exterior algebra on generator(s) $x, \dots$ and $P(x, \dots)$ the polynomial algebra. $TP_h(x)$ denotes the truncated polynomial algebra $P(x)/(x^{p^h})$.

1 The Main Results

1.1 Splittings of H-spaces

We regard the standard generator u_k of $P(n)^*(S^k)$ as a map $u_k\colon S^k \to \underline{P(n)}_k$. We consider spaces X that satisfy the following axioms.

Axioms 1.1

(A) X is a connected H-space of *finite type* (meaning that each homotopy group $\pi_i(X)$ is finitely generated);
(B) $k(n)_*(X)$ is a free $k(n)_*$-module (equivalently, has no v_n-torsion);
(C) For any $k > 0$, any map $S^k \to X$ factors through the map u_k to give a map $\underline{P(n)}_k \to X$.

Our first theorem classifies these spaces.

Theorem 1.2 *Given $n > 0$, the spaces X that satisfy Axioms* 1.1 *have the following properties.*

(i) *For each $k > 0$, there is (up to homotopy) a unique $(k-1)$-connected (but not k-connected) example Y_k that does not decompose as a product of spaces.*

(ii) *Every X is homotopy equivalent to some product $Y = \prod_i Y_{k_i}$, where the number of copies of each Y_k is finite and is uniquely determined by X.*
(iii) *Every retract of X is another example.*
(iv) *Every product of examples is an example, provided it has finite type.*
(v) *The loop space ΩX is another example, provided X is simply connected.*

Shortly, in Definition 1.5, we shall reveal the spaces Y_k explicitly.

Remark The above decompositions and equivalences are *not* as H-spaces. Nevertheless, no information is lost, because in (ii) for example, the given multiplication on X corresponds to *some* multiplication on Y; as we (shall) have complete information on the possible maps $Y \times Y \to Y$, we can in principle detect which of them are H-space multiplications.

Part (iii) is clear. So is (iv), with the help of the Künneth formula for $k(n)$-homology (as in [Bo95, Theorem 4.2]). Part (v) will follow immediately from (ii) and Theorem 1.7. We prove (i) and (ii) in Section 3.

1.2 Towers Built from $k(n)$

Although Axiom 1.1(C) is technically convenient, it lacks intuitive content. Here, we replace it by a more appealing axiom. This makes Theorem 1.2 analogous to the results of [Wi75], as we discuss later in this section.

We consider spaces that are built from the spaces $\underline{k(n)}_i$ in a particularly nice way, using only v_n-torsion invariants. We recall that $k(n)_* = \mathbb{F}_p[v_n]$, where v_n has degree $2N = 2(p^n - 1)$.

Definition 1.3 Given a space Y, we call a map $z\colon Y \to \underline{k(n)}_{q+1}$ a *v_n-torsion map* if, considered as an element of $k(n)^*(Y)$, it satisfies $v_n^c z = 0$ for some c. (We assume $q \geq 0$. Indeed, z must be zero unless $q \geq 2N + 1 = 2p^n - 1$.)

We call a space X a *$k(n)$-tower with v_n-free homotopy* if it is the homotopy limit of a sequence of spaces and maps

$$\cdots \to X_3 \to X_2 \to X_1 \to X_0 = \text{point} \tag{1.1}$$

in which each map $X_i \to X_{i-1}$ (for $i > 0$) is the homotopy fibre of some v_n-torsion map $z_i\colon X_{i-1} \to \underline{k(n)}_{q(i)+1}$. (We allow the possibility of a finite tower, $X = X_m$ for some m, or even a tower having only one stage, $X = X_1 = \underline{k(n)}_{q(1)}$, as well as the degenerate case where X is contractible.)

A v_n-torsion map $z\colon Y \to \underline{k(n)}_{q+1}$ necessarily induces the zero homomorphism on homotopy. Then for each $i > 0$ (assuming X is connected, so that $q(i) \geq 1$), the homotopy long exact sequence of z_i reduces to the short exact sequence of groups

$$0 \to \Sigma^{q(i)}\mathbb{F}_p[v_n] \to \pi_*(X_i) \to \pi_*(X_{i-1}) \to 0. \tag{1.2}$$

Thus $\pi_*(X)$ is an iterated extension of suspensions of $\mathbb{F}_p[v_n]$. (Our terminology is abusive to the extent that we do not have a natural action of v_n on $\pi_*(X_i)$ for $i > 1$.)

We study such towers in more detail in Section 4 and prove the following equivalence.

Theorem 1.4 *If we replace Axiom* 1.1(C) *by the axiom*

(C′) *X is a k(n)-tower with v_n-free homotopy,*

we obtain the same class of H-spaces. Thus Theorem 1.2 *remains valid.*

1.3 Examples Based on $P(n)$

The prime ideal $I_n = (p, v_1, v_2, \dots, v_{n-1}) \subset \mathrm{BP}_* = \mathbb{Z}_{(p)}[v_1, v_2, v_3, \dots]$ is invariant and therefore of particular interest. (We set $v_0 = p$, and take $I_1 = (p)$ and $I_0 = (0)$.) The spectrum $P(n)$ is constructed (see Section 2) to have the quotient ring

$$P(n)_* = \mathrm{BP}_* / I_n = \mathbb{F}_p[v_n, v_{n+1}, v_{n+2}, \dots]$$

as its homotopy. In particular, $P(0) = \mathrm{BP}$, and $P(1)$ is just BP mod p.

Further, given $m \geq n$, we kill off the ideal

$$J_m = (v_{m+1}, v_{m+2}, v_{m+3}, \dots) \subset P(n)_* \tag{1.3}$$

to produce the spectrum we call $P(n, m)$ (but known as $\mathrm{BP}\langle n, m+1\rangle$ to Yosimura [Yo76], and as $\mathrm{BP}\langle p, v_1, \dots, v_{n-1}, v_{m+1}, \dots\rangle$ to Yagita [Ya76]), with homotopy

$$P(n, m)_* = P(n)_* / J_m = \mathbb{F}_p[v_n, v_{n+1}, \dots, v_m].$$

It comes equipped with a canonical map $\rho(m)\colon P(n) \to P(n, m)$. These spectra are intimately connected with the spectra $E(n, m) = v_m^{-1} P(n, m)$, which are essential in Ravenel–Wilson–Yagita [RWY98]. We recognize $P(n, n)$ as $k(n)$.

Remark Unlike I_n, the ideal J_m is not at all canonical, as it depends on the choice of the generators v_i of $P(n)_*$. Nevertheless, our results are independent of this choice, as we are concerned only with the additive structure of $P(n, m)$.

The behavior of these spectra depends on the numerical function

$$g(n, m) = 2(p^n + p^{n+1} + \cdots + p^m), \tag{1.4}$$

where it is reasonable to define $g(n, n-1) = 0$.

Definition 1.5 Given $k > 0$, we define the H-space $Y_k = \underline{P(n, m)}_k$, where the integer m is defined in terms of (1.4) by

$$g(n, m-1) < k \leq g(n, m). \tag{1.5}$$

For convenience, we also define $Y_0 = \mathbb{F}_p$, viewed as a discrete group.

These are the spaces Y_k that appear in Theorem 1.2. In particular, $Y_k = \underline{k(n)}_k$ for $0 < k \leq 2p^n$. As the spaces $\underline{P(n)}_k$ satisfy Axioms 1.1, they must decompose according to Theorem 1.2(ii). We establish the following splittings in Section 3.

Theorem 1.6 *Assume $k \geq 0$. If $k > 0$, define m by* (1.5)*; if $k = 0$, take $m = n - 1$. Then we have homotopy decompositions*

$$\underline{P(n)}_k \simeq Y_k \times \prod_{j>m} Y_{k+2(p^j-1)} \tag{1.6}$$

and, for any $h > m$,

$$\underline{P(n,h)}_k \simeq Y_k \times \prod_{j=m+1}^{h} Y_{k+2(p^j-1)}. \tag{1.7}$$

These are equivalences of H-spaces **[***except in the extreme case when $p = 2$ and $k = g(n,m)$***]**.

We showed in [BW01, Theorem 1.1] that such splittings exist, though without making them explicit as we do here in Section 3. They are patterned after the splittings of the spaces $\underline{\mathrm{BP}}_k$ in [Wi75], which were recovered explicitly in [BJW95] and are reviewed below.

We note that (1.7) reduces to Definition 1.5 when $h = m$.

Remark No such result holds for $\underline{P(n,m)}_k$ when $k > g(n,m)$, as Axiom 1.1(B) definitely *fails* (otherwise this space would contradict Theorem 1.2(ii)).

We use (1.7) to decompose $\Omega Y_k = \underline{P(n,m)}_{k-1}$ explicitly.

Theorem 1.7 *The loop space ΩY_k is given for all $k > 0$ as follows:*

(i) *If k does not have the form $g(n,q) + 1$ for any q, then $\Omega Y_k \simeq Y_{k-1}$.*
(ii) *If $k = g(n,q) + 1$, where $q \geq n - 1$, then $\Omega Y_k \simeq Y_{k-1} \times Y_{k-1+2(p^{q+1}-1)}$.*

Since Ω is a right adjoint functor and so preserves products, this gives part (v) of Theorem 1.2. We leave it as an exercise to decompose the negative spaces $\underline{P(n)}_{-k}$ for $k > 0$, by writing them as $\Omega^{k+1}\underline{P(n)}_1$, and similarly for $\underline{P(n,m)}_{-k}$.

1.4 Some History

For $n = 0$, the results differ slightly. Recall that $k(0) = H(\mathbb{Z}_{(p)})$, $P(0) = \mathrm{BP}$, and (see [Wi75]) $P(0,m) = \mathrm{BP}\langle m\rangle$ has $\mathrm{BP}\langle m\rangle_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_m]$. Axioms 1.1 (with (C) replaced by (C′) as in Theorem 1.4) then yield connected H-spaces X whose homotopy groups $\pi_k(X)$ and homology groups $H_k(X)$ are all free $\mathbb{Z}_{(p)}$-modules of finite rank. The Postnikov k-invariants of such spaces are necessarily torsion elements. Theorem 1.2 remains valid exactly as stated, with m still defined by (1.5). However, Theorem 1.6 gives H-space equivalences only for $g(0,m-1) < k < g(0,m)$; for

$k = g(0, m)$, we have merely a homotopy equivalence. (Of course, $Y_0 = \mathbb{Z}_{(p)}$ rather than $\mathbb{F}_p$.) These are the main results of [Wi75] or [BJW95, Theorem 1.16], and form the motivation for this work.

1.5 The Structure of $P(n)$-Cohomology

We extend Quillen's theorem on complex cobordism to $P(n)$.

Theorem 1.8 *For any space X, the cohomology $P(n)^*(X)$ is generated as a $P(n)_*$-module (topologically if X is infinite) by elements of $P(n)^i(X)$ for $i > 0$ together with one element of $P(n)^0(X)$ for each component of X.*

This result is essential for the calculations in Ravenel–Wilson–Yagita [RWY98]. One version was stated [Ya84, Theorem 1.11] without proof (although the approach suggested is now known not to work). In Section 8, our machinery of additive unstable operations provides a very short direct proof in terms of an explicit formula.

We also refine Landweber's filtration theorem. Yosimura [Yo76, Theorem 3.4] and Yagita [Ya76] both observed that Landweber's theorem generalizes to stable $P(n)$-cohomology comodules M. The only finitely generated invariant prime ideals in $P(n)_*$ are $I_m = (v_n, v_{n+1}, \dots, v_{m-1})$ for $n \le m < \infty$ (where I_n is interpreted as (0)). We find in Section 8 that an unstable comodule structure on M (in the sense of [BJW95, Definition 6.32]) restricts the possible Landweber factors as follows.

Lemma 1.9 *Let M be a $P(n)_*$-module with a single generator $x \in M^k$ (in homology degree $-k$) and annihilator ideal $\mathrm{Ann}(x) = I_m$, where $n \le m < \infty$, so that*

$$M \cong \Sigma^{-k} P(m)_* \cong \Sigma^k P(m)^* \cong \Sigma^k (P(n)^*/I_m).$$

Then M admits an unstable $P(n)$-cohomology comodule structure if and only if k satisfies the appropriate condition (depending on m and p):

(i) $k \ge 0$ *if* $m = n$;
(ii) $k \ge g(n, n) - 1$ *if* $m = n + 1$;
(iii) $k \ge g(n, m-1) - 1$ *if* $m \ge n + 2$ *and p is odd;*
(iv) $k \ge g(n, m-1) - 2$ *if* $m \ge n + 2$ *and* $p = 2$;

and this comodule structure is unique.

This leads directly to the filtration theorem.

Theorem 1.10 *Let M be an unstable $P(n)$-cohomology comodule of finite type (each M^i a finitely generated $\mathbb{F}_p$-module) and bounded above ($M^i = 0$ for all $i > i_0$). Then M admits a finite filtration by subcomodules $0 = M_0 \subset M_1 \subset \cdots \subset M_h = M$ in which each quotient M_i/M_{i-1} is a monogenic comodule $\Sigma^{-k_i} P(m_i)_*$ with generator x_i, as listed in Lemma* 1.9. *In particular, M is a finitely presented $P(n)_*$-module.*

If, in addition, M is a $P(n)_*$-algebra of any of the forms:

(i) $M = P(n)^*(X)$, *for a finite complex X;*

(ii) $M = \mathrm{Im}[f^*\colon P(n)^*(Y) \to P(n)^*(X)]$, *for a map of spaces* $f\colon X \to Y$, *where* X *is a finite complex;*
(iii) *A spacelike (see [BJW95, Definition 7.14]) unstable* $P(n)_*$*-cohomology algebra;*

then we may take each M_i *to be an invariant ideal in* M. *At the last stage, we may take* $x_h = 1$ *and* $m_h = n$.

Our proof in Section 8 quotes the method of proof in [BJW95, Theorem 20.11]. However, here we *prove* that M is finitely presented, instead of assuming it. (Of course, it has long been known that for finite X, $P(n)^*(X)$ is a coherent $P(n)_*$-module and hence finitely presented.) In [BJW95, Theorem 20.11], we overlooked the fact that this modification applies equally well to $\mathrm{BP} = P(0)$, as follows. (Again, (i) is not new. However, (ii) is non-trivial and new when $\mathrm{BP}^*(Y)$ has phantom classes.)

Theorem 1.11 *Let* M *be an unstable* BP*-cohomology comodule of finite type (each* M^i *a finitely generated* $\mathbb{Z}_{(p)}$*-module) and bounded above, for example:*

(i) $M = \mathrm{BP}^*(X)$, *for a finite complex* X;
(ii) $M = \mathrm{Im}[f^*\colon \mathrm{BP}^*(Y) \to \mathrm{BP}^*(X)]$, *for a map of spaces* $f\colon X \to Y$, *where* X *is a finite complex.*

Then M *is a finitely presented* BP_**-module.*

1.6 Homological Dimension

Our starting point is the Conner–Floyd theorem [CF66, Theorem 10.1] that the map of ring spectra from the unitary Thom spectrum MU to the K-theory spectrum K determined by the Todd genus induces for finite X an isomorphism of cohomology theories

$$K_* \otimes_{MU_*} MU^*(X) \xrightarrow{\cong} K^*(X).$$

A far-reaching analogue is the result

$$E(n,m)_* \otimes_{P(n)_*} P(n)^*(X) \cong E(n,m)^*(X), \tag{1.8}$$

where $E(n,m) = v_m^{-1}P(n,m)$. A key ingredient of such results is knowledge of the homological dimension of various (co)homology modules.

The case $m = n$ of (1.8) is due to Morava [Mo85] as part of his structure theorem, and is quoted and reproved in [JW75], as well as by Yagita [Ya76]. The case $n = 0$, along with results on the homological dimension of $\mathrm{BP}_*(X)$, was proved by Johnson–Wilson [JW73, Remark 5.13] by means of the splitting theorem for BP in [Wi75]. Shortly afterwards, Landweber [La76] reproved this case by using cohomology operations instead of the splitting, establishing his exact functor theorem in the process; however, he was unable to recover Corollary 4.4 of [JW73], which gave an upper bound on the homological dimension of $\mathrm{BP}_*(X)$. Later, Morava and Yagita [Ya77, Theorem 3.11] showed that $P(n)^*(X)$ is a $\mathrm{BP}^*(\mathrm{BP})$-module. Yagita and Yosimura [Yo76] both used this fact to generalize the exact functor theorem to $P(n)$, which fully includes (1.8), and obtain homological dimension results for $P(n)_*(X)$.

We have now gone full circle, and with our splitting for $P(n)$ in hand, can use the techniques of [JW73] to recover these results as well as (1.8) with the added benefit of the following estimate, which we establish in Section 5.

Theorem 1.12 *Assume that X is a finite complex of dimension less than $g(n,m)/2$. Then the homological dimension of the $P(n)_*$-module $P(n)_*(X)$ is at most $m-n$.*

Although the exact functor theorem does not apply, $\rho(m)\colon P(n) \to P(n,m)$ still induces a natural homomorphism of $P(n,m)_*$-modules

$$\overline{\rho(m)}\colon P(n,m)_* \otimes_{P(n)_*} P(n)^*(X) \to P(n,m)^*(X).$$

This is an isomorphism when X is a point, but not in general, as the left side is not a cohomology theory. Classically, as in [JW73], one then asks for which X it *is* an isomorphism. Instead, we show in Section 5 that it is *always* an isomorphism in a certain range of degrees **[**with no modification if $p=2$**]**. Explicitly, its components are

$$\overline{\rho(m)}\colon P(n)^h(X)\Big/\sum_{j>m} v_j P(n)^{h+2(p^j-1)}(X) \to P(n,m)^h(X). \tag{1.9}$$

Theorem 1.13 *Assume that X is finite-dimensional and that $m \geq n > 0$. Then* (1.9) *is an isomorphism for all $h \leq g(n,m)$, and therefore a $P(n,m)_*$-module isomorphism in this range.*

In particular, for $m=n$ we have the isomorphism

$$\overline{\rho(n)}\colon P(n)^h(X)\Big/\sum_{j>n} v_j P(n)^{h+2(p^j-1)}(X) \cong k(n)^h(X)$$

for all $h \leq 2p^n$, which preserves the v_n-action in this range.

2 The Ring Spectrum $P(n)$

As the literature is somewhat conflicting **[**especially when $p=2$**]**, we review the construction of $P(n)$ in fair detail. In this section, we work entirely in the graded stable homotopy category Stab_*.

The spectrum $P(n)$, so named by Johnson–Wilson [JW75], was based on work of Morava. It may conveniently be constructed directly from the Thom spectrum MU by applying Sullivan–Baas theory [Ba73] to kill off the unwanted generators of MU_*, as well as p (with no need for localization). (As stable $P(n)$-cohomology operations act faithfully on $P(n)$-homology, no information is lost by working in homology.)

It is automatically a BP-module spectrum, with an action map $\lambda\colon \mathrm{BP}\wedge P(n) \to P(n)$ that satisfies the usual two module axioms, and the canonical map $\mathrm{BP} \to P(n)$ is BP-linear. It comes equipped with an exterior algebra $E(Q_0, Q_1, \ldots, Q_{n-1})$ of BP-linear operations, where Q_i has homology degree $-(2p^i-1)$; we write the monomial basis elements as $Q^I = Q_0^{i_0}\circ Q_1^{i_1}\circ\cdots\circ Q_{n-1}^{i_{n-1}}$ for each multi-index $I=(i_0,i_1,\ldots,i_{n-1})$, where each i_r is 0 or 1.

2.1 The Multiplication

The canonical map $\eta\colon S^0 \to \mathrm{BP} \to P(n)$ serves as the unit map of $P(n)$, where S^0 denotes the sphere spectrum, but there is no obvious multiplication on $P(n)$. It is known that for $p \neq 2$, there is a unique multiplication $\phi\colon P(n) \wedge P(n) \to P(n)$ having the following properties:

Axioms 2.1

(A) ϕ is BP-bilinear;
(B) $\mathrm{BP} \to P(n)$ is multiplicative;
(C) ϕ has $\eta\colon S^0 \to P(n)$ as two-sided unit;
(D) ϕ is commutative;
(E) ϕ is associative;
(F) each $Q_i\colon P(n) \to P(n)$ is a derivation, in the sense that

$$Q_i \circ \phi = \phi \circ (Q_i \wedge \mathrm{id}) + \phi \circ (\mathrm{id} \wedge Q_i)\colon P(n) \wedge P(n) \to P(n). \tag{2.1}$$

Historically, three quite different approaches have been used. First, for $p \neq 2$, Morava [Mo79] used averaging over the symmetric group Σ_2 to produce idempotent operations in (co)bordism with repeated singularities. These operations yield a canonical multiplication ϕ on $P(n)$ that is automatically commutative (*cf.* Mironov [Mi78, Theorem 4.2]). Associativity by this method involves averaging over Σ_3 and requires $p \geq 5$ [Mi78, Theorem 4.1].

The second method is heavily geometric. Mironov [Mi75] and Shimada–Yagita [SY76] constructed (roughly equivalent) explicit multiplications on $P(n)$ in the Baas bordism context for *any* prime p. These apparently depend on a sequence of choices of Morava manifolds. They automatically satisfy axioms 2.1(A)–(C). Moreover, Shimada–Yagita [SY76, Theorem 5.25] and Mironov [Mi78, Theorem 2.4] both show that the obstructions to associativity lie in groups that vanish, and also obtain (F). The disadvantage of this approach is that uniqueness is difficult to handle.

Third, Würgler [Wü77] developed an entirely algebraic cohomological approach in terms of comodules, which leads to the existence of ϕ and the following results.

Lemma 2.2 *In the graded stable homotopy category* Stab_**:*

(i) *Any* BP*-linear map* $P(n) \to P(n)$*, of any degree, can be uniquely written in the form*

$$\sum_I c_I Q^I\colon P(n) \to P(n), \tag{2.2}$$

with coefficients $c_I \in P(n)_*$ *of the appropriate degrees;*

(ii) *Any* BP*-bilinear map* $P(n) \wedge P(n) \to P(n)$*, of any degree, can be uniquely written in the form*

$$\sum_{I,J} c_{I,J}\phi \circ (Q^I \wedge Q^J)\colon P(n) \wedge P(n) \to P(n), \tag{2.3}$$

with coefficients $c_{I,J} \in P(n)_*$ *of the appropriate degrees;*

(iii) *Any* BP*-trilinear map* $P(n) \wedge P(n) \wedge P(n) \to P(n)$*, of any degree, can be uniquely written in the form*

$$\sum_{I,J,K} c_{I,J,K}\phi \circ (\phi \wedge \mathrm{id}) \circ (Q^I \wedge Q^J \wedge Q^K)\colon P(n) \wedge P(n) \wedge P(n) \to P(n), \tag{2.4}$$

with coefficients $c_{I,J,K} \in P(n)_*$ *of the appropriate degrees.*

Proof Part (i) is a strengthened form of Würgler [Wü77, Proposition 3.5], (ii) is [Wü77, Proposition 4.12], and (iii) is entirely analogous. ∎

Lemma 2.3 *The canonical map* $\rho\colon P(n) \to P(n+1)$ *is a map of ring spectra.*

Proof By a slight generalization of (2.3) (also proved by Würgler), any BP-bilinear map $P(n) \wedge P(n) \to P(n+1)$, in particular $\phi \circ (\rho \wedge \rho)$, can be written

$$\sum_{I,J} c_{I,J}\, \rho \circ \phi \circ (Q^I \wedge Q^J)\colon P(n) \wedge P(n) \to P(n+1),$$

with coefficients $c_{I,J} \in P(n+1)_*$. Since $\phi \circ (\rho \wedge \rho) \circ (\eta \wedge \eta) = \eta$, the sparseness of $P(n+1)_*$ leaves $\rho \circ \phi$ as the only candidate for $\phi \circ (\rho \wedge \rho)$. **[**This works even for $p = 2$, regardless of the choices of multiplication on $P(n)$ and $P(n+1)$.**]** ∎

If we write $\phi \circ (\eta \wedge \mathrm{id})$ in the form (2.2), the sparseness of $P(n)_*$ yields Axiom 2.1(C) **[**even for $p = 2$**]**, since we know $\phi \circ (\eta \wedge \eta) = \eta$. Then (B) is a formal consequence of (A), (C), and the BP-linearity of the map $\mathrm{BP} \to P(n)$.

Since any BP-bilinear multiplication can be written in the form (2.3), the sparseness of $P(n)_*$ ensures that ϕ is unique, *as long as* $p \geq 3$. Further, (D) holds, since $\phi \circ T$ also satisfies (A) and (C), where $T\colon P(n) \wedge P(n) \to P(n) \wedge P(n)$ denotes the switch map.

We may similarly deduce the associativity of ϕ, *provided* $p \geq 5$, by writing $\phi \circ (\mathrm{id} \wedge \phi)$ in the form (2.4). We also obtain (F), provided $p \geq 3$, by writing $Q_i \circ \phi$ in the form (2.3); since $(Q_i \circ \phi) \circ (\eta \wedge \mathrm{id}) = Q_i = (Q_i \circ \phi) \circ (\mathrm{id} \wedge \eta)$, the only candidate is (2.1).

Finally, we should mention that there is now a fourth approach, the brave new ring context of Elmendorf–Kriz–Mandell–May. See [EKMM96] for p odd **[**or Strickland [St99] for $p = 2$**]**.

The Case $p = 2$**:** It is well known that there is *no* commutative multiplication on $P(n)$ when $p = 2$. Instead, we see [Bo] that there are exactly *two* multiplications that satisfy all of Axioms 2.1 except (D). To make $P(n)$ a ring spectrum, we arbitrarily choose one of the two good multiplications as ϕ; then the other is its opposite, $\overline{\phi} = \phi \circ T$, which defines the *opposite* ring spectrum $\overline{P(n)}$. Nassau [Na02, Theorem 3] shows that complex conjugation defines an isomorphism of ring spectra $\Xi\colon P(n) \cong \overline{P(n)}$.

Mironov [Mi78, Theorem 4.7] computed $\bar{\phi}$ explicitly in the form (2.3) as

$$\bar{\phi} = \phi \circ T = \phi + v_n\phi \circ (Q_{n-1} \wedge Q_{n-1})\colon P(n) \wedge P(n) \to P(n). \tag{2.5}$$

From now on, we write $Q = Q_{n-1}$, in view of its frequent occurrence.

2.2 Products in Homology and Cohomology

We review briefly the various products in $P(n)$-(co)homology. Their properties are familiar enough **[**except when $p = 2$**]**. We remind the reader that the operations Q_i act on both homology and cohomology.

Given $x \in P(n)^*(X)$ and $y \in P(n)^*(Y)$, we have the *cohomology cross product* $x \times y \in P(n)^*(X \times Y)$; by taking $Y = X$ and using the diagonal map of X, we deduce the *cup product* $xy \in P(n)^*(X)$, which makes $P(n)^*(X)$ a ring. Given $a \in P(n)_*(X)$ and $b \in P(n)_*(Y)$, we have the *homology cross product* $a \times b \in P(n)_*(X \times Y)$. All three products are associative. For $p \neq 2$, they are also commutative in the sense that $T^*(y \times x) = \pm x \times y$, $yx = \pm xy$, and $T_*(b \times a) = \pm a \times b$. By (2.1), each Q_i is a derivation for all three products.

By taking X as a one-point space, $P(n)^*(Y)$ and $P(n)_*(Y)$ become $P(n)_*$-modules, and both cross products are $P(n)_*$-bilinear **[**even for $p = 2$; see below**]**.

There is also the *scalar* or *Kronecker product* $\langle x, a\rangle \in P(n)_*$ of $x \in P(n)^*(X)$ and $a \in P(n)_*(X)$, which is $P(n)_*$-bilinear **[**even for $p = 2$; see [Bo]**]**.

The Case $p = 2$: There are of course no signs, but the noncommutativity of ϕ forces us to watch carefully for any shuffling of copies of $P(n)$. Nevertheless, we find [Bo] that the Künneth and duality formulae continue to hold, exactly as stated in [Bo95].

It is immediate from (2.5) that

$$T^*(y \times x) = x \mathbin{\overline{\times}} y = x \times y + v_n Qx \times Qy \quad \text{in } P(n)^*(X \times Y),$$

where $x \mathbin{\overline{\times}} y$ denotes the *twisted* cross product formed using the opposite multiplication $\bar{\phi}$ on $P(n)$. For cup products, this implies

$$yx = xy + v_n(Qx)(Qy) \quad \text{in } P(n)^*(X),$$

so that $P(n)^*(X)$ is *not* commutative in general in the ordinary sense. Alternatively, these products are T_Q-*commutative* if we replace the standard commutativity isomorphism $T\colon A \otimes B \cong B \otimes A$ everywhere by $T_Q\colon A \otimes B \cong B \otimes A$, defined by

$$T_Q(a \otimes b) = b \otimes a + v_n Qb \otimes Qa \quad \text{in } B \otimes A. \tag{2.6}$$

Similarly, homology is also T_Q-commutative, in the sense that

$$T_*(b \times a) = a \mathbin{\overline{\times}} b = a \times b + v_n Qa \times Qb \quad \text{in } P(n)_*(X \times Y).$$

Taking X to be a point shows that the $P(n)_*$-actions on $P(n)^*(Y)$ and $P(n)_*(Y)$ are independent of the choice of ϕ. In [Bo], we find that $\langle x, a\rangle$ is also independent of this choice.

There is one surprise, on account of the hidden shuffling, proved in [Bo].

Proposition 2.4 *Given $x \in P(n)^*(X)$, $y \in P(n)^*(Y)$, $a \in P(n)_*(X)$, and $b \in P(n)_*(Y)$, we have*

$$\langle x \times y, a \times b\rangle = \langle x, a\rangle\langle y, b\rangle + v_n\langle x, Qa\rangle\langle Qy, b\rangle.$$

If instead we mix the products, we find

$$\langle x \times y, a \overline{\times} b\rangle = \langle x, a\rangle\langle y, b\rangle. \tag{2.7}$$

3 Proofs of the Main Theorems

In this section, we establish Theorems 1.2 and 1.6. More precisely, we reduce them to two key lemmas: Lemma 3.1 provides our main splitting, and Lemma 3.4 will imply that our splittings are best possible.

3.1 Splittings

All our splittings are derived from the following splitting.

Lemma 3.1 *For $k \le g(n, m)$, where $m \ge n$, there is a map $\overline{\theta(m)}\colon \underline{P(n, m)}_k \to \underline{P(n)}_k$ that splits the canonical map $\rho(m)\colon \underline{P(n)}_k \to \underline{P(n, m)}_k$, i.e., $\rho(m) \circ \overline{\theta(m)} \simeq \mathrm{id}$. It is a map of H-spaces* **[***except when $p = 2$ and $k = g(n, m)$***]**.

We express this in terms of idempotent $P(n)$-cohomology operations in Section 5.

A short direct proof of Lemma 3.1 is presented in [BW01], based on the bar spectral sequence. For such k, we show that $E_*(\underline{P(n, m)}_k)$ is a quotient of $E_*(\underline{P(n)}_k)$, first for $E = P(n)$, then for $E = P(n, m)$, and that these are free E_*-modules. It follows by duality that $\overline{\theta(m)}$ exists, but its status as an H-map is left unclear.

We deduce other useful splittings. The canonical map

$$\rho(m-1, m)\colon P(n, m) \to P(n, m-1),$$

which kills v_m, fits into the exact triangle of spectra

$$P(n, m) \xrightarrow{v_m} P(n, m) \xrightarrow{\rho(m-1,m)} P(n, m-1) \xrightarrow{\delta} P(n, m). \tag{3.1}$$

On homotopy groups, this induces the obvious short exact sequence

$$0 \to \mathbb{F}_p[v_n, v_{n+1}, \dots, v_m] \xrightarrow{v_m} \mathbb{F}_p[v_n, v_{n+1}, \dots, v_m] \to \mathbb{F}_p[v_n, v_{n+1}, \dots, v_{m-1}] \to 0.$$

Unstably, we have the H-space fibration

$$\underline{P(n, m)}_{k+2(p^m-1)} \xrightarrow{v_m} \underline{P(n, m)}_k \xrightarrow{\rho(m-1,m)} \underline{P(n, m-1)}_k.$$

For $k \le g(n, m-1)$, the composite

$$\underline{P(n, m-1)}_k \xrightarrow{\overline{\theta(m-1)}} \underline{P(n)}_k \xrightarrow{\rho(m)} \underline{P(n, m)}_k \tag{3.2}$$

automatically splits $\rho(m-1,m)$, to yield the decomposition

$$\underline{P(n,m)}_k \simeq \underline{P(n,m-1)}_k \times \underline{P(n,m)}_{k+2(p^m-1)}, \tag{3.3}$$

where the two injections are (3.2) and v_m. This is a decomposition of H-spaces **[**except when $p=2$ and $k=g(n,m-1)$**]**.

Proof of Theorem 1.6 This is completely analogous to the proof of [BJW95, Theorem 1.16]. Everything we need is contained in the commutative diagram

$$\begin{array}{ccccc} \underline{P(n)}_{k+2(p^j-1)} & \xrightarrow{v_j} & \underline{P(n)}_k & \xrightarrow{\rho(m)} & \underline{P(n,m)}_k \\ \downarrow{\scriptstyle \rho(j)} & & \downarrow{\scriptstyle \rho(j)} & \nearrow{\scriptstyle \rho(m,j)} & \\ \underline{P(n,j)}_{k+2(p^j-1)} & \xrightarrow{v_j} & \underline{P(n,j)}_k & & \end{array} \tag{3.4}$$

of H-spaces and canonical H-maps, where $j>m$.

With m given by (1.5), we observe that the spaces Y_k and $Y_{k+2(p^j-1)}$ appear in the diagram disguised as $\underline{P(n,m)}_k$ and $\underline{P(n,j)}_{k+2(p^j-1)}$. We insert the splittings $\overline{\theta(m)}$ and $\overline{\theta(j)}$ from Lemma 3.1 to produce the desired decomposition of $\underline{P(n)}_k$, as suggested by the decomposition of abelian groups

$$\mathbb{F}_p[v_n, v_{n+1}, v_{n+2}, \dots] = \mathbb{F}_p[v_n, v_{n+1}, \dots, v_m] \oplus \bigoplus_{j>m} v_j \mathbb{F}_p[v_n, v_{n+1}, \dots, v_j].$$

(But we warn that our splittings cannot be expected to induce exactly this decomposition of the coefficient ring $P(n)_*$, and it seems likely that they *never* do.)

In detail, we map Y_k into $\underline{P(n)}_k$ by $\overline{\theta(m)}$, which is an H-map **[**unless $p=2$ and $k=g(n,m)$**]**, and $Y_{k+2(p^j-1)}$, for each $j>m$, by the H-map (in all cases)

$$Y_{k+2(p^j-1)} \xrightarrow{\overline{\theta(j)}} \underline{P(n)}_{k+2(p^j-1)} \xrightarrow{v_j} \underline{P(n)}_k.$$

We multiply these together, using the H-space structure of $\underline{P(n)}_k$, to form a map $f\colon W \to \underline{P(n)}_k$ from the *restricted* direct product W (the union of all finite subproducts) of the based Y-spaces mentioned.

We filter $P(n)_*$ by the ideals J_j. We note that $v_j \circ \overline{\theta(j)}$ induces a homomorphism $P(n,j)_* \to J_{j-1}$ on homotopy groups that induces the quotient isomorphism

$$P(n,j)_* = \mathbb{F}_p[v_n, v_{n+1}, \dots]/J_j \xrightarrow{v_j} J_{j-1}/J_j.$$

This is enough to guarantee that f induces an isomorphism on homotopy groups and is thus a homotopy equivalence. Because the connectivities of the Y-spaces increase, W is homotopy-equivalent to the desired full product, and we have (1.6).

The same method applies to $\underline{P(n,h)}_k$, with the simplification that the product W is now finite. (One can also produce decompositions like (1.7) directly from the splittings (3.3) by induction on h, though the resulting maps are different and far more complicated.)

For $k=0$, the splitting $\overline{\theta(n-1)}\colon Y_0 = \mathbb{F}_p \to \underline{P(n)}_0$ is obvious and unique up to homotopy. We can still use diagram (3.4). ∎

3.2 Indecomposability

On the other hand, we need to know that Y_k does not split.

Lemma 3.2 *A map $f\colon Y_k \to Y_k$ is a homotopy equivalence if and only if it induces an isomorphism on the bottom homotopy group $\pi_k(Y_k) \cong \mathbb{F}_p$.*

Corollary 3.3 *The space Y_k does not decompose as a product.*

In Section 12, we prove the following about $P(n)$ and deduce Lemma 3.2 from it.

Lemma 3.4 *Represent an unstable operation $r\colon P(n)^k(-) \to P(n)^m(-)$, where $k > 0$ and $m > 0$, by the map $r\colon \underline{P(n)}_k \to \underline{P(n)}_m$. Then the induced homomorphism on homotopy groups*

$$r_*\colon \Sigma^k P(n)_* \cong \pi_*(\underline{P(n)}_k) \xrightarrow{\pi_*(r)} \pi_*(\underline{P(n)}_m) \cong \Sigma^m P(n)_* \tag{3.5}$$

has the properties, for any element $v \in P(n)_*$:

(i) $r_*\Sigma^k(v_n v) = v_n r_* \Sigma^k v$;
(ii) $r_*\Sigma^k(v_q v) \equiv v_q r_* \Sigma^k v \bmod I_q = (v_n, v_{n+1}, \dots, v_{q-1})$, *provided* $k > g(n, q-1)$.

3.3 Construction of Maps

Our strategy for proving Theorem 1.2 is to construct enough maps to and from the spaces $\underline{P(n)}_k$.

Lemma 3.5 *If X is a space for which $k(n)_*(X)$ is a free $k(n)_*$-module, then $P(n)_*(X)$ is a free $P(n)_*$-module.*

Proof Lemmas 4.7 (with $k = m = n$) and 2.1 of Yosimura [Yo76] show that $P(n)_*(X)$ is a flat $P(n)_*$-module. Such modules are free by [Yo76, Proposition 1.5]. ∎

Lemma 3.6 *Let X be a $(k-1)$-connected space with $\pi_k(X)$ a nonzero finite abelian p-group and suppose $k(n)_*(X)$ is a free $k(n)_*$-module. Then there exists a map $f\colon X \to \underline{P(n)}_k$ that induces a nonzero homomorphism $f_*\colon \pi_k(X) \to \pi_k(\underline{P(n)}_k) \cong \mathbb{F}_p$ on the bottom homotopy groups.*

Proof Since $P(n)_*(X)$ is a free $P(n)_*$-module by Lemma 3.5, the universal coefficient theorem [Bo95, Theorem 4.14] gives $P(n)^*(X) \cong \operatorname{Hom}^*_{P(n)_*}(P(n)_*(X), P(n)_*)$. As X is $(k-1)$-connected, $P(n)_k(X) \cong H_k(X; \mathbb{F}_p) \cong \pi_k(X) \otimes \mathbb{F}_p \neq 0$, and it is clear that suitable cohomology classes $f \in P(n)^k(X)$, *i.e.,* maps $f\colon X \to \underline{P(n)}_k$, exist. ∎

Proof of Theorem 1.2(i) and (ii) We first note that for $k > 0$, the space $\underline{P(n)}_k$ satisfies Axioms 1.1. Axiom (A) is clear. Axiom (B) holds by [RW96]. Axiom (C) is easy. Take any element $\Sigma^k v \in \Sigma^k P(n)_* \cong \pi_*(\underline{P(n)}_k)$, where $v \in P(n)_h$. Viewed as a cohomology class, it is $vu_{k+h} \in P(n)^*(S^{k+h})$. Multiplication by v on $P(n)^*(-)$ is represented by the map we want, $v\colon \underline{P(n)}_{k+h} \to \underline{P(n)}_k$.

Then Y_k, being a retract of $\underline{P(n)}_k$, also satisfies the axioms. By Corollary 3.3, it is indecomposable. Uniqueness of Y_k and our decompositions will follow from (ii), under the assumption that all our H-spaces have finite type.

For the induction step in (ii) given any $(k-1)$-connected space X that satisfies the axioms, define m by (1.5). Then Lemma 3.6 provides a map

$$h\colon X \to \underline{P(n)}_k \xrightarrow{\rho(m)} \underline{P(n,m)}_k = Y_k$$

that induces a nonzero homomorphism $h_*\colon \pi_k(X) \to \pi_k(Y_k) \cong \mathbb{F}_p$. Choose $\alpha \in \pi_k(X)$ such that $h_*\alpha = 1 \in \mathbb{F}_p$; then Axiom 1.1(C) provides a map

$$f\colon Y_k = \underline{P(n,m)}_k \xrightarrow{\overline{\theta(m)}} \underline{P(n)}_k \to X$$

that induces $f_*1 = \alpha$. By Lemma 3.2, $h \circ f\colon Y_k \to Y_k$ is a homotopy equivalence. We use the homotopy fibre $j\colon F \to X$ of h and the multiplication μ on X to construct a homotopy equivalence

$$Y_k \times F \xrightarrow{f \times j} X \times X \xrightarrow{\mu} X.$$

Then F, being a retract of X, again satisfies the axioms.

We begin the induction with Z_0 as the given space, and find a sequence of equivalences $Z_i \simeq Y_{k_i} \times Z_{i+1}$ for $i \geq 0$. By finiteness, the spaces Z_i become more and more highly connected as i increases, and we deduce $Z_0 \simeq \prod_i Y_{k_i}$ as required. ∎

4 $k(n)$-Towers with v_n-Free Homotopy

In this section, we prove Theorem 1.4. We must show that the original Axiom 1.1(C) is equivalent (in the presence of the other axioms) to Axiom (C′), which asserts that X is a $k(n)$-tower with v_n-free homotopy. Lemma 4.1 shows that (C′) implies (C), while Lemma 4.3 gives the converse.

Lemma 4.1 *Suppose the connected H-space X is a $k(n)$-tower of finite type with v_n-free homotopy. Then Axiom* (C) *holds: given $k > 0$, any map $S^k \to X$ factors through the standard map $u_k\colon S^k \to \underline{P(n)}_k$ to yield a map $\underline{P(n)}_k \to X$.*

We first show that it does not matter how far up the tower we can lift.

Lemma 4.2 *In diagram* (1.1)*, any map $f\colon \underline{P(n)}_k \to X_{i-1}$ lifts to a map $\underline{P(n)}_k \to X$.*

Proof With z_i as in Definition 1.3, we note that $v_n^c(z_i \circ f) = f^*(v_n^c z_i) = 0$ in $k(n)^*(\underline{P(n)}_k)$. But by [RW96], $k(n)_*(\underline{P(n)}_k)$ and hence $k(n)^*(\underline{P(n)}_k)$ contain no v_n-torsion; therefore $z_i \circ f \simeq 0$ and f lifts to $f'\colon \underline{P(n)}_k \to X_i$.

By induction and limits, f lifts all the way to X. ∎

Proof of Lemma 4.1 For *any* connected space Y and $k > 0$, let us call an element $\alpha \in \pi_k(Y)$, or map $\alpha\colon S^k \to Y$, *extendable* if it extends over u_k to a map $\underline{P(n)}_k \to Y$.

All elements of $\pi_*(\underline{k(n)}_q) \cong \Sigma^q \mathbb{F}_p[v_n]$ are obviously extendable. It follows from diagram (1.2) that every element in $\mathrm{Ker}[\pi_*(X_i) \to \pi_*(X_{i-1})]$ is extendable.

By Lemma 4.2, any extendable element of $\pi_*(X_{i-1})$ lifts in diagram (1.1) to some extendable element of $\pi_*(X)$.

The sum of any two extendable elements of $\pi_k(X)$ is again extendable: given $f_1, f_2\colon \underline{P(n)}_k \to X$, we use the given multiplication μ on X to construct the map

$$\underline{P(n)}_k \xrightarrow{\Delta} \underline{P(n)}_k \times \underline{P(n)}_k \xrightarrow{f_1 \times f_2} X \times X \xrightarrow{\mu} X.$$

Together, these facts imply that every element of $\pi_*(X)$ is extendable. ∎

4.1 The Space Y_k

A countable product of $k(n)$-towers with v_n-free homotopy is another such tower (provided it has finite type). In view of Theorem 1.2(ii), it suffices to prove the following.

Lemma 4.3 *For each $k > 0$, the space Y_k is a $k(n)$-tower with v_n-free homotopy.*

We first destabilize the Johnson–Wilson construction [JW75, §4] of a filtration of the spectrum $P(n)$ whose subquotients are suspensions of $k(n)$, and adapt it for $P(n, m)$. The result will be a tower

$$\cdots \to W_3 \to W_2 \to W_1 \to W_0 = \underline{P(n,m)}_k \tag{4.1}$$

with trivial homotopy limit, where each W_i is the homotopy fibre of a map $W_{i-1} \to \underline{k(n)}_{q(i)}$ that is epic on homotopy groups. This depends on the following lemma, where we recall that $\pi_*(\underline{P(n,m)}_k) \cong \Sigma^k P(n,m)_*$ etc.

Lemma 4.4 *Given $v \in P(n,m)_h$ and $k \le g(n,m)$, where $v \neq 0$, there exist an integer c and stable $P(n)$-operation r such that the composite*

$$s\colon \underline{P(n,m)}_k \xrightarrow{\overline{\theta(m)}} \underline{P(n)}_k \xrightarrow{r} \underline{P(n)}_{k+h-2cN} \xrightarrow{\rho(n)} \underline{k(n)}_{k+h-2cN}$$

induces $s_*\Sigma^k v = \Sigma^{k+h-2cN} v_n^c$ on homotopy groups.

Proof Lemma 1.12 of [JW75], viewed unstably, supplies c and r. ∎

We construct the tower (4.1) by induction, starting from $W_0 = \underline{P(n,m)}_k$. Given $j\colon W_{i-1} \to \underline{P(n,m)}_k$, where W_{i-1} is $(k+h-1)$-connected and $j_*\colon \pi_*(W_{i-1}) \to \pi_*(\underline{P(n,m)}_k)$ is monic, we choose a bottom nonzero element $u \in \pi_{k+h}(W_{i-1})$ to kill. Lemma 4.4 provides a map $s\colon \underline{P(n,m)}_k \to \underline{k(n)}_{k+h-2cN}$ such that $s_* j_* u = \Sigma^{k+h-2cN} v_n^c$. For dimensional reasons, $s \circ j$ factors through

$$v_n^c\colon \underline{k(n)}_{k+h} \to \underline{k(n)}_{k+h-2cN}$$

to produce the desired map $W_{i-1} \to \underline{k(n)}_{k+h}$, with fibre W_i.

This is the wrong kind of tower for Definition 1.3. To correct it, we could take the homotopy fibre X_i of each map $W_i \to \underline{P(n)}_k$, to express $\Omega\underline{P(n,m)}_k = \underline{P(n,m)}_{k-1}$ as a $k(n)$-tower with v_n-free homotopy. This approach *fails* to produce a suitable tower for $\underline{P(n,m)}_k$ when $k = g(n,m)$. Our solution is to observe that it is inefficient to deloop and then take fibres; instead, we prove only what we actually need.

Lemma 4.5 *Given a $(k+h)$-connected map $q\colon \underline{P(n,m)}_k \to X$ and any map*

$$s\colon \underline{P(n,m)}_k \to \underline{k(n)}_{k+h-2cN},$$

there exists a v_n-torsion map $z\colon X \to \underline{k(n)}_{k+h+1}$ such that s is one value of the following Toda bracket:

$$s \in \langle v_n^c, z, q\rangle\colon \underline{P(n,m)}_k \to \underline{k(n)}_{k+h-2cN}.$$

Proof We are using the adjoint (but equivalent) description of a Toda bracket in terms of loop spaces instead of suspensions. We build the commutative diagram Figure 1 in which the two rows are fibration sequences and $l = k+h$. We start

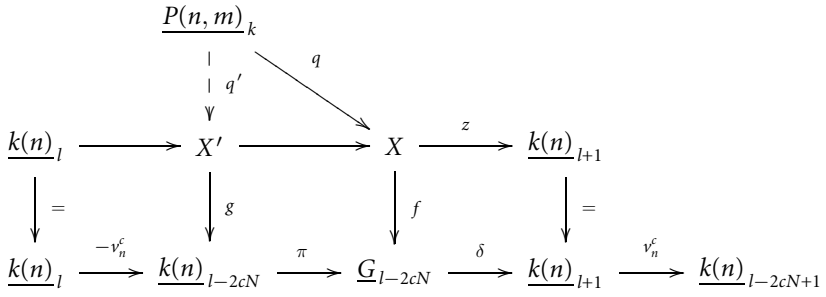

Figure 1: Construction of the Toda bracket $\langle v_n^c, z, q\rangle$

with the obvious fibration as the bottow row, where (stably) G denotes the cofibre of $v_n^c\colon k(n) \to k(n)$, with homotopy $\mathbb{F}_p[v_n]/(v_n^c)$. By the connectivity of q, $q^*\colon G^j(X) \to$

$G^j(\underline{P(n,m)}_k)$ is an isomorphism for $j \le k+h-2cN+2N-1$, so that $\pi \circ s$ factors uniquely through q to yield a map f such that $f \circ q = \pi \circ s$. We put $z = \delta \circ f$, which automatically satisfies $v_n^c z = 0$. We define X' as the homotopy fibre of z, and fill in g to form a morphism of fibrations.

Since X' may be constructed as a pullback, we can fill in q' to lift q and satisfy $g \circ q' = s$. (Equivalently, $v_n^c \circ [\underline{P(n,m)}_k, \underline{k(n)}_{k+h}]$ is part of the indeterminacy of the Toda bracket.) Then by definition, $g \circ q' = s$ is one value of the Toda bracket. ■

Proof of Lemma 4.3 We build the desired tower for $\underline{P(n,m)}_k$ by induction, starting from a point as X_0. Suppose we have constructed a map $q_{i-1}\colon \underline{P(n,m)}_k \to X_{i-1}$ that induces a surjection $q_{i-1*}\colon \Sigma^k P(n,m)_* \to \pi_*(X_{i-1})$ on homotopy groups, with kernel K an $\mathbb{F}_p[v_n]$-submodule. We choose a bottom nonzero element $\Sigma^k v \in K_{k+h}$ to kill, where $K_j = 0$ for $j < k+h$. Then Lemma 4.4 provides a map $s\colon \underline{P(n,m)}_k \to \underline{k(n)}_{k+h-2cN}$. We use Lemma 4.5 to build Figure 1, taking q_{i-i} as q and X_{i-1} as X.

We next take homotopy groups of Figure 1. By Lemma 3.4(i) applied to $r \circ \overline{\theta(m)} \circ \rho(m)\colon \underline{P(n)}_k \to \underline{P(n)}_{k+h-2cN}$, s_* is a homomorphism of $\mathbb{F}_p[v_n]$-modules. By exactness and the hypothesis that $q_*\Sigma^k(v_n^t v) = 0$, $q'_*(v_n^t v)$ must lift to $-\Sigma^{k+h}v_n^t \in \pi_*(\underline{k(n)}_{k+h})$. It now follows that q'_* is also epic, with kernel

$$K' = \mathrm{Ker}\big[s_*|K\colon K \to \Sigma^{k+h-2cN}\mathbb{F}_p[v_n]\big] \subset K,$$

a strictly smaller $\mathbb{F}_p[v_n]$-submodule of $\Sigma^k P(n,m)_*$. We take X' as X_i and q' as q_i.

The kernels K become more and more highly connected as i increases, hence $\underline{P(n,m)}_k$ is the homotopy limit of the spaces X_i. ■

5 Splittings of $P(n)$-Cohomology

In this section, we translate the H-space splittings in Section 3 into splittings of $P(n)$-cohomology. We also deduce Theorems 1.12 and 1.13.

We have yet to prove Lemmas 3.1, 3.2 and 3.4. Lemma 3.1 is equivalent to the following statement for the represented functors. (We do not mention Lemmas 3.2 and 3.4 again until Section 12.)

Lemma 5.1 *Assume that $k \le g(n,m)$, where $m \ge n$. Then there is a splitting*

$$\overline{\theta(m)}\colon P(n,m)^k(X) \to P(n)^k(X)$$

of $\rho(m)\colon P(n)^k(X) \to P(n,m)^k(X)$ that satisfies $\rho(m) \circ \overline{\theta(m)} = \mathrm{id}$ and is natural for spaces X. It is additive **[***except when $p = 2$ and $k = g(n,m)$***]**.

This we actually prove in Section 9 **[**except the nonadditive case; see [Bo]**]**, by constructing an idempotent cohomology operation $\theta(m)$ in $P(n)^k(X)$. Unlike the case of BP, the use of nonadditive operations yields no further splittings **[**unless $p = 2$**]**.

We next translate equation (3.3).

Corollary 5.2 *For $k \leq g(n, m-1)$, where $m > n$, we have the natural short exact sequence of abelian groups*

$$0 \to P(n,m)^{k+2(p^m-1)}(X) \xrightarrow{v_m} P(n,m)^k(X) \xrightarrow{\rho(m-1,m)} P(n,m-1)^k(X) \to 0.$$

This splits naturally **[***unless $p = 2$ and $k = g(n, m-1)$* **]**.

This implies our homological dimension bound, by the methods of [JW73].

Proof of Theorem 1.12 Following Yosimura [Yo76, Theorem 4.8], we need to show that

$$\rho(m-1,m)\colon P(n,m)_i(X) \to P(n,m-1)_i(X) \tag{5.1}$$

is epic for all i. For $i \leq 2(p^m-1)$, this is trivial, by the exact sequence

$$P(n,m)_i(X) \xrightarrow{\rho(m-1,m)} P(n,m-1)_i(X) \xrightarrow{\delta} P(n,m)_{i-2(p^m-1)-1}(X)$$

arising from the exact triangle (3.1).

For $i > 2(p^m-1)$, we embed X in $\mathbb{R}^{2q+1}$, where q is the dimension of X, and take a regular neighborhood V of X. By Poincaré duality, (5.1) is equivalent to

$$\rho(m-1,m)\colon P(n,m)^{2q+1-i}(V,\partial V) \to P(n,m-1)^{2q+1-i}(V,\partial V).$$

This is epic by Corollary 5.2, because by hypothesis

$$2q+1-i \leq (g(n,m)-2)+1-(2(p^m-1)+1) = g(n,m-1). \qquad \blacksquare$$

We also translate Theorem 1.6, using the splittings made explicit in Section 3, and finally deduce Theorem 1.13. (Decompositions like (5.3) also follow directly from Corollary 5.2 by induction on h, though the resulting homomorphisms are different.)

Theorem 5.3 *Let X be any space and suppose that $m \geq n > 0$.*

(i) *If $k \leq g(n,m)$* **[***replaced by $k < g(n,m)$ if $p = 2$***]**, *we have the natural abelian group decomposition*

$$P(n)^k(X) \cong P(n,m)^k(X) \oplus \prod_{j>m} P(n,j)^{k+2(p^j-1)}(X), \tag{5.2}$$

where the first factor on the right is injected by $\overline{\theta(m)}$, and the others by

$$P(n,j)^{k+2(p^j-1)}(X) \xrightarrow{\overline{\theta(j)}} P(n)^{k+2(p^j-1)}(X) \xrightarrow{v_j} P(n)^k(X).$$

Hence, by composition with $\rho(h)\colon P(n)^k(X) \to P(n,h)^k(X)$ for any $h > m$,

$$P(n,h)^k(X) \cong P(n,m)^k(X) \oplus \bigoplus_{j=m+1}^{h} P(n,j)^{k+2(p^j-1)}(X). \tag{5.3}$$

These decompositions are maximal if $k > g(n,m-1)$ **[***also for $k = g(n,m-1)$ if $p = 2$***]**. *(They are in no sense decompositions as $P(n)_*$-modules.)*

(ii) *If $p = 2$ and $k = g(n, m)$, we replace* (5.2) *and* (5.3) *by the natural short exact sequences*

$$0 \to \prod_{j>m} P(n, j)^{k+2^{j+1}-2}(X) \to P(n)^k(X) \xrightarrow{\rho(m)} P(n, m)^k(X) \to 0,$$

$$0 \to \bigoplus_{j=m+1}^{h} P(n, j)^{k+2^{j+1}-2}(X) \to P(n, h)^k(X) \xrightarrow{\rho(m,h)} P(n, m)^k(X) \to 0.$$

Because $P(n, n) = k(n)$ is so familiar, we break out the special case $m = n$. For $h < 2(p^n - 1)$, we can even replace $k(n)$ by the periodic Morava K-theory $K(n)$.

Corollary 5.4 *For $h \leq 2p^n$, where $n > 0$, we have, for all spaces X, the natural abelian group decomposition*

$$P(n)^h(X) \cong k(n)^h(X) \oplus \prod_{j>n} P(n, j)^{h+2(p^j-1)}(X),$$

except that if $p = 2$ and $h = 2^{n+1}$, we have only the natural short exact sequence

$$0 \to \prod_{j>n} P(n, j)^{h+2^{j+1}-2}(X) \to P(n)^h(X) \xrightarrow{\rho(n)} k(n)^h(X) \to 0.$$

Remark All the splittings exhibited above depend on the choice of $\theta(m)$, which is not canonical and does not respect multiplication by v_j.

Proof of Theorem 1.13 As X is finite-dimensional, the sum in (1.9) is essentially finite. Lemma 5.1 shows that $\overline{\rho(m)}$ is epic. It is clear from Theorem 5.3 that Ker $\overline{\rho(m)}$ is contained in the sum, and must therefore be the sum. ■

6 Stable Operations in $P(n)$-Cohomology

In this section, we describe the stable operations in $P(n)$-cohomology $P(n)^*(-)$ in the style of [Bo95]. The results are old and well known **[**except for $p = 2$**]**, but we include them for completeness and ease of reference; more importantly, they serve as a pattern for Sections 7 and 10.

6.1 Monoidal Structure

(For the language of monoidal categories and functors, see [Ma71, Ch. VII].) Since $P(n)_*$ is a commutative ring **[**even if $p = 2$**]**, the graded category $(\mathsf{FMod}_*, \widehat{\otimes}, P(n)_*)$ of complete Hausdorff filtered $P(n)_*$-modules is a symmetric monoidal category, with all (completed) tensor products taken over $P(n)_*$. The cross product makes $P(n)$-cohomology a monoidal functor,

$$P(n)^*(-)\colon (\mathrm{Ho}^{\mathrm{op}}, \times, \mathrm{point}) \to (\mathsf{FMod}, \widehat{\otimes}, P(n)_*).$$

(Conveniently, $P(n)^*(X)$ has no phantom classes and so is already complete Hausdorff.) For homology, we similarly have the monoidal functor

$$P(n)_*(-)\colon (\mathsf{Ho}, \times, \text{point}) \to (\mathsf{Mod}, \otimes, P(n)_*),$$

with values in the category Mod of *discrete* $P(n)_*$-modules. Both functors are symmetric for $p \neq 2$.

The cohomology version for spectra and graded maps is

$$P(n)^*(-, o)\colon (\mathsf{Stab}_*^{\mathrm{op}}, \wedge, S^0) \to (\mathsf{FMod}_*, \widehat{\otimes}, P(n)_*),$$

and similarly for homology. (We include the basepoint subspectrum o in our notation as a reminder that all stable (co)homology is reduced, and to distinguish it from the (co)homology of a space, which here will generally be absolute.)

6.2 Operations

Because $\Gamma = P(n)_*(P(n), o)$ is a free $P(n)_*$-module, we may identify its dual $P(n)_*$-module $D\Gamma$ with $\mathcal{A} = P(n)^*(P(n), o)$, the algebra of all stable operations in $P(n)$-cohomology, and have available all the stable machinery and results of [Bo95]. In particular, we have the monoidal functor

$$S\colon (\mathsf{FMod}_*, \widehat{\otimes}, P(n)_*) \to (\mathsf{FMod}_*, \widehat{\otimes}, P(n)_*)$$

defined by $SM = \mathsf{FMod}_*(\mathcal{A}, M)$. If M is filtered by submodules F^aM, we filter SM by the submodules $F^aSM = SF^aM$; as in [Bo95], SM is again complete Hausdorff. The ring spectrum structure of $P(n)$ gives S its monoidal structure (see diagram (6.4) below), which is symmetric for $p \neq 2$. (As in [Bo95], care is needed in keeping track of the many $P(n)_*$-module actions, some of which are not obvious.)

The action of stable $P(n)$-cohomology operations is visibly encoded in the monoidal natural transformation

$$\rho_X\colon P(n)^*(X) \to S(P(n)^*(X)) = \mathsf{FMod}_*(P(n)^*(P(n), o), P(n)^*(X)) \tag{6.1}$$

defined by $\rho_X x = x^*$, where we treat $x \in P(n)^*(X)$ as a map of spectra $x\colon X_+ \to P(n)$ and X_+ denotes the disjoint union of X and a (new) basepoint.

6.3 The Coaction

To convert the action of $\mathcal{A}$ into a coaction by Γ, we recall the natural isomorphism [Bo95, (11.4)]

$$\theta M\colon S'M = M \mathbin{\widehat{\otimes}} \Gamma \cong \mathsf{FMod}_*(D\Gamma, M) \cong \mathsf{FMod}_*(\mathcal{A}, M) = SM,$$

given on $x \in M$, $c \in \Gamma$, and $r \in \mathcal{A} \cong D\Gamma$ by

$$((\theta M)(x \otimes c))r = \pm\langle r, c\rangle x, \tag{6.2}$$

with the expected sign. We use it to transfer all the structure from the functor S to S' and replace (6.1) by the equivalent natural transformation

$$\rho_X\colon P(n)^*(X) \to S'P(n)^*(X) = P(n)^*(X) \mathbin{\widehat{\otimes}} \Gamma. \tag{6.3}$$

6.4 The Monoid

The resulting monoidal structure on S' is necessarily induced by a monoid structure on the $P(n)_*$-module Γ (as we see by naturality from the case $M = N = P(n)_*$ in diagram (6.4), below), and conversely. We simply need to compute it.

Lemma 6.1 *The following monoid structure on* Γ*, which is inherited from the monoidal functor S, makes the natural transformation* (6.3) *monoidal:*

(i) *If p is odd, the multiplication on* Γ *is the obvious one,*

$$\Gamma \otimes \Gamma = P(n)_*(P(n), o) \otimes P(n)_*(P(n), o) \xrightarrow{\times} P(n)_*(P(n) \wedge P(n), o) \xrightarrow{\phi_*} P(n)_*(P(n), o) = \Gamma,$$

as inferred by writing $\Gamma = P(n)_*(P(n), o)$*. The unit homomorphism of* Γ *is*

$$P(n)_* = P(n)_*(S^0, o) \xrightarrow{\eta_*} P(n)_*(P(n), o) = \Gamma.$$

(ii) *If* $p = 2$*, the multiplication is instead*

$$\Gamma \otimes \Gamma = P(n)_*(P(n), o) \otimes P(n)_*(P(n), o) \xrightarrow{\overline{\times}} P(n)_*(P(n) \wedge P(n), o) \xrightarrow{\phi_*} P(n)_*(P(n), o) = \Gamma,$$

which is better suggested by writing $\Gamma = \overline{P(n)}_*(P(n), o)$*. The unit is unaffected.*

Proof The multiplication ϕ on $P(n)$ induces

$$\phi^* \colon D\Gamma \cong P(n)^*(P(n), o) \to P(n)^*(P(n) \wedge P(n), o) \cong D\Gamma \mathbin{\widehat{\otimes}} D\Gamma,$$

with the help of the Künneth formula [Bo95, Theorem 4.19]. The natural transformations $\zeta(M, N)$ for S' and S form the left and right sides of the commutative diagram

(6.4)
$$\begin{array}{ccc}
(M \mathbin{\widehat{\otimes}} \Gamma) \mathbin{\widehat{\otimes}} (N \mathbin{\widehat{\otimes}} \Gamma) & \xrightarrow{\theta M \otimes \theta N} & \mathsf{FMod}_*(D\Gamma, M) \mathbin{\widehat{\otimes}} \mathsf{FMod}_*(D\Gamma, N) \\
\downarrow \cong & & \downarrow \cong \\
M \mathbin{\widehat{\otimes}} N \mathbin{\widehat{\otimes}} (\Gamma \otimes \Gamma) & & \mathsf{FMod}_*(D\Gamma \mathbin{\widehat{\otimes}} D\Gamma, M \mathbin{\widehat{\otimes}} N) \\
\downarrow {\scriptstyle M \otimes N \otimes \phi} & & \downarrow {\scriptstyle \mathsf{FMod}_*(\phi^*, \mathrm{id})} \\
M \mathbin{\widehat{\otimes}} N \mathbin{\widehat{\otimes}} \Gamma & \xrightarrow{\theta(M \mathbin{\widehat{\otimes}} N)} & \mathsf{FMod}_*(D\Gamma, M \mathbin{\widehat{\otimes}} N)
\end{array}$$

which features the multiplication $\phi\colon \Gamma \otimes \Gamma \to \Gamma$. We evaluate on $x \otimes c \otimes y \otimes d$, where $x \in M$, $y \in N$, and $c, d \in \Gamma$. By (6.2), the lower route gives the element $r \mapsto \pm\langle r, cd\rangle x \otimes y$ of $\mathsf{FMod}_*(D\Gamma, M \mathbin{\widehat{\otimes}} N)$. The upper route gives

$$r \otimes s \mapsto \pm\langle r, c\rangle\langle s, d\rangle x \otimes y$$

in $\mathsf{FMod}_*(D\Gamma \mathbin{\widehat{\otimes}} D\Gamma, M \mathbin{\widehat{\otimes}} N)$. *Assuming* $p \neq 2$, we can rewrite this as $\pm\langle r \times s, c \times d\rangle x \otimes y$; then in $\mathsf{FMod}_*(D\Gamma, M \mathbin{\widehat{\otimes}} N)$ we find

$$r \mapsto \pm\langle \phi^* r, c \times d\rangle x \otimes y = \pm\langle r, \phi_*(c \times d)\rangle x \otimes y.$$

Thus $cd = \phi_*(c \times d)$ (with no sign) as expected, which is (i).

[If $p = 2$, this calculation is *false*; we must use (2.7) instead, which states that $\langle r, c\rangle\langle s, d\rangle = \langle r \times s, c \mathbin{\overline{\times}} d\rangle$. Then $cd = \phi_*(c \mathbin{\overline{\times}} d)$, for (ii).**]**

The unit $z\colon P(n)_* \to SP(n)_*$ takes $1 = \eta \in P(n)_*$ to the homomorphism

$$D\Gamma \cong P(n)^*(P(n), o) \xrightarrow{\eta^*} P(n)^*(S^0, o) \cong P(n)_*,$$

in other words, $r \mapsto \langle \eta^* r, 1\rangle = \langle r, \eta_* 1\rangle$. Comparison with (6.2) shows that the corresponding element of $S'P(n)_* = P(n)_* \otimes \Gamma \cong \Gamma$ is $\eta_* 1$. ■

If X is a point in (6.3), we find the *right unit* ring homomorphism

$$\eta_R\colon P(n)_* \to S'P(n)_* = P(n)_* \otimes \Gamma \cong \Gamma, \tag{6.5}$$

which is used to make Γ a right $P(n)_*$-module (hence a bimodule). Since ρ is monoidal, this action makes (6.3) a homomorphism of $P(n)_*$-modules.

6.5 The Hopf Algebroid

Now we add the algebra structure of $\mathcal{A}$. Exactly as in [Bo95, §10], composition of operations and the identity operation induce natural transformations $\psi\colon S \to SS$ and $\epsilon\colon S \to I$. These make S a monoidal comonad in the category FMod, and (6.1) makes $P(n)^*(X)$ an S-coalgebra.

We transfer this structure too to S'. The resulting monoidal comonad structure on S' is necessarily induced by a Hopf algebroid structure on Γ (as we see by taking $M = P(n)_*$), and conversely. This structure consists of a coassociative comultiplication $\psi_S\colon \Gamma \to \Gamma \otimes \Gamma$ with counit $\epsilon_S\colon \Gamma \to P(n)_*$. These behave exactly as in Adams [Ad74] or [Bo95, Theorem 11.35]; in particular, ψ_S and ϵ_S are homomorphisms of $P(n)_*$-bimodules and algebras. **[**This all works without change for $p = 2$; see [Bo].**]**

Proposition 6.2 *The stable operations in $P(n)$-cohomology are encoded in the Hopf algebroid $\Gamma = P(n)_*(P(n), o)$* **[***replaced by $\Gamma = \overline{P(n)}_*(P(n), o)$ for $p = 2$***]**.

The discussion of the structure of Γ carries over from the case $K(n)$ in [Bo95] with little change **[**except that we allow $p = 2$**]**. We even use the same test spaces.

The One-Point Space We already discussed this in (6.5). The coaction ρ reduces to the ring homomorphism η_R, which is determined by the elements

$$w_k = \eta_R v_k \in \Gamma_{2(p^k-1)} \quad \text{for } k \geq n.$$

6.6 Complex Orientation

Our next test space is complex projective space $\mathbb{C}P^\infty$. As $P(n)$ inherits a complex orientation from BP (or MU), we have $P(n)^*(\mathbb{C}P^\infty) = P(n)_*[[x]]$, the formal power series ring generated by the Chern class $x = x(\xi)$ of the Hopf line bundle ξ over $\mathbb{C}P^\infty$, filtered by powers of the ideal (x).

The coaction ρ for $\mathbb{C}P^\infty$ defines elements $b_j \in \Gamma_{2j-2}$ by the formula [Bo95, (13.2)]

$$\rho x = b(x) = \sum_{j=1}^{\infty} x^j \otimes b_j \quad \text{in } P(n)^*(\mathbb{C}P^\infty) \mathbin{\widehat{\otimes}} \Gamma \cong \Gamma[[x]]. \tag{6.6}$$

Here, $b(x)$ is a useful formal abbreviation for the right side. As always in the stable context [Bo95, Proposition 13.4], $b_1 = 1$ and $b_0 = 0$.

Further, the comultiplication ψ_S is given on b_i as the coefficient of x^i in

$$\psi_S b(x) = \sum_{j=1}^{\infty} b(x)^j \otimes b_j \quad \text{in } (\Gamma \otimes \Gamma)[[x]], \tag{6.7}$$

and $\epsilon_S b_j = 0$ for all $j > 1$.

Since $P(n)$ is p-local, we need only the accelerated elements $b_{(j)} = b_{p^j} \in \Gamma_{2(p^j-1)}$ for $j \geq 0$, where $b_{(0)} = 1$; the other b's are expressible in terms of these and the v's and w's by [Bo95, Lemma 13.7].

The p-th power map $\zeta\colon \mathbb{C}P^\infty \to \mathbb{C}P^\infty$, whose bundle interpretation is $\zeta^*\xi = \xi^{\otimes p}$, induces in cohomology

$$\zeta^* x = [p](x) = \sum_{i=N}^{\infty} g_i x^{i+1} \quad \text{in } P(n)^*(\mathbb{C}P^\infty) = P(n)_*[[x]]$$

for certain coefficients $g_i \in P(n)_{2i}$. This formal power series is known as the *p-series* for $P(n)$. There are no lower terms as $g_0 = p = 0$ in $P(n)_0$. (The elements g_i are traditionally written a_i, but we rename them in order to avoid confusion with other elements, also named a_i, that appear shortly.)

We need only one standard fact [RW77, Theorem 3.11(b)] about the p-series:

$$[p](x) \equiv v_k x^{p^k} \qquad \mod (v_n, \dots, \widehat{v_k}, \dots) \tag{6.8}$$

for any $k \geq n$, where the ideal is generated by all the v's except v_k. In words, $[p](x)$ contains terms $v_k x^{p^k}$ but not $\lambda v_k^i x^q$ for any $i > 1$. In particular,

$$[p](x) = v_n x^{p^n} + v_{n+1} x^{p^{n+1}} + \text{higher terms}. \tag{6.9}$$

Hence as k varies, we have

$$[p](x) \equiv \sum_{k=n}^{\infty} v_k x^{p^k} \mod V^2, \tag{6.10}$$

where V denotes the maximal ideal $(v_n, v_{n+1}, v_{n+2}, \dots) \subset P(n)_*$.

Naturality of ρ with respect to the map ζ yields the identity [Bo95, (13.11)]

$$b\big([p](x)\big) = [p]_R\big(b(x)\big) = \sum_{i=N}^{\infty} b(x)^{i+1} \eta_R g_i \quad \text{in } \Gamma[[x]]. \tag{6.11}$$

The lowest power of x that occurs is x^{p^n}.

Definition 6.3 For each $k \geq n$, we define the *k-th main stable relation* $(\mathcal{R}_k)$ as the coefficient of x^{p^k} in (6.11).

Since $b_{(0)} = 1$, the first relation $(\mathcal{R}_n)$ is simply $v_n 1 = w_n$, which implies that every stable operation is v_n-linear. For $k > n$, (6.10) shows that (6.11) has a term $w_k x^{p^k}$ on the right, and $(\mathcal{R}_k)$ becomes an inductive formula for w_k in terms of the v's and b's and lower w's.

6.7 Cohomology of a Lens Space, for p Odd

Our final test space is the $2N$-skeleton L of the lens space $K(\mathbb{Z}/p, 1)$. Geometrically, L is the orbit space of the standard $\mathbb{Z}/p$ -action on the unit sphere $S^{2N+1} \subset \mathbb{C}^{N+1}$ given as complex multiplication by $\mathbb{Z}/p \subset S^1 \subset \mathbb{C}$, with the top cell omitted by requiring the last coordinate to be real non-negative, up to the action of $\mathbb{Z}/p$. (Retaining the top cell, as in [Bo95], adds some extra complication but offers little benefit.)

Following [Bo95, §14], its cohomology is

$$P(n)^*(L) = \big(E(u) \otimes TP_n(x)\big) \big/ (ux^N), \tag{6.12}$$

because the Atiyah–Hirzebruch spectral sequence can support no differential. Here, x is induced from the Chern class of the Hopf line bundle on $\mathbb{C}P^N$, which is a quotient space of L, and u is uniquely defined as restricting to the standard generator $u_1 \in P(n)^*(S^1)$, where we recognize the 1-skeleton L^1 of L as the circle S^1.

Since x is a Chern class, the coaction ρ_L is given on x by naturality as $\rho_L x = b(x)$. Although L is not an H-space, there are, as in [Bo95, (14.31)], partial multiplications $L^{2k} \times L^{2m} \to L$ on the skeletons whenever $k + m = N$, which imply that

$$\rho_L u = u \otimes 1 + \sum_{i=0}^{n-1} x^{p^i} \otimes a_{(i)} \quad \text{in } P(n)^*(L) \otimes \Gamma \tag{6.13}$$

for certain elements $a_{(i)} \in \Gamma_{2p^i-1}$ that this equation defines. (We warn that these generators differ from Würgler's [Wü77] and Yagita's [Ya77] generators a_i by the

conjugation in Γ; as a result, certain formulae become transposed. Our generators are chosen for compatibility with [Bo95, Wi84], because they destabilize properly in Sections 7 and 10.) The element $a_{(n)}$ does not exist because u fails to lift to the $2p^n$-skeleton of the lens space. As in [Bo95, Theorem 14.32], the coalgebra structure is given by

$$\psi_S a_{(k)} = a_{(k)} \otimes 1 + \sum_{i=0}^{k-1} b_{(k-i)}^{p^i} \otimes a_{(i)} + 1 \otimes a_{(k)} \tag{6.14}$$

and $\epsilon_S a_{(k)} = 0$.

6.8 Cohomology of Real Projective Space, for $p = 2$

Here, the same test space L is better known as real projective space $\mathbb{R}P^{2N}$. It remains true that the Atiyah–Hirzebruch spectral sequence can support no differential, so that

$$P(n)^*(\mathbb{R}P^{2N}) = P(n)_*[t]/(t^{2N+1}), \tag{6.15}$$

generated by the unique nonzero element $t \in P(n)^1(\mathbb{R}P^{2N})$. As above, we find that

$$\rho t = t \otimes 1 + \sum_{i=0}^{n-1} t^{2^{i+1}} \otimes a_{(i)}, \tag{6.16}$$

which defines elements $a_{(i)} \in \Gamma_{2^{i+1}-1}$. Indeed, this formula is identical to (6.13), since $x = t^2$ is the Chern class of the complexified real Hopf line bundle. Thus (6.14) remains valid for $p = 2$.

6.9 Summary

Würgler [Wü77] and Yagita [Ya77] both proved that we now have enough elements of Γ to handle all stable operations.

Theorem 6.4 *The stable operations in $P(n)$-cohomology are dual to the Hopf algebroid $\Gamma = P(n)_*(P(n), o)$* **[***replaced by $\overline{P(n)}_*(P(n), o)$ if $p = 2$***]**, *which is generated as a $P(n)_*$-algebra by the elements $b_{(j)}$ and $a_{(i)}$ defined by* (6.6) *and* (6.13) **[***replaced by* (6.16) *if $p = 2$***]**.

(i) *For odd p, as a $P(n)_*$-algebra,*

$$\Gamma = P(n)_*(P(n), o) = E(a_{(0)}, a_{(1)}, \dots, a_{(n-1)}) \otimes P(b_{(1)}, b_{(2)}, b_{(3)}, \dots).$$

(ii) *For $p = 2$, as a $P(n)_*$-algebra,*

$$\Gamma = \overline{P(n)}_*(P(n), o) = P(a_{(0)}, a_{(1)}, \dots, a_{(n-1)}, b_{(n+1)}, b_{(n+2)}, \dots),$$

and the elements $b_{(j)}$ for $j \le n$ are given by the relations

$$a_{(i)}^2 = b_{(i+1)} \quad \text{for } 0 \le i \le n-1. \tag{6.17}$$

(iii) *As a left $P(n)_*$-module, Γ is free with a basis consisting of all monomials*

$$a^I b^J = a_{(0)}^{i_0} a_{(1)}^{i_1} \cdots a_{(n-1)}^{i_{n-1}} b_{(1)}^{j_1} b_{(2)}^{j_2} b_{(3)}^{j_3} \cdots ,$$

with multi-indices $I = (i_0, i_1, \dots, i_{n-1})$ and $J = (j_1, j_2, \dots)$ in which each $i_r = 0$ or 1.

(iv) *The right $P(n)_*$-action on Γ is given by multiplication by the elements $w_k = \eta_R v_k$, where $w_n = v_n 1$ and w_k is determined inductively for $k > n$ by the main relation $(\mathcal{R}_k)$ (see Definition 6.3).*

(v) *The comultiplication $\psi_S\colon \Gamma \to \Gamma \otimes \Gamma$ is the $P(n)_*$-algebra homomorphism given on the generators by (6.7) and (6.14).*

(vi) *The counit $\epsilon_S\colon \Gamma \to P(n)_*$ is the $P(n)_*$-algebra homomorphism given on the generators by $\epsilon_S a_{(i)} = 0$ for all i and $\epsilon_S b_{(j)} = 0$ for $j > 0$.*

Proof What survives intact from Würgler [Wü77, Theorem 2.13] and Yagita [Ya77, Lemma 3.5], even for $p = 2$, is (iii) (using the conjugate generators to the $a_{(i)}$). Parts (i), (iv), (v) and (vi) need no further comment.

[In (ii), commutativity is *not* trivial; see Nassau [Na02, Bo]. Since t^2 is a Chern class, $(\rho t)^2 = \rho(t^2) = b(t^2)$. Comparing the coefficients of $t^{2^{i+2}}$ with the help of (6.6) and (6.16), we deduce (6.17) for $i < n-1$.

This argument *fails* for $i = n-1$, as $t^{2^{n+1}} = 0$; nevertheless, the result still holds by [Na02, Theorem 2], which corrects [KW87]. Alternatively, the map of ring spectra $P(n) \to P(n+1)$ in Lemma 2.3 sends each generator of $\Gamma(n) = \Gamma$ to its namesake in $\Gamma(n+1)$. As $a_{(n-1)}^2 = b_{(n)}$ in $\Gamma(n+1)$, the only candidates for $a_{(n-1)}^2$ in $\Gamma(n)$ are $b_{(n)}$ and $b_{(n)} + v_n 1$. Since $\epsilon_S(a_{(n-1)}^2) = (\epsilon_S a_{(n-1)})^2 = 0$, we must choose $b_{(n)}$.**]** ∎

7 Additive Operations in $P(n)$-Cohomology

In this section, we describe the additive unstable operations in $P(n)$-cohomology in the style of [BJW95], in terms of a certain bigraded algebra Q_*^*, which, like Γ, is a $P(n)_*$-bimodule equipped with a coalgebra structure (ψ_A, ϵ_A) (called $(Q(\psi), Q(\epsilon))$ in [BJW95]) that encodes the composition of operations and the identity operation. Although the results bear a strong formal resemblance to the stable results in Section 6, the stable proofs do not carry over; instead, one has to compute the whole Hopf ring in Section 11 and then take the indecomposables.

For p odd, we define $Q_*^* = QP(n)_*(\underline{P(n)}_*)$, the algebra of indecomposables in the Hopf ring $P(n)_*(\underline{P(n)}_*)$. Specifically, Q_i^k denotes the group of indecomposables in degree i of the Hopf algebra $P(n)_*(\underline{P(n)}_k)$; its elements have *total* degree $i-k$ in Q_*^* (and this is the degree that governs signs). The multiplication and unit in Q_*^* are induced from $\circ$-multiplication and the element $[1]$ in the Hopf ring by the homomorphisms (10.1). The left $P(n)_*$-module action is induced from the Hopf ring: if $v \in P(n)_j$ and $c \in Q_i^k$, we have $vc \in Q_{j+i}^k$. **[**When $p = 2$, it should be no surprise after Proposition 6.2 that the correct Hopf ring to consider is not $P(n)_*(\underline{P(n)}_*)$ but $\overline{P(n)}_*(\underline{P(n)}_*)$; in this case, we set $Q_*^* = Q\overline{P(n)}_*(\underline{P(n)}_*)$. This is the same left $P(n)_*$-module as $QP(n)_*(\underline{P(n)}_*)$, but with slightly different multiplication.**]**

By [RW96, Corollary 1.5], both Q_*^* and the Hopf ring are free $P(n)_*$-modules. These conditions ensure [BJW95, Lemma 4.16(a)] that the dual module to Q_*^* is indeed the module of all additive unstable operations on $P(n)$-cohomology, and make available all the machinery and results on additive operations. We thus identify

(i) The *additive unstable operation* $r\colon P(n)^k(-) \to P(n)^m(-)$.
(ii) The *primitive cohomology class* $r\iota_k \in P(n)^m(\underline{P(n)}_k)$.
(iii) The representing *H-map of H-spaces* $r\colon \underline{P(n)}_k \to \underline{P(n)}_m$, up to homotopy.
(iv) The $P(n)_*$*-linear functional* $\langle r, -\rangle\colon Q_*^k \to P(n)_*$, of degree $k-m$.

The action of additive operations on $P(n)^*(X)$ is encoded in coactions

$$\rho_X\colon P(n)^k(X) \to P(n)^*(X) \mathbin{\widehat{\otimes}} Q_*^k \tag{7.1}$$

(one for each k), which are monoidal as k varies **[**even if $p = 2$**]**.

To construct the generators of Q_*^*, we use the same test spaces as stably in Section 6, together with the circle. We record the values of ψ_A and ϵ_A on each generator.

7.1 Cohomology of a Point

The *right unit* ring homomorphism $\eta_R\colon P(n)_* \to Q_0^*$ is just the coaction ρ for the one-point space, and so is determined by the elements

$$w_k = \eta_R v_k \in Q_0^{-2(p^k-1)} \quad \text{for } k \geq n. \tag{7.2}$$

We use η_R to make Q_*^* a right $P(n)_*$-module and the coactions ρ_X in (7.1) into a $P(n)_*$-module homomorphism.

7.2 Cohomology of a Circle

The coaction for the circle S^1 defines the *suspension* element $e \in Q_1^1$ by

$$\rho u_1 = u_1 \otimes e \quad \text{in } P(n)^*(S^1) \otimes Q_*^1 = E(u_1) \otimes Q_*^1. \tag{7.3}$$

As in [BJW95, Proposition 12.3(d)], $\psi_A e = e \otimes e$ and $\epsilon_A e = 1$.

Then for any $j > 0$, the coaction for the j-sphere S^j is given by

$$\rho u_j = u_j \otimes e^j \quad \text{in } P(n)^*(S^j) \otimes Q_*^j = E(u_j) \otimes Q_*^j.$$

Given any additive operation $r\colon P(n)^k(-) \to P(n)^m(-)$, represented by the map $r\colon \underline{P(n)}_k \to \underline{P(n)}_m$, where $k, m > 0$, we use $P(n)^k(S^j) \cong \pi_j(\underline{P(n)}_k) \cong \Sigma^k P(n)_{j-k}$ to rewrite the induced homomorphism on homotopy groups as

$$r_*\colon \Sigma^k P(n)_* \cong \pi_*(\underline{P(n)}_k) \xrightarrow{\pi_*(r)} \pi_*(\underline{P(n)}_m) \cong \Sigma^m P(n)_*.$$

By [BJW95, Corollary 12.4], this is given on $\Sigma^k v$, where $v \in P(n)_i$, by the formula

$$r_*(\Sigma^k v) = \Sigma^m \langle r, e^{k+i}(\eta_R v)\rangle. \tag{7.4}$$

7.3 Complex Orientation

The coaction for $\mathbb{C}P^\infty$ defines elements $b_j \in Q^2_{2j}$ by

$$\rho x = b(x) = \sum_{j=1}^{\infty} x^j \otimes b_j \quad \text{in } P(n)^*(\mathbb{C}P^\infty) \mathbin{\widehat{\otimes}} Q^2_* \cong Q^2_*[[x]], \tag{7.5}$$

which is formally identical to (6.6), except that now $b_1 = e^2$ by [BJW95, Proposition 14.4(a)]. As in [BJW95], $\psi_A b_i$ is the coefficient of x^i in

$$\psi_A b(x) = \sum_{j=1}^{\infty} b(x)^j \otimes b_j \quad \text{in } (Q^*_* \otimes Q^2_*)[[x]],$$

and $\epsilon_A b_j = 0$ for $j > 1$.

Again [BJW95, Lemma 14.6], we need only the accelerated elements $b_{(j)} = b_{p^j}$ for $j \geq 0$, so $b_{(0)} = e^2$. The additive version of (6.11) also looks the same,

$$b([p](x)) = [p]_R(b(x)) = \sum_{i=N}^{\infty} b(x)^{i+1} \eta_R g_i \quad \text{in } Q^2_*[[x]]. \tag{7.6}$$

Definition 7.1 For each $k \geq n$, we define the *k-th main additive relation* $(\mathcal{R}_k)$ as the coefficient of x^{p^k} in (7.6).

In view of (6.9), the first two main relations are simply

$$(\mathcal{R}_n) \qquad b^{p^n}_{(0)} w_n = v_n b_{(0)} \quad \text{in } Q^2_* \tag{7.7}$$

and

$$(\mathcal{R}_{n+1}) \quad b^{p^n}_{(1)} w_n + b^{p^{n+1}}_{(0)} w_{n+1} = v_{n+1} b_{(0)} + v_n^p b_{(1)} \quad \text{in } Q^2_*. \tag{7.8}$$

We shall find in (10.6) that $(\mathcal{R}_n)$ desuspends once to

$$(\mathcal{R}'_n) \qquad e b^N_{(0)} w_n = v_n e \quad \text{in } Q^1_*. \tag{7.9}$$

By (6.10), the general main relation for $k \geq n$ has the form

$$(\mathcal{R}_k) \quad \sum_{i=n}^{k} b^{p^i}_{(k-i)} w_i \equiv 0 \quad \text{in } Q^2_* \quad \text{mod } \mathfrak{V} + \mathfrak{W}^2, \tag{7.10}$$

where $\mathfrak{V} = (v_n, v_{n+1}, v_{n+2}, \dots)$ and $\mathfrak{W} = (w_n, w_{n+1}, w_{n+2}, \dots)$ denote ideals in Q^*_*.

7.4 Cohomology of a Lens Space

Our last test space is the lens space skeleton L, whose cohomology is given by (6.12), assuming p is odd. We already know $\rho_L x = b(x)$ from (7.5). For u, we find, as in [BJW95, (16.21)], that

$$\rho_L u = u\otimes e + \sum_{i=0}^{n-1} x^{p^i} \otimes a_{(i)} \quad \text{in } P(n)^*(L) \otimes Q^1_* \tag{7.11}$$

for certain elements $a_{(i)} \in Q^1_{2p^i}$ that this equation defines. We deduce that

$$\psi_A a_{(k)} = a_{(k)} \otimes e + \sum_{i=0}^{k} b^{p^i}_{(k-i)} \otimes a_{(i)} \tag{7.12}$$

and $\epsilon_A a_{(k)} = 0$.

[If $p = 2$, $L = \mathbb{R}P^{2N}$ has different cohomology (6.15), and we replace (7.11) by

$$\rho t = t \otimes e + \sum_{i=0}^{n-1} t^{2^{i+1}} \otimes a_{(i)} \quad \text{in } P(t)/(t^{2N+1}) \otimes Q^1_*. \tag{7.13}$$

Nevertheless, (7.12) and $\epsilon_A a_{(k)} = 0$ remain valid for $p = 2$. By [Bo], (6.17) destabilizes in the obvious way, to

$$a^2_{(i)} = b_{(i+1)} \quad \text{for } 0 \le i \le n-1.\text{]} \tag{7.14}$$

7.5 More Relations

We shall find in (10.8) that one more suspension factor can be squeezed out of (7.9) if we first multiply by $a_{(0)}$, to give

$$(\mathcal{R}''_n) \qquad a_{(0)} b^N_{(0)} w_n = v_n a_{(0)} \quad \text{in } Q^1_*. \tag{7.15}$$

[When $p = 2$, we can multiply this by another $a_{(0)}$ and use (7.14) to obtain the unexpected formula

$$b^N_{(0)} b_{(1)} w_n = v_n b_{(1)}. \tag{7.16}$$

This is not all; if we multiply $(\mathcal{R}_{n+1})$ (7.8) by $b^N_{(0)}$, we obtain the reduction formula

$$b^{N+2^{n+1}}_{(0)} w_{n+1} = v_n^2 b^N_{(0)} b_{(1)} + v_n b^{2^n}_{(1)} + v_{n+1} b^{2^n}_{(0)}, \tag{7.17}$$

by using (7.16) to simplify one of the terms.**]**

7.6 Summary

We have the additive version of the Hopf algebroid Γ.

Theorem 7.2 *The additive unstable operations in $P(n)$-cohomology are dual to the $P(n)_*$-algebra $Q^*_* = QP(n)_*(\underline{P(n)}_*)$* **[***replaced by $Q\overline{P(n)}_*(\underline{P(n)}_*)$ if $p = 2$***]***, *which has the following properties:*

(i) *Q^*_* is the commutative bigraded $P(n)_*$-algebra generated by the elements*

- (a) *$w_k \in Q_0^{-2(p^k-1)}$ for $k \geq n$, defined by η_R in* (7.2)*;*
- (b) *$e \in Q^1_1$, the suspension element, defined by* (7.3)*;*
- (c) *$b_{(j)} \in Q^2_{2p^j}$ for $j \geq 0$, defined by* (7.5)*;*
- (d) *$a_{(i)} \in Q^1_{2p^i}$ for $0 \leq i < n$, defined by* (7.11) **[***replaced by* (7.13) *if $p = 2$***]***;*

subject to the relations $e^2 = b_{(0)}$, the main relations $(\mathcal{R}_k)$ for $k > n$ (see Definition 7.1*), and the two variants* (7.9) *and* (7.15) *of $(\mathcal{R}_n)$* **[***also* (7.14) *if $p = 2$***]***;*

(ii) *Q^*_* is a free left $P(n)_*$-module;*

(iii) *multiplication by the elements w_k makes Q^*_* a right $P(n)_*$-module;*

(iv) *the comultiplication $\psi_A\colon Q^*_* \to Q^*_* \otimes Q^*_*$ is the homomorphism of algebras and of $P(n)_*$-bimodules given on each generator as noted above;*

(v) *the counit $\epsilon_A\colon Q^*_* \to P(n)_*$ is the $P(n)_*$-algebra homomorphism given on generators by $\epsilon_A e = 1$, $\epsilon_A a_{(i)} = 0$, $\epsilon_A b_{(j)} = 0$ for $j > 0$, $\epsilon_A b_{(0)} = 1$, and $\epsilon_A w_k = v_k$.*

Parts (iii)–(v) need no further comment. Part (ii) is included in Theorem 8.2. Part (i) can be read off from Theorem 11.1. **[**For commutativity when $p = 2$, we refer to [Bo].**]**

We recall [BJW95, (6.3)] the stabilization homomorphism $Q(\sigma)\colon Q^k_* \to \Gamma$, which has degree zero. We may use it to recover the structure on Γ in Theorem 6.4 from Q^*_* simply by setting $e = 1$. The coalgebra structure (ψ_A, ϵ_A) stabilizes to (ψ_S, ϵ_S).

8 Relations for Additive Operations

We noted in Theorem 7.2 that Q^*_* is a free $P(n)_*$-module, which is not at all obvious from the generators and relations given. In this section, we exhibit a basis of Q^*_* and prove in Lemma 8.3 that it spans the module.

We also establish some direct applications of additive operations.

8.1 The Ravenel–Wilson Basis

Since $e^2 = b_{(0)}$ and $a^2_{(i)} = 0$ trivially if p is odd **[**replaced by $a^2_{(i)} = b_{(i+1)}$ if $p = 2$, from (7.14)**]**, any monomial in the listed generators of the $P(n)_*$-algebra Q^*_* can be written in the abbreviated form

$$e^\epsilon a^I b^J w^K = e^\epsilon a^{i_0}_{(0)} a^{i_1}_{(1)} \cdots a^{i_{n-1}}_{(n-1)} b^{j_0}_{(0)} b^{j_1}_{(1)} b^{j_2}_{(2)} \cdots w_n^{k_n} w_{n+1}^{k_{n+1}} w_{n+2}^{k_{n+2}} \cdots , \tag{8.1}$$

with multi-indices $I = (i_0, i_1, \dots, i_{n-1})$, $J = (j_0, j_1, \dots)$, and $K = (k_n, k_{n+1}, \dots)$, where each i_r, also ϵ, is 0 or 1. (We keep the w's to the right, as a reminder that they define the right action of $P(n)_*$ on Q_*^*.) We introduce the following parameters:

- The *b-length* is $\sum_r j_r$, the total number of factors of the form $b_{(j)}$;
- The *w-length* is $\sum_r k_r$, the total number of factors of the form w_k.

As with BP in [RW77], it is easier to specify which monomials are not wanted in forming the basis than those which are. **[**For $p = 2$, the basis is not written out in detail in [RW96], and contains some surprises.**]** There are *two* variants; we shall need the second in Sections 10 and 11.

Definition 8.1 We call the monomial (8.1) *Q-allowable* if it does *not* have any of the following forms **[**note that (iv) and (v) apply only if $p = 2$**]**:

(i) $b_{(d_n)}^{p^n} b_{(d_{n+1})}^{p^{n+1}} \cdots b_{(d_q)}^{p^q} w_q c$, with $0 \le d_n \le d_{n+1} \le \cdots \le d_q$, $q \ge n$;
(ii) $e b_{(0)}^N b_{(d_{n+1})}^{p^{n+1}} \cdots b_{(d_q)}^{p^q} w_q c$, with $0 \le d_{n+1} \le \cdots \le d_q$, $q \ge n$;
(iii) $a_{(0)} b_{(0)}^N b_{(d_{n+1})}^{p^{n+1}} \cdots b_{(d_q)}^{p^q} w_q c$, with $0 \le d_{n+1} \le \cdots \le d_q$, $q \ge n$;
(iv) $b_{(0)}^N b_{(1)} b_{(d_{n+1})}^{2^{n+1}} \cdots b_{(d_q)}^{2^q} w_q c$, where $p = 2$, with $0 \le d_{n+1} \le \cdots \le d_q$, $q \ge n$;
(v) $b_{(0)}^{N+2^{n+1}} b_{(d_{n+2})}^{2^{n+2}} \cdots b_{(d_q)}^{2^q} w_q c$, where $p = 2$, with $0 \le d_{n+2} \le \cdots \le d_q$, $q \ge n+1$;

where c is any monomial ($c = 1$ is permitted) in the generators e, $a_{(i)}$, $b_{(j)}$, and w_k.

More generally, we call the monomial *allowable* if it is not of the form (i) or (ii).

Remark In [RW96], a monomial is called *n-allowable* (lies in $\mathcal{A}_n$) if it is not of the form (i). If it contains a factor e or $a_{(0)}$, it is called *n-plus allowable* (lies in $\mathcal{A}_n^+$) if it is not of the form (i), (ii) or (iii).

From [RW96, Theorem 1.3], we have the *Ravenel–Wilson basis* of Q_*^*.

Theorem 8.2 (Ravenel–Wilson) *The Q-allowable monomials* (8.1) *form a basis of the free* $P(n)_*$*-module* $Q_*^* = QP(n)_*(\underline{P(n)}_*)$ **[***or* $Q\overline{P(n)}_*(\underline{P(n)}_*)$ *if* $p = 2$**]**.

Later in this section, we shall reprove half the theorem.

Lemma 8.3 *The relations* $e^2 = b_{(0)}$, *the main relations* $(\mathcal{R}_k)$ *for* $k > n$, **[***relation* (7.14) *if* $p = 2$,**]** *and the variants* (7.9) *and* (7.15) *of* $(\mathcal{R}_n)$ *imply that the Q-allowable monomials* (8.1) *span the* $P(n)_*$*-module* $Q_*^* = QP(n)_*(\underline{P(n)}_*)$ **[***or* $Q\overline{P(n)}_*(\underline{P(n)}_*)$ *if* $p = 2$**]**.

8.2 Generators of Cohomology

Just as in [BJW95, Theorem 20.2], Theorem 1.8 follows directly from the fact that the additive operations on $P(n)^{-k}(-)$ form the $P(n)_*$-dual of the free $P(n)_*$-module Q_*^{-k}, whose generators all lie in groups Q_j^{-k} with $j \geq 0$.

We combine the following two lemmas, which correspond to Theorem 20.3 and Lemma 20.5 of [BJW95]. We study the linear functional $\epsilon_A = \langle \iota_{-k}, - \rangle \colon Q_*^{-k} \to P(n)_*$ defined by the identity operation ι_{-k} on $P(n)^{-k}(-)$, which is plainly additive.

Lemma 8.4 *Given any integer $k > 0$, there exist:*

(i) *a sequence of additive unstable operations $r_i \colon P(n)^{-k}(-) \to P(n)^{m(i)}(-)$ with $m(i) \geq 0$,*

(ii) *a sequence of elements $v(i) \in P(n)_*$ with $\deg(v(i)) \to \infty$,*

*such that in any additively unstable $P(n)$-cohomology comodule M (*e.g., *$P(n)^*(X)$ for any space X) any $x \in M^{-k}$ decomposes as the (topological infinite) sum $x = \sum_i v(i) r_i x$.*

Proof Let $\{c_1, c_2, c_3, \dots\}$ be the Ravenel–Wilson (or any other) basis of the free $P(n)_*$-module Q_*^{-k}, with $c_i \in Q_{m(i)}^{-k}$. Trivially, $m(i) \geq 0$. For fixed $x \in M^{-k}$ and any additive operation r, the linearity of rx in r may be expressed, as in [BJW95, (6.39)], by the formula

$$rx = \sum_i \langle r, c_i \rangle r_i x, \tag{8.2}$$

where r_i denotes the operation dual to c_i. We take $r = \iota_{-k}$, put $v(i) = \langle \iota_{-k}, c_i \rangle$, and note that $\deg(v(i)) = m(i) + k \to \infty$. ∎

Remark The coefficients are readily computed: $v(i) = \epsilon_A c_i = v^K$ if the monomial c_i has the form $e^\epsilon b_{(0)}^j w^K$, and $v(i) = 0$ otherwise. Thus many terms are zero.

To get the more precise information for Theorem 1.8, we write the space X as the disjoint union of its components and reduce to the case when X is connected.

Lemma 8.5 *Let M be a connected* (see [BJW95, Definition 7.14]) *additively unstable $P(n)$-cohomology algebra (*e.g., *$P(n)^*(X)$ for any connected space X). Then as a topological $P(n)_*$-module, M is generated by $1_M \in M^0$ and elements of M^i for $i > 0$. The generator 1_M is never redundant.*

Proof Let L be the submodule generated (topologically) by the elements of all the M^i for $i > 0$. By Lemma 8.4, we need only consider $x \in M^0$. We choose a basis $\{c_1, c_2, c_3, \dots\}$ of Q_*^0 with $c_1 = 1$.

We recall from [BJW95, Definition 7.13] the *collapse* operation κ_j on $P(n)^j(-)$ for any j; since M is connected, on any $x \in M^j$ it satisfies $\kappa_j x = v 1_M$ for some $v \in P(n)_{-j}$. But (8.2) gives $\kappa_0 x \equiv r_1 x \bmod L$ and also $x = \iota_0 x \equiv r_1 x \bmod L$. Thus $x \equiv \kappa_0 x = \lambda 1_M \bmod L$ for some $\lambda \in \mathbb{F}_p$.

Since $\kappa L = 0$ and $\kappa_0 1_M = 1_M$, 1_M never lies in L. ∎

8.3 Higher-Order Relations

The proof of Lemma 8.3 resembles that of [BJW95, Theorem 18.16]. The Nakayama Lemma [Bo95, §15] (which is easier for $P(n)_*$ than for BP_*, as $p = 0$) allows us to work throughout modulo the ideal $\mathfrak{V} \subset Q_*^*$. We also work modulo powers of $\mathfrak{W}$. (These ideals were introduced in (7.10), which displays the w-linear terms in the relation $(\mathcal{R}_k)$.)

When $q = n$ and $c = 1$, we observe that Definition 8.1(i) is the first term in $(\mathcal{R}_{d_n+n})$, and is thus expressible by (7.10) in terms of Q-allowable monomials mod $\mathfrak{V} + \mathfrak{W}^2$. Equation (7.9) shows that $(\mathcal{R}'_n)$ takes care of (ii), while (7.15) shows that $(\mathcal{R}''_n)$ takes care of (iii). **[**If $p = 2$, we use (7.16) and (7.17) to handle (iv) and (v).**]**

Otherwise, the relations $(\mathcal{R}_k)$ are not at all transparent. We handle the general disallowed monomial Definition 8.1(i) by eliminating the $q - n$ variables $w_n, w_{n+1}, \dots, w_{q-1}$ from the $q - n + 1$ relations $(\mathcal{R}_{d_n+n})$, $(\mathcal{R}_{d_{n+1}+n+1}), \dots, (\mathcal{R}_{d_q+q})$, expressed in the form (7.10), to obtain the *higher-order derived relation*

$$\Delta_q w_q + \sum_{r>q} \Delta_r w_r \equiv 0 \mod \mathfrak{V} + \mathfrak{W}^2, \tag{8.3}$$

for certain determinants Δ_r. Explicitly, for any $r \geq q$,

$$\Delta_r = \sum_{\pi} \epsilon_\pi b^{p^{\pi n}}_{(d_n+n-\pi n)} \cdots b^{p^{\pi(q-1)}}_{(d_{q-1}+q-1-\pi(q-1))} b^{p^{\pi r}}_{(d_q+q-\pi r)}, \tag{8.4}$$

where we sum over all permutations π of $\{n, \dots, q-1, r\}$, write ϵ_π for the sign of π, and adopt the convention that meaningless factors $b_{(j)}$ with $j < 0$ are taken as 0.

We order the b-monomials *lexicographically* ($b^J < b^K$ if and only if there exists $t \geq 0$ such that $j_r = k_r$ for all $r < t$, and $j_t < k_t$).

Lemma 8.6 *For any $r \geq q$, the determinant Δ_r in* (8.4) *has the form*

$$\Delta_r = b^{p^n}_{(d_n)} b^{p^{n+1}}_{(d_{n+1})} \cdots b^{p^{q-1}}_{(d_{q-1})} b^{p^r}_{(d_q+q-r)} \quad + \textit{higher terms.}$$

Proof The displayed term is the diagonal term with $\pi = \mathrm{id}$. For any other permutation π, there is a first index t such that $\pi t > t$, so that $n \leq t \leq q-1$ and $\pi k = k$ for all $k < t$. The corresponding term $\epsilon_\pi b^{p^n}_{(d_n)} \cdots b^{p^{t-1}}_{(d_{t-1})} b^{p^{\pi t}}_{(d_t+t-\pi t)} \cdots$ in (8.4) is higher, because $d_t + t - \pi t < d_t$. ∎

Proof of Lemma 8.3 We show that each Q-disallowed monomial in Definition 8.1 is a linear combination mod $\mathfrak{V}$ of higher monomials with the same w-length, and monomials of greater w-length, where we partially order all monomials according to the factor b^J (and ignore e, $a_{(i)}$, and w_k). Since there are only finitely many monomials in each bidegree, the result follows.

For Definition 8.1(i), Lemma 8.6 shows how to use (8.3) to express $b^{p^n}_{(d_n)} \cdots b^{p^q}_{(d_q)} w_q$ as a linear combination mod $\mathfrak{V}$ of higher monomials and monomials with w-length

at least 2, since for $r > q$, the diagonal term of Δ_r is higher than the diagonal term of Δ_q. Multiplication by c preserves the ordering.

For (ii), (iii) **[**and (iv), if $p = 2$**]**, we modify (8.3) by eliminating the variables $w_{n+1}, \dots, w_{q-1}$ from the relations $(\mathcal{R}_{d_{n+1}+n+1}), \dots, (\mathcal{R}_{d_q+q})$ to obtain

$$\Delta'_n w_n + \Delta'_q w_q + \sum_{r>q} \Delta'_r w_r \equiv 0 \mod \mathfrak{V} + \mathfrak{W}^2.$$

When we multiply this by $eb^N_{(0)}c$, the first term drops out by (7.9). Lemma 8.6, slightly modified (or with n replaced by $n+1$), shows that (ii) is the lowest of the remaining terms. If we multiply by $a_{(0)}b^N_{(0)}c$ instead and use (7.15), we obtain (iii). **[**For (iv), we multiply by $b^N_{(0)}b_{(1)}c$ and use (7.16). For (v), we eliminate the variables $w_{n+2}, \dots, w_{q-1}$ from the relations $(\mathcal{R}_{d_{n+2}+n+2}), \dots, (\mathcal{R}_{d_q+q})$ to obtain a higher-order relation

$$\Delta''_n w_n + \Delta''_{n+1} w_{n+1} + \Delta''_q w_q + \sum_{r>q} \Delta''_r w_r \equiv 0 \mod \mathfrak{V} + \mathfrak{W}^2.$$

When we multiply this by $b^{N+2^{n+1}}_{(0)}c$, the first two terms drop out by (7.7) and (7.17). The diagonal term in the determinant Δ''_q gives (v).**]** ∎

8.4 The First Higher-Order Relation

The first relation for a given q, where we eliminate w_n, $w_{n+1}, \dots, w_{q-1}$ from $(\mathcal{R}'_n), (\mathcal{R}_{n+1}), \dots, (\mathcal{R}_q)$, is particularly important. The additive version for $P(n)$ of Bendersky's lemma [Be86, Theorem 6.2] (or see [BJW95, Lemma 18.23]) gives more precise information than our proof of Lemma 8.3, and follows immediately from Lemma 12.1.

We recall the ideal $I_q = (v_n, v_{n+1}, \dots, v_{q-1}) \subset P(n)_*$ (where $I_n = (0)$).

Lemma 8.7 *In* $Q^*_* = QP(n)_*(\underline{P(n)}_*)$ **[***replaced by* $Q\overline{P(n)}_*(\underline{P(n)}_*)$ *if* $p = 2$**]**, *we have the relation*

$$e^{g(n,q)-1}w_q \equiv v_q e^{g(n,q-1)+1} \bmod I_q Q^*_* \quad \textit{for } q \geq n.$$

[*If* $p = 2$*, this is almost superseded by the relation*

$$e^{g(n,q)-2}w_q \equiv v_q e^{g(n,q-1)} \bmod I_q Q^*_* \quad \textit{for } q \geq n+1.\textbf{]}$$

8.5 Primitive Elements

Let M be an unstable $P(n)$-cohomology comodule (in the sense of [BJW95, Definition 6.32]). An element $x \in M^k$ is called (additively unstably) *primitive* if the coaction ρ_M has the value $\rho_M x = x \otimes e^k$ on x. Then for any $v \in P(n)_*$,

$$\rho_M(vx) = x \otimes e^k(\eta_R v). \tag{8.5}$$

Of course, all this requires $k \geq 0$, but more is true, as in [BJW95, Lemma 20.8].

Lemma 8.8 *Let $x \in M^k$ be a nonzero primitive element of the unstable $P(n)$-cohomology comodule M, and assume $q \geq n$.*

(i) *If $I_q x = 0$ and k satisfies the condition (depending on p and q)*

$$\begin{aligned} k &\geq g(n,q) - 1 \quad \text{if } p \text{ is odd or } q = n, \\ k &\geq g(n,q) - 2 \quad \text{if } p = 2 \text{ and } q \geq n+1, \end{aligned} \tag{8.6}$$

then $v_q x$ is primitive (possibly zero).

(ii) *If k does not satisfy the condition* (8.6)*, then for all $i > 0$, $v_q^i x$ is nonzero and is not primitive.*

Proof For (i) (8.5) gives $\rho(v_q x) = x \otimes e^k w_q$. By Lemma 8.7, this is the same as $x \otimes v_q e^{k-2(p^q-1)} = v_q x \otimes e^{k-2(p^q-1)}$, since $I_q x = 0$.

For (ii), we have $\rho(v_q^i x) = x \otimes e^k w_q^i$. Here, $e^k w_q^i$ is Q-allowable by Definition 8.1 and hence a basis element of Q_*^*, which shows that $v_q^i x$ is not primitive. ■

Proof of Lemma 1.9 We must have $\rho x = x \otimes e^k$. If $m > n$, we have $v_{m-1} x = 0$, and case (ii) of Lemma 8.8 with $q = m-1$ does not apply; hence the lower bound on k.

Conversely, (8.5) specifies the coaction on all of M, and Lemma 8.7 shows it is well defined. ■

Proof of Theorem 1.10 We build an increasing sequence

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M$$

of subcomodules of M. For each $i > 0$, just as in the proof of [BJW95, Theorem 20.11], we construct a primitive element $x_i \in M/M_{i-1}$ with $\mathrm{Ann}(x_i) = I_{m_i}$ for some m_i, using Lemma 8.8 in place of of [BJW95, Lemma 20.8]. We take $M_i/M_{i-1} \subset M/M_{i-1}$ as the $P(n)_*$-submodule generated by x_i. Lemma 1.9 describes M_i/M_{i-1}.

Because each $k_i \geq 0$ in Theorem 1.10 and each M^k is a finitely generated $\mathbb{F}_p$-module, this sequence must terminate after finitely many steps. We deduce that M is a finitely presented $P(n)_*$-module. ■

9 Idempotent Operations

Lemma 9.1 delivers the promised additive idempotent operations $\theta(m)$ in $P(n)$-cohomology that we need for Lemma 5.1, which is equivalent to Lemma 3.1. In fact, we find a large class of $\theta(m)$, among which none seems to be preferred. The rest of this section applies the work in Section 8 to prove Lemma 9.1.

Lemma 9.1 *Assume that $k \leq g(n,m)$* **[***replaced by $k \leq g(n,m) - 1$ if $p = 2$***]***, where $m \geq n$. Then there exists an additive idempotent operation $\theta(m)$ on $P(n)^k(-)$ having the following properties:*

(i) *The image of the operation $\theta(m)$ is represented by the space $\underline{P(n,m)}_k$.*

(ii) *The map* $\theta(m)\colon \underline{P(n)}_k \to \underline{P(n)}_k$ *factors to yield an* H*-space splitting* $\overline{\theta(m)}\colon \underline{P(n,m)}_k \to \underline{P(n)}_k$ *of the canonical* H*-map* $\rho(m)\colon \underline{P(n)}_k \to \underline{P(n,m)}_k$.

(iii) *For all spaces* X, $\overline{\theta(m)}$ *naturally embeds* $P(n,m)^*(X) \subset P(n)^*(X)$ *as a summand, in the sense of abelian groups (but not as* $P(n)_*$*-modules).*

Remark Exactly as in [BJW95, Remark, p. 817], we can make the splittings $\theta(m)$ compatible as k and m vary if we wish. The decomposition factors of $\underline{P(n)}_k$ resulting from this approach must of course be the same as in Theorem 1.6, according to Theorem 1.2(ii), but the injection maps are different, in general. However, we emphasize that the splitting theorems as stated in Sections 1 and 5 do not require any compatibility.

9.1 The ideals $\mathfrak{I}_m$

As in [BJW95], the ideal $J_m = (v_{m+1}, v_{m+2}, \dots) \subset P(n)_*$, introduced in (1.3), gives rise to an analogous ideal for the right action of $P(n)_*$ on Q_*^*.

Definition 9.2 Given any $m \geq n$, we define the ideal

$$\mathfrak{I}_m = (w_{m+1}, w_{m+2}, w_{m+3}, \dots) \subset Q_*^*.$$

We need to know how $\mathfrak{I}_m$ sits inside Q_*^*. As in [BJW95], the answer is remarkably clean, in a certain range.

Lemma 9.3 *For* $k \leq g(n,m)$ **[***replaced by* $k \leq g(n,m) - 1$ *if* $p = 2$**]**, $Q_*^k \cap \mathfrak{I}_m$ *is the left* $P(n)_*$*-submodule of* Q_*^k *spanned by all the* Q*-allowable monomials* (8.1) *that lie in it and contain an explicit factor* w_q *for some* $q > m$.

Remark By Lemma 8.7, $v_{m+1} e b_{(0)}^{g(n,m)/2} - z$ lies in $\mathfrak{I}_m$, where $z \in I_{m+1} Q_*^*$, so the result definitely fails for $k = g(n,m) + 1$ **[**also for $k = g(n,m)$ if $p = 2$**]**.

Proof Any monomial that contains w_h with $h > m$ visibly lies in $\mathfrak{I}_m$. To show the converse, we fix k and i_0 and prove by *downward* induction on h that for all $i \leq i_0$, all elements in Q_i^k of the form cw_h lie in the indicated $P(n)_*$-submodule. This statement is trivial for sufficiently large h (depending on k and i_0).

We therefore choose $q > m$, assume the statement holds for all $h > q$, and prove it for $h = q$. Take $cw_q \in Q_i^k$, where $i \leq i_0$, so that $c \in Q_i^{k+2(p^q-1)}$. By Lemma 8.3, we may reduce to the case where c is a Q-allowable monomial. We note that in Definition 8.1, the Q-disallowed monomials (i) and (iv) have b-length $\frac{1}{2}g(n,q)$, while (ii), (iii) and (v) have b-length $\frac{1}{2}g(n,q) - 1$.

Case 1: c has no factor e, $a_{(0)}$, *or* w_j. For odd p, the b-length of c is at most

$$\tfrac{1}{2}\big(k + 2(p^q - 1)\big) \leq \tfrac{1}{2}\big(g(n,m) + 2p^q - 2\big) < \tfrac{1}{2}g(n,q),$$

which makes cw_q also Q-allowable, as only rule (i) of Definition 8.1 is relevant. **[**If $p = 2$, we need to assume $k \leq g(n,m) - 1$ to get the stronger bound $\frac{1}{2}g(n,q) - 1$.**]**

Case 2: $c = ey$ or $c = a_{(0)}y$, where y has no factor w_j. In this case, the b-length of c is at most

$$\tfrac{1}{2}\big(k - 1 + 2(p^q - 1)\big) \leq \tfrac{1}{2}\big(g(n,m) + 2p^q - 3\big) < \tfrac{1}{2}g(n,q) - 1,$$

which makes cw_q automatically Q-allowable.

Case 3: $c = yw_j$, where $j \leq q$. Then cw_q remains Q-allowable, by the form of Definition 8.1.

Case 4: $c = yw_j$, where $j > q$. By induction, $cw_q = (yw_q)w_j$ lies in the indicated submodule. ■

9.2 Linear Functionals

To establish Lemma 9.1, we actually construct the associated $P(n)_*$-linear functional $\langle \theta(m), - \rangle \colon Q^k_* \to P(n)_*$.

Lemma 9.4 *Assume the linear functional $\langle \theta(m), - \rangle \colon Q^k_* \to P(n)_*$ corresponding to the additive operation $\theta(m) \colon P(n)^k(-) \to P(n)^k(-)$ satisfies the conditions:*

$$\langle \theta(m), Q^k_* \cap \mathfrak{J}_m \rangle = 0, \tag{9.1}$$

$$\langle \theta(m), c \rangle \equiv \epsilon_A c \bmod J_m \quad \textit{for all } c \in Q^k_*, \tag{9.2}$$

where $\epsilon_A \colon Q^k_* \to P(n)_*$ is the augmentation. Then

(i) *The homology homomorphism $Q(\theta(m)_*) \colon Q^k_* \to Q^k_*$ induced by the representing map $\theta(m) \colon \underline{P(n)}_k \to \underline{P(n)}_k$ satisfies*

(a) $Q(\theta(m)_*)(Q^k_* \cap \mathfrak{J}_m) = 0$,
(b) $Q(\theta(m)_*) \equiv \mathrm{id} \colon Q^k_* \to Q^k_* \bmod \mathfrak{J}_m$.

(ii) *$Q(\theta(m)_*)$ induces a splitting of the short exact sequence*

$$0 \to Q^k_* \cap \mathfrak{J}_m \to Q^k_* \to Q^k_*/(Q^k_* \cap \mathfrak{J}_m) \to 0$$

of left $P(n)_*$-modules.

(iii) *The operation $\theta(m)$ is idempotent and has the properties listed in Lemma* 9.1.

Proof The proof is patterned after [BJW95, Lemma 22.2]. We require the commutative diagram

$$\begin{array}{ccccccc}
Q^k_* & \xrightarrow{\psi_A} & Q^*_* \otimes Q^k_* & \xrightarrow{\mathrm{id} \otimes \langle \theta(m), - \rangle} & Q^*_* \otimes P(n)_* & \xrightarrow{\lambda_R} & Q^*_* \\
\downarrow{\scriptstyle q'} & & \downarrow & \nearrow & \downarrow & & \downarrow{\scriptstyle q'} \\
Q^k_*/\mathfrak{J}_m & \xrightarrow{\overline{\psi_A}} & Q^*_* \otimes Q^k_*/\mathfrak{J}_m & \xrightarrow{\mathrm{id} \otimes \overline{\epsilon_A}} & Q^*_* \otimes P(n)_*/J_m & \xrightarrow{\overline{\lambda_R}} & Q^*_*/\mathfrak{J}_m
\end{array}$$

of $P(n)_*$-module homomorphisms, where $\overline{\psi_A}$, $\overline{\epsilon_A}$ and $\overline{\lambda_R}$ denote quotients of ψ_A, ϵ_A, and the right action λ_R of $P(n)_*$ on Q_*^*, $Q_*^k/\mathfrak{J}_m$ is really $Q_*^k/(Q_*^k \cap \mathfrak{J}_m)$, and the vertical arrows are the obvious projections. The conditions (9.1) and (9.2) on $\langle\theta(m), -\rangle$ are exactly what we need to fill in the diagonal.

By [BJW95, Lemma 6.51(c)], the top row gives the homology homomorphism $Q(\theta(m)_*)$, while by [BJW95, (6.31)], the bottom row reduces to the identity homomorphism of $Q_*^k/\mathfrak{J}_m$. Thus the diagonal provides a splitting we call $j'\colon Q_*^k/\mathfrak{J}_m \to Q_*^k$ that satisfies $j' \circ q' = Q(\theta(m)_*)$ and $q' \circ j' = \mathrm{id}$ and so yields (i). Part (ii) is merely a restatement of (i).

It follows by faithfulness that $\theta(m)$ is an idempotent operation, so that the image $h(-) = \theta(m)P(n)^k(-) \subset P(n)^k(-)$ is an ungraded cohomology theory. By [Bo95, Theorem 3.6], $h(-)$ is represented (on Ho) by some H-space Y, and the additive operations $h(-) \subset P(n)^k(-)$ and $\theta(m)\colon P(n)^k(-) \to h(-)$ are represented by H-maps $j\colon Y \to \underline{P(n)}_k$ and $q\colon \underline{P(n)}_k \to Y$, respectively, that satisfy $j \circ q = \theta(m)$ and $q \circ j = \mathrm{id}$.

To finish (iii), we apply the homotopy group functor $\pi_*(-)$ to obtain homomorphisms $q_*\colon \pi_*(\underline{P(n)}_k) \to \pi_*(Y)$ and $j_*\colon \pi_*(Y) \to \pi_*(\underline{P(n)}_k)$ that satisfy $q_* \circ j_* = \mathrm{id}$ and $j_* \circ q_* = \theta(m)_*$. Recall that $\pi_*(\underline{P(n)}_k) \cong \Sigma^k P(n)_*$. Given $v \in P(n)_i$, (7.4) evaluates $\theta(m)_*\Sigma^k v = \Sigma^k\langle\theta(m), e^{k+i}(\eta_R v)\rangle$. Then (9.1) yields $\theta(m)_*\Sigma^k v = 0$ if $v \in J_m$, while for any v, (9.2) gives $\theta(m)_*\Sigma^k v \equiv \Sigma^k v \bmod J_m$. It follows that $\rho(m) \circ j\colon Y \to \underline{P(n,m)}_k$ induces an isomorphism of homotopy groups and is therefore a homotopy equivalence. To establish the properties listed in Lemma 9.1, we put $\overline{\theta(m)} = j \circ g$, where $g\colon \underline{P(n,m)}_k \to Y$ is a homotopy inverse to $\rho(m) \circ j$. ■

Proof of Lemma 9.1 Lemma 9.3 makes it obvious that linear functionals $\langle\theta(m), -\rangle$ exist that satisfy the conditions (9.1) and (9.2), so that Lemma 9.4 applies. ■

Remark As an explicit example, choose $\langle\theta(m), -\rangle$ on the Ravenel–Wilson basis as $\langle\theta(m), c\rangle = v^K$ if c has the form $e^\epsilon b_{(0)}^j w^K$ but contains no factor w_k with $k > m$, and $\langle\theta(m), c\rangle = 0$ otherwise. To determine $\langle\theta(m), c\rangle$ for c not in the basis, we must first express c in terms of the basis.

10 Unstable Operations in $P(n)$-Cohomology

In this section, we use *all* unstable operations in $P(n)$-cohomology to obtain generators and relations for the Hopf ring $P(n)_*(\underline{P(n)}_*)$, in the style of [BJW95]. The two multiplications are $c*d = \mu_*(c \times d)$ and $c \circ d = \phi_*(c \times d)$, induced respectively by the maps $\mu\colon \underline{P(n)}_k \times \underline{P(n)}_k \to \underline{P(n)}_k$ and $\phi\colon \underline{P(n)}_k \times \underline{P(n)}_m \to \underline{P(n)}_{k+m}$ that represent addition and multiplication in $P(n)$-cohomology, and 1_k will denote the $*$-identity element of $P(n)_*(\underline{P(n)}_k)$. **[**If $p = 2$, we use the Hopf ring $\overline{P(n)}_*(\underline{P(n)}_*)$ instead, replacing $c \times d$ by $c \overline{\times} d$ in both multiplications.**]** We still assume that $0 < n < \infty$.

We deduce the results of Section 7 on additive operations by applying the homo-

morphism

$$q_k\colon P(n)_*(\underline{P(n)}_k) \to Q^k_*, \tag{10.1}$$

which neglects 1_k and decomposables, shifts degrees by $-k$, and (as k varies) takes $\circ$-products to products (*with a sign*, on account of the degree shift). However, the Hopf ring structure maps ψ and ϵ are unrelated to ψ_A and ϵ_A.

Since the Hopf ring is a free $P(n)_*$-module by [RW96, Corollary 1.5], [BJW95, Theorem 4.14] allows us to identify the following:

(i) the cohomology *operation* $r\colon P(n)^k(-) \to P(n)^m(-)$;
(ii) the cohomology *class* $r(\iota_k) \in P(n)^m(\underline{P(n)}_k)$;
(iii) the representing *map of spaces* $r\colon \underline{P(n)}_k \to \underline{P(n)}_m$, up to homotopy;
(iv) the $P(n)_*$*-linear functional* $\langle r, -\rangle\colon P(n)_*(\underline{P(n)}_k) \to P(n)_*$ of degree $-m$ **[**or $\langle r, -\rangle\colon \overline{P(n)}_*(\underline{P(n)}_k) \to P(n)_*$ if $p = 2$**]**.

10.1 Hopf Rings for $p = 2$

When $p = 2$, $P(n)_*(\underline{P(n)}_*)$ and $\overline{P(n)}_*(\underline{P(n)}_*)$ are not Hopf rings in the ordinary sense (though $H_*(\underline{P(n)}_*; \mathbb{F}_2)$ is one, and is described in [BW01] and after Theorem 11.3). (A few things are simpler: there are no signs and χ is the identity.) Because $P(n)$ is not commutative, all Hopf ring axioms that shuffle factors must be modified to use the commutativity isomorphism T_Q of (2.6), which results in extra terms; see Section 2 or [Bo] for details. Neither multiplication is commutative in the ordinary sense, nor is ψ cocommutative.

The Hopf ring $\overline{P(n)}_*(\underline{P(n)}_*)$ is identical to $P(n)_*(\underline{P(n)}_*)$ as a $P(n)_*$-module. The choice of multiplication on $P(n)$ does not affect the $\overline{P(n)}_*$-module structure on $\overline{P(n)}_*(\underline{P(n)}_*)$, nor does it affect the $\circ$-generators that we construct below. However, switching to the other good multiplication on $P(n)$ replaces $c \circ d = \phi_*(c \overline{\times} d)$ by

$$\bar{\phi}_*(c \times d) = \phi_* T_*(c \times d) = \phi_*(d \overline{\times} c) = d \circ c,$$

which is different in general; and similarly for $c * d$.

10.2 The Cartan Formulae

Assume first that p is odd. Given a cohomology class $x \in P(n)^k(X)$, we encode the action of operations on x by a formula of the form

$$r(x) = \sum_\alpha \langle r, c_\alpha \rangle x_\alpha \quad \text{for all } r,$$

for suitable choices $c_\alpha \in P(n)_*(\underline{P(n)}_k)$ and $x_\alpha \in P(n)^*(X)$. (Here and elsewhere, we mean all operations r that have the correct domain degree. The sum may be infinite if X is not finite-dimensional.) Similarly, given $y \in P(n)^m(X)$, suppose

$$r(y) = \sum_\beta \langle r, d_\beta \rangle y_\beta \quad \text{for all } r.$$

Then the two Cartan formulae [BJW95, (10.23), (10.36)] are:

$$r(x+y) = \sum_{\alpha}\sum_{\beta}(-1)^{\deg(x_\alpha)\deg(y_\beta)}\langle r, c_\alpha * d_\beta\rangle x_\alpha y_\beta,$$

$$r(xy) = \sum_{\alpha}\sum_{\beta}(-1)^{\deg(x_\alpha)\deg(y_\beta)}\langle r, c_\alpha \circ d_\beta\rangle x_\alpha y_\beta.$$

We use them repeatedly without further reference.

The Case $p = 2$ Examination reveals that the proof of the Cartan formulae in [BJW95] relies on the identity $\langle x\times y, a\times b\rangle = \pm\langle x,a\rangle\langle y,b\rangle$, which is false for $p=2$; we must replace $a\times b$ by $a\overline{\times}b$ and use (2.7) instead. When we use the Hopf ring $\overline{P(n)}_*(\underline{P(n)}_*)$, both Cartan formulae remain valid as stated.

10.3 Cohomology of a Point

Our first test space is the one-point space. For each $v \in P(n)_q$, the Hopf ring element $[v] \in P(n)_0\left(\underline{P(n)}_{-q}\right)$ **[**or $\overline{P(n)}_0\left(\underline{P(n)}_{-q}\right)$ if $p=2$**]** is defined by the identity

$$r(v) = \langle r, [v]\rangle \quad \text{in } P(n)^*(\text{point}) = P(n)_*, \text{ for all } r. \tag{10.2}$$

The properties of these elements were listed in [BJW95, Proposition 11.2]. As $[v+v'] = [v] * [v']$ and $[vv'] = [v]\circ[v']$, we are primarily interested in the elements $[v_k] \in P(n)_0(\underline{P(n)}_{-2(p^k-1)})$ for $k \geq n$ **[**or in $\overline{P(n)}_0(\underline{P(n)}_{-2(2^k-1)})$ if $p=2$**]**. Then (10.1) maps $[v_k]$ to w_k.

We have the important relation $[1]^{*p} = [p] = [0_0] = 1_0$.

10.4 Cohomology of a Circle

Our second test space is the circle S^1. The *suspension* element $e = e_1 \in P(n)_1(\underline{P(n)}_1)$ **[**or $\overline{P(n)}_1(\underline{P(n)}_1)$ if $p=2$**]** is defined by the action of operations r on the standard generator $u_1 \in P(n)^1(S^1)$,

$$r(u_1) = \langle r, 1_1\rangle 1_S + \langle r, e\rangle u_1 \quad \text{in } P(n)^*(S^1) = E(u_1), \text{ for all } r. \tag{10.3}$$

The properties of e were listed in [BJW95, Proposition 13.7].

10.5 Complex Orientation

Our third test space is $\mathbb{C}P^\infty$. The Hopf ring elements $b_j \in P(n)_{2j}(\underline{P(n)}_2)$ **[**or $\overline{P(n)}_{2j}(\underline{P(n)}_2)$ if $p=2$**]** for $j \geq 0$ are defined by the identity

$$r(x) = \langle r, b(x)\rangle = \sum_{j=0}^{\infty}\langle r, b_j\rangle x^j \quad \text{in } P(n)^*(\mathbb{C}P^\infty) \cong P(n)_*[[x]], \text{ for all } r, \tag{10.4}$$

where $b(x)$ is a convenient formal abbreviation for $\sum_j b_j x^j$. Their properties were listed in [BJW95, Proposition 15.3]. In particular, $b_0 = 1_2$ is now nonzero and $b_1 = -e \circ e$. Again, the accelerated elements $b_{(j)} = b_{p^j} \in P(n)_{2p^j}(\underline{P(n)}_2)$ **[**or in $\overline{P(n)}_{2^{j+1}}(\underline{P(n)}_2)$ if $p = 2$**]** suffice, as [BJW95, Lemma 15.9] shows how to express the other b's inductively in terms of these and the v's and $[v]$'s.

Naturality of (10.4) with respect to the p-th power map $\zeta\colon \mathbb{C}P^\infty \to \mathbb{C}P^\infty$, with massive use of the Cartan formulae, yields the identity

$$b([p](x)) = \mathop{\ast}_{i=N}^{\infty} \{b(x)^{\circ i+1} \circ [g_i]\} \quad \text{in } P(n)_*(\underline{P(n)}_2)[[x]] \tag{10.5}$$

[or in $\overline{P(n)}_*(\underline{P(n)}_2)[[x]]$ if $p = 2$**]**, as in [BJW95, (15.14)]. The lowest power of x that occurs is still x^{p^n}, apart from the term 1_2 on each side.

Definition 10.1 For each $k \geq n$, we define the *k-th main unstable relation* $(\mathcal{R}_k)$ as the coefficient of x^{p^k} in (10.5).

The first relation is simply

$$(\mathcal{R}_n) \qquad v_n b_{(0)} = b_{(0)}^{\circ p^n} \circ [v_n] \quad \text{in } P(n)_*(\underline{P(n)}_2)$$

[or in $\overline{P(n)}_*(\underline{P(n)}_2)$ if $p = 2$**]**. By [RW96, Proposition 2.1(j)], it desuspends once to

$$(\mathcal{R}'_n) \qquad v_n e = e \circ b_{(0)}^{\circ N} \circ [v_n] \quad \text{in } P(n)_*(\underline{P(n)}_1) \tag{10.6}$$

[or in $\overline{P(n)}_*(\underline{P(n)}_1)$ if $p = 2$**]**. The second relation is almost as easy, in view of (6.9):

$$(\mathcal{R}_{n+1}) \qquad b_{(1)}^{\circ p^n} \circ [v_n] + b_{(0)}^{\circ p^{n+1}} \circ [v_{n+1}] = v_n^p b_{(1)} + v_{n+1} b_{(0)}.$$

10.6 Cohomology of a Lens Space, for p Odd

Our final test space is the lens space skeleton L, whose cohomology (6.12) has two generators u and x. As x is a Chern class, (10.4) gives $r(x)$ by naturality. We define Hopf ring elements a_i and c_i by the identity

$$r(u) = \sum_{i=0}^{N} \langle r, a_i \rangle x^i + \sum_{i=0}^{N-1} \langle r, c_i \rangle u x^i \quad \text{in } P(n)^*(L), \text{ for all } r. \tag{10.7}$$

Not by coincidence, their formal properties are exactly the same as in the case $E = K(n)$ of [BJW95]. The formal abbreviation $a(x) = \sum_i a_i x^i$ is convenient.

Proposition 10.2 *For p odd, the Hopf ring elements $a_i \in P(n)_{2i}(\underline{P(n)}_1)$ (for $0 \leq i < p^n$), $a_{(i)} = a_{p^i} \in P(n)_{2p^i}(\underline{P(n)}_1)$ (for $0 \leq i < n$), and $c_i \in P(n)_{2i+1}(\underline{P(n)}_1)$ (for $0 \leq i \leq p^n - 2$) defined by* (10.7) *have the following properties:*

(i) $a_0 = 1_1$ *and* $c_0 = e$;
(ii) $\psi a_k = \sum_{i+j=k} a_i \otimes a_j$;
(iii) $\epsilon a_i = 0$ *for all* $i > 0$, *in particular,* $\epsilon a_{(i)} = 0$ *for all* i;
(iv) $a_i * a_j = \binom{i+j}{i} a_{i+j}$, *provided* $i + j < p^n$;
(v) $a_{(i)}^{*p} = 0$ *for* $0 \le i < n-1$;
(vi) $\chi a_i = (-1)^i a_i$, *in particular,* $\chi a_{(i)} = -a_{(i)}$;
(vii) $c_i = e * a_i$;
(viii) $a_{(i)} \circ a_{(j)} = -a_{(j)} \circ a_{(i)}$;
(ix) $a_{(i)} \circ a_{(i)} = 0$;
(x) *for all* r, $r_* a_k$ *is the coefficient of* x^k *in the formal identity*

$$r_* a(x) = \mathop{\ast}_{i=0}^{N} \{b(x)^{\circ i} \circ [\langle r, a_i \rangle]\} * \mathop{\ast}_{i=0}^{N-1} \{a(x) \circ b(x)^{\circ i} \circ [\langle r, c_i \rangle]\}$$

in $P(n)_*(\underline{P(n)}_*)[x]/(x^{p^n})$.

Proof The statement and proof are identical to [BJW95, Proposition 17.16], except that we offer a simpler proof of (vi) (and could have also in [BJW95]; compare the divided power Hopf algebra $\Gamma(a_1)$).

If m is odd, say $m = 2k+1$, we can write the defining equation for χa_m as

$$\chi a_m + \sum_{i=1}^{k} (\chi a_{m-i} * a_i + \chi a_i * a_{m-i}) + a_m = 0.$$

By induction, the terms in the sum cancel in pairs, as $m-i$ and i have opposite parity.

If m is even, (iv) decomposes a_m as a $*$-product, and we again use induction. ∎

We emphasize that (v) is *not* valid for $i = n-1$; instead, [RW96, Proposition 2.1(i)] shows that the unstable analogue of (7.15) is

$$(\mathcal{R}_n'') \qquad a_{(n-1)}^{*p} = v_n a_{(0)} - a_{(0)} \circ b_{(0)}^{\circ N} \circ [v_n] \quad \text{in } P(n)_*(\underline{P(n)}_1). \tag{10.8}$$

10.7 Cohomology of Real Projective Space, for $p = 2$

In this case, $L = \mathbb{R}P^{2N}$, with cohomology (6.15). We define Hopf ring elements f_i by the identity

$$r(t) = \sum_{i=0}^{2N} \langle r, f_i \rangle t^i \quad \text{in } P(n)^*(\mathbb{R}P^{2N}) = P(n)_*[t]/(t^{2N+1}), \text{ for all } r. \tag{10.9}$$

Again, we mimic (10.7) by writing $a_i = f_{2i}$, $a_{(i)} = a_{2^i} = f_{2^{i+1}}$, and $c_i = f_{2i+1}$. We make the obvious changes to Proposition 10.2 and write $f(t) = \sum_i f_i t^i$. We warn that the analogy is not perfect; ψa_k acquires many extra terms. Also, (iv) now requires proof; see [Bo].

Proposition 10.3 *For $p = 2$, the Hopf ring elements $f_i \in \overline{P(n)}_i(\underline{P(n)}_1)$ (for $0 \le i \le 2N$) and $a_{(i)} = f_{2^{i+1}} \in \overline{P(n)}_{2^{i+1}}(\underline{P(n)}_1)$ (for $0 \le i \le n-1$) defined by* (10.9) *have the following properties:*

(i) $f_0 = 1_1$ *and* $f_1 = e$;
(ii) $\psi f_k = \sum_{i+j=k} f_i \otimes f_j$;
(iii) $\epsilon f_i = 0$ *for all* $i > 0$, *in particular,* $\epsilon a_{(i)} = 0$ *for all* i;
(iv) $a_{(i)} \circ a_{(j)} = a_{(j)} \circ a_{(i)}$;
(v) $f_i * f_j = \binom{i+j}{i} f_{i+j}$, *provided* $i + j \le 2N$;
(vi) $a_{(i)} * a_{(i)} = 0$ *for* $0 \le i < n-1$;
(vii) *for all* r, $r_* f_k$ *is the coefficient of* t^k *in the formal identity*

$$r_* f(t) = \mathop{\ast}_{i=0}^{2N} \{ f(t)^{\circ i} \circ [\langle r, f_i \rangle] \} \quad \text{in } \overline{P(n)}_*(\underline{P(n)}_*)[t] \Big/ (t^{2N+1}).$$

Again, for $i = n-1$, (vi) is replaced by (10.8), now taken in $\overline{P(n)}_*(\underline{P(n)}_1)$. Finally, we prove in [Bo] that (7.14) lifts in the obvious way.

Lemma 10.4 *In the Hopf ring $\overline{P(n)}_*(\underline{P(n)}_*)$ for $p = 2$, we have*

$$a_{(i)} \circ a_{(i)} = b_{(i+1)} \quad \text{for } 0 \le i \le n-1. \tag{10.10}$$

Remark There is a case for writing e here as $a_{(-1)}$, so that the identity $e \circ e = b_{(0)}$ becomes a natural extension of (10.10).

11 Structure of the Hopf Ring

In this section, we present two descriptions of the Hopf ring $P(n)_*(\underline{P(n)}_*)$ **[**replaced by $\overline{P(n)}_*(\underline{P(n)}_*)$ if $p = 2$**]**: a clean concise description in terms of the generators and relations developed in Section 10, and a concrete computational description that specifies exactly what the elements of the Hopf ring are. (This relies heavily on the technical work of Ravenel–Wilson [RW96], and in no way replaces it.)

Theorem 11.1 (Ravenel–Wilson) *The Hopf ring $P(n)_*(\underline{P(n)}_*)$* **[***which is replaced by $\overline{P(n)}_*(\underline{P(n)}_*)$ if $p = 2$***]** *over $P(n)_*$ has the $\circ$-generators:*

(i) $[v_k] \in P(n)_0(\underline{P(n)}_{-2(p^k-1)})$ *for* $k \ge n$, *defined by* (10.2);
(ii) $e \in P(n)_1(\underline{P(n)}_1)$, *defined by* (10.3);
(iii) $b_{(j)} = b_{p^j} \in P(n)_{2p^j}(\underline{P(n)}_2)$ *for* $j \ge 0$, *defined by* (10.4);
(iv) $a_{(i)} = a_{p^i} \in P(n)_{2p^i}(\underline{P(n)}_1)$ *for* $0 \le i < n$, *defined by* (10.7) **[***replaced by* (10.9) *if* $p = 2$**]**;

subject to the relations $[1]^{*p} = 1_0$, $e \circ e = -b_{(0)}$, *the main relations* $(\mathcal{R}_k)$ *for* $k > n$ *(see Definition* 10.1*) and the two variants* (10.6) *and* (10.8) *of* $(\mathcal{R}_n)$ **[***also* (10.10) *if* $p = 2$**]**.

11.1 Allowable Monomials

For our second description of the Hopf ring, we reinterpret the general monomial (8.1) as the $\circ$-monomial

$$(11.1)\quad e^{\circ\epsilon}\circ a^{\circ I}\circ b^{\circ J}\circ[v^K]$$
$$= e^{\circ\epsilon}\circ a_{(0)}^{\circ i_0}\circ a_{(1)}^{\circ i_1}\circ\cdots\circ a_{(n-1)}^{\circ i_{n-1}}\circ b_{(0)}^{\circ j_0}\circ b_{(1)}^{\circ j_1}\circ b_{(2)}^{\circ j_2}\circ\cdots\circ[v_n]^{\circ k_n}\circ[v_{n+1}]^{\circ k_{n+1}}\circ\cdots$$

(We adopt the usual convention [RW77] that $d^{\circ 0} = [1] - 1_0$ for any element d with $\epsilon d = 0$, so that $d^{\circ 0}\circ d = d$ holds. We also set $[v_k]^{\circ 0} = [v_k^0] = [1]$.)

We define it to be *allowable* or *Q-allowable* exactly as in Definition 8.1.

A direct description of the allowable monomials is useful, to replace the indirectness of Definition 8.1. As in [RW96], Δ_0 denotes the multi-index $(1, 0, 0, \dots)$.

Proposition 11.2 *Any allowable $\circ$-monomial c in the Hopf ring can be written uniquely in one of the standard forms*

$$(11.2a)\qquad c = a^{\circ I}\circ b^{\circ G+L}\circ[v^K] \quad \textit{if } c \textit{ does not involve } e;,$$

$$(11.2b)\qquad c = e\circ a^{\circ I}\circ b^{\circ G+L-\Delta_0}\circ[v^K] \quad \textit{if } c \textit{ involves } e;,$$

where the multi-index G is defined by

$$(11.3)\qquad b^{\circ G} = b_{(d_n)}^{\circ p^n}\circ b_{(d_{n+1})}^{\circ p^{n+1}}\circ\cdots\circ b_{(d_{q-1})}^{\circ p^{q-1}},$$

$L = (l_0, l_1, l_2, \dots)$, and the indices satisfy

(i) $q \geq n$;
(ii) $0 \leq d_n \leq d_{n+1} \leq \cdots \leq d_{q-1}$;
(iii) $0 \leq l_t < p^r$ *for all* $t < d_r$, *for* $n \leq r < q$;
(iv) $0 \leq l_t < p^q$ *for all* t;
(v) $k_r = 0$ (i.e., v^K *contains no factor* v_r) *for all* $r < q$;
(vi) *in Case* (11.2b), $d_n = 0$ *or* $l_0 > 0$.

Conversely, any such monomial is allowable.

Proof If the allowable monomial c does not involve e, we choose each d_r in turn as small as possible, so that (iii) holds; moreover, (iii) requires this choice of d_r. (If we cannot even start, $q = n$, $c = a^{\circ I}\circ b^{\circ L}\circ[v^K]$, and (ii), (iii) and (v) become vacuous.) We continue as long as possible, until (iv) holds. In view of Definition 8.1(i), c does not contain $[v_r]$ for any $r < q$, and (v) holds.

If c has the form $e\circ c'$, we note that $e\circ c = b_{(0)}\circ c'$ remains allowable, and apply case (11.2a) to it. Here, we need (vi) so that Δ_0 can be subtracted off.

Conversely, the monomials (11.2) are easily seen to be allowable. ∎

11.2 The Algebra Structure

We recall that a *simple system of generators* of a graded algebra A with multiplication $*$ over a graded ring R of characteristic p is a set of elements $z_1, z_2, z_3, \dots$ such that the finite products

$$z^{*M} = z_1^{*m_1} * z_2^{*m_2} * z_3^{*m_3} * \cdots, \tag{11.4}$$

where $0 \le m_r < p$ for each z_r of even degree and $m_r = 0$ or 1 for each z_r of odd degree, form a set of free R-module generators of A.

The following description is also essentially included in [RW96, Theorems 1.3, 1.4] **[**except that for $p = 2$, (iv) was not written out explicitly and contains the surprise (c), below**]**. For $I \neq (1, 1, \dots, 1)$, $\rho(I)$ denotes the smallest t such that $i_{n-t} = 0$.

Theorem 11.3 (Ravenel–Wilson) *Assume $0 < n < \infty$, and let k be any integer. Then*

(i) *The Hopf algebra $P(n)_*(\underline{P(n)}_k)$ **[**or $\overline{P(n)}_*(\underline{P(n)}_k)$ if $p = 2$**]** has as a simple system of $*$-generators the set of all allowable $\circ$-monomials* (11.1) *(that lie in it).*

(ii) *The Q-allowable $\circ$-monomials form a minimal set of algebra $*$-generators of $P(n)_*(\underline{P(n)}_*)$ **[**or $\overline{P(n)}_*(\underline{P(n)}_*)$ if $p = 2$**]**.*

(iii) *For p odd, $P(n)_*(\underline{P(n)}_k)$ is the tensor product of the following subalgebras, one for each Q-allowable $\circ$-monomial (that lies in it):*

- (a) *$TP_{\rho(I)}(a^{\circ I} \circ b^{\circ J} \circ [v^K])$ for $I \neq (1, 1, \dots, 1)$;*
- (b) *$P(a^{\circ I} \circ b^{\circ J} \circ [v^K])$ for $I = (1, 1, \dots, 1)$;*
- (c) *$E(e \circ a^{\circ I} \circ b^{\circ J} \circ [v^K])$.*

(iv) *For $p = 2$, $\overline{P(n)}_*(\underline{P(n)}_k)$ contains the following subalgebras, one for each Q-allowable $\circ$-monomial (that lies in it), and is additively (but not multiplicatively) isomorphic to their tensor product:*

- (a) *$TP_{\rho(I)}(e^{\circ\epsilon} \circ a^{\circ I} \circ b^{\circ J} \circ [v^K])$ for $I \neq (1, 1, \dots, 1)$;*
- (b) *$P(a^{\circ I} \circ b^{\circ J} \circ [v^K])$ for $I = (1, 1, \dots, 1)$;*
- (c) *$TP_{n+1}(e \circ a^{\circ I} \circ b^{\circ J} \circ [v^K])$ for $I = (1, 1, \dots, 1)$.*

Remark For $p = 2$, the quotient algebra

$$H_*\big(\underline{P(n)}_k; \mathbb{F}_2\big) \cong \mathbb{F}_2 \otimes_{P(n)_*} \overline{P(n)}_*(\underline{P(n)}_k)$$

is the tensor product of the subalgebras listed in (iv), interpreted as $\mathbb{F}_2$-algebras.

To complete this description, we need the structure maps $*$, $\circ$, ψ, ϵ, and χ, which are all (bi)linear. We know ψ, ϵ, and χ on each generator e, $a_{(i)}$, $b_{(j)}$ and $[v_k]$; then the Hopf ring laws determine these operations in general.

11.3 Reduction to Standard Form

We reprove part of Theorem 11.1 by showing that we have enough relations to reduce any Hopf ring expression to a $P(n)_*$-linear combination of $*$-products (11.4) of allowable $\circ$-monomials.

For $\circ$, we need to know how to $\circ$-multiply any two $\circ$-monomials (11.1); then the distributive laws for $(a*b)\circ c$ and $a\circ(b*c)$ **[**modified if $p=2$**]** take care of general $*$-monomials z^{*M} as in (11.4). As the $\circ$-generators $\circ$-commute up to sign **[**even for $p=2$**]**, all we need is a reduction formula for each non-allowable $\circ$-monomial (11.1).

The relation $e\circ e=-b_{(0)}$ takes care of $e^{\circ 2}$. If p is odd, $a_{(i)}^{\circ 2}=0$ is automatic, by Proposition 10.2(ix). **[**If $p=2$, we use $a_{(i)}^{\circ 2}=b_{(i+1)}$ instead, from (10.10).**]**

For the disallowed monomials (i) and (ii) of Definition 8.1, we use the same relations as in Lemma 8.3, now working modulo $*$-decomposables as well. These use only the relations $(\mathcal{R}_k)$ for $k>n$ and (10.6), which implies $(\mathcal{R}_n)$.

For the $*$-product of two $*$-monomials (11.4), we shuffle the $\circ$-monomials into the desired order (with the appropriate sign), and deal with excess $*$-powers of any $\circ$-monomial. **[**If $p=2$, shuffling introduces extra terms, but the process quickly terminates, because the $*$-commutator $c*d-d*c$ of any two $\circ$-monomials is $*$-central; see [Bo] for details.**]**

11.4 The Frobenius Operator

To finish the reduction to standard form, we need a formula for the Frobenius operator $Fc=c^{*p}=c*c*\cdots*c$ on each allowable $\circ$-monomial c of even degree **[**or any degree if $p=2$**]**.

We start from the relation $[1]^{*p}=1_0$, which we rewrite as $F([1]-1_0)=0$. We next reverse the identity [BJW95, (15.13)] as

$$F(c\circ b^{\circ J})=(Fc)\circ b^{\circ 0,J}, \tag{11.5}$$

where $0, J$ denotes the extended multi-index $(0, j_0, j_1, j_2, \dots)$. The proof used only the property $\psi b_k=\sum_{i+j=k} b_i\otimes b_j$. Since a_k has the same property when p is odd, according to Proposition 10.2(ii), we similarly have $F(a^{\circ I,0}\circ c)=a^{\circ 0,I}\circ Fc$ for any multi-index $I=(i_0,i_1,i_2,\dots,i_{n-2})$. **[**For $p=2$, Proposition 10.3 delivers the same result, and also $F(e\circ c)=a_{(0)}\circ Fc$.**]** For $a_{(n-1)}$, we rewrite the relation (10.8) as $Fa_{(n-1)}=v_n a_{(0)}-a_{(0)}\circ b_{(0)}^{\circ N}\circ[v_n]$. Since applying $-\circ[v^K]$ preserves $*$-multiplication, we immediately have $F(c\circ[v^K])=(Fc)\circ[v^K]$.

Combining these, we find the general formulae $F(a^{\circ I,0}\circ b^{\circ J}\circ[v^K])=0$ and (with attention to the shuffles needed and the resulting signs)

$$F(a^{\circ I,1}\circ b^{\circ J}\circ[v^K])=(-1)^{|I|+1}a^{\circ 1,I}\circ b^{\circ N,J}\circ[v_n v^K]+(-1)^{|I|}v_n a^{\circ 1,I}\circ b^{\circ 0,J}\circ[v^K], \tag{11.6}$$

where $|I|=\sum_r i_r$.

[If $p=2$, we need also the formulae involving e, which are

$$F(e\circ a^{\circ I,0}\circ b^{\circ J}\circ[v^K])=0,$$

(11.7) $F(e \circ a^{\circ I,1} \circ b^{\circ J} \circ [v^K]) =$

$$a^{\circ 0,I} \circ b_{(1)} \circ b^{\circ N,J} \circ [v_n v^K] + v_n a^{\circ 0,I} \circ b_{(1)} \circ b^{\circ 0,J} \circ [v^K],$$

in which we make use of $a_{(0)} \circ a_{(0)} = b_{(1)}$. For example,

$$F(e \circ a_{(n-1)} \circ b_{(0)}^{\circ N}) = b_{(0)}^{\circ N+2^{n+1}} \circ [v_{n+1}] + v_n b_{(1)}^{\circ 2^n} + v_n^2 b_{(0)}^{\circ N} \circ b_{(1)} + v_{n+1} b_{(0)}^{\circ 2^n}$$

after reduction to standard form, which recovers (7.17).**]**

11.5 A Reduction Formula

There is a difficulty with (11.6) which obscures the algebraic structure of the Hopf ring. Even in the simple case

$$F(a_{(n-1)} \circ b^{\circ G}) = -a_{(0)} \circ b^{\circ N,G} \circ [v_n] + v_n a_{(0)} \circ b^{\circ 0,G},$$

with G as in (11.3), the first term on the right is visibly not allowable (unless $q = n$, so that $G = 0$). What we need is a reduction formula for

$$b^{\circ 0,G} \circ [v_n] = b_{(d_n+1)}^{\circ p^n} \circ b_{(d_{n+1}+1)}^{\circ p^{n+1}} \circ \cdots \circ b_{(d_{q-1}+1)}^{\circ p^{q-1}} \circ [v_n],$$

which is essentially [RW96, Lemma 3.8]. It involves the p-th $\circ$-power of $b^{\circ G}$,

$$(b^{\circ G})^{\circ p} = b^{\circ pG} = b_{(d_n)}^{\circ p^{n+1}} \circ b_{(d_{n+1})}^{\circ p^{n+2}} \circ \cdots \circ b_{(d_{q-1})}^{\circ p^q}.$$

Lemma 11.4 *Using only the main relations* $(\mathcal{R}_k)$*, the* $\circ$*-monomial* $b^{\circ 0,G} \circ [v_n]$*, with* G *as in* (11.3)*, reduces to an allowable monomial by a formula of the form*

$$b^{\circ 0,G} \circ [v_n] \equiv (-1)^{q-n} b^{\circ pG} \circ [v_q] + \cdots,$$

where the omitted terms do not involve any $a_{(i)}$ *and either* (i) *have the form* $b^{\circ J} \circ [v_k]$ *with* $b^{\circ J}$ *lexicographically higher than* $b^{\circ pG}$ *(see Section 8),* (ii) *lie in the ideal* $\mathfrak{V} = (v_n, v_{n+1}, \ldots)$*,* (iii) *have* $[v]$*-length at least 2, or* (iv) *are* $*$*-decomposable.*

We apply this to (11.6) **[**also (11.7) if $p = 2$**]**.

Corollary 11.5 *For the general allowable* $\circ$*-monomial* (11.2a) *without e, we have*

(11.8) $F(a^{\circ I,1} \circ b^{\circ G+L} \circ [v^K]) \equiv (-1)^{q-n+|I|+1} a^{\circ 1,I} \circ \{b^{\circ N,L} \circ b^{\circ pG} \circ [v_q v^K] + \cdots\}.$

[*If* $p = 2$*, we similarly obtain*

(11.9) $F(e \circ a^{\circ I,1} \circ b^{\circ G+L-\Delta_0} \circ [v^K]) \equiv a^{\circ 0,I} \circ \{b^{\circ N,L} \circ b^{\circ 2G} \circ [v_q v^K] + \ldots\}$

from (11.2b).**]**

The leading term on the right in (11.8) is always allowable: written in standard form (11.2), it is $a^{\circ 1,I} \circ b^{\circ G+L'} \circ [v^{K'}]$, with the same G, $v^{K'} = v_q v^K$, and $b^{\circ L'} = b^{\circ N,L} \circ b^{\circ (p-1)G}$. Careful bookkeeping shows that as the indices vary, it runs through all the Q-disallowed $\circ$-monomials of type Definition 8.1(iii) that are nevertheless allowable, once each. **[**Similarly, (11.9) accounts for types (iv) and (v).**]**

It follows that F never kills anything unexpected. Now we can read off parts (iii) and (iv) of Theorem 11.3.

Proof of Lemma 11.4 For $n \le r \le q$, we set $c_r = b_{(d_n)}^{\circ p^{n+1}} \circ b_{(d_{n+1})}^{\circ p^{n+2}} \circ \cdots \circ b_{(d_{r-1})}^{\circ p^r}$, so that (conventionally) $c_n = b^{\circ 0}$ and $c_q = b^{\circ pG}$.

We show first that $c_r \circ [v_s] \equiv 0$ whenever $n \le s < r \le q$, by induction on s. We $\circ$-multiply $(\mathcal{R}_{d_s+s})$ by c_s; by (7.10), the k-th term is $c_s \circ b_{(d_s+s-k)}^{\circ p^k} \circ [v_k]$. If $k < s$, this term is neglected by induction. (If $s = n$, there are no such terms.) If $k > s$, we have $d_s + s - k < d_s$, and this term is lexicographically higher. If $k = s$, we have $c_s \circ b_{(d_s)}^{\circ p^s} \circ [v_s]$, which gives $c_{s+1} \circ [v_s] \equiv 0$ when we $\circ$-multiply by $b_{(d_s)}^{\circ p^{s+1}-p^s}$; hence $c_r \circ [v_s] \equiv 0$ for any $r > s$, if we $\circ$-multiply by further factors.

Then we show that $c_s \circ b_{(d_s+1)}^{\circ p^s} \circ [v_s] \equiv -c_{s+1} \circ [v_{s+1}]$ for $n \le s < q$, from which the result follows by induction, starting from $c_n = b^{\circ 0}$. We $\circ$-multiply $(\mathcal{R}_{d_s+s+1})$ by c_s. The k-th term is $c_s \circ b_{(d_s+s+1-k)}^{\circ p^k} \circ [v_k]$, which we have just shown is negligible if $k < s$. If $k > s+1$, we have $d_s + s + 1 - k < d_s$, and the term is lexicographically higher. The two remaining terms, with $k = s$ and $k = s+1$, are the desired terms. ∎

12 Effect on Homotopy Groups

Given an unstable operation $r\colon P(n)^k(-) \to P(n)^m(-)$, where $k, m > 0$, consider the homomorphism of homotopy groups $r_*\colon \Sigma^k P(n)_* \to \Sigma^m P(n)_*$ (see diagram (3.5)) induced by the representing map $r\colon \underline{P(n)}_k \to \underline{P(n)}_m$. By [BJW95, Lemma 13.9], it is given on $\Sigma^k v$, where $v \in P(n)_i$, by the unstable analogue of (7.4), namely

$$r_* \Sigma^k v = \Sigma^m \langle r, e_{k+i} \circ [v] \rangle, \tag{12.1}$$

where $e_{2j} = b_{(0)}^{\circ j}$ and $e_{2j+1} = e \circ b_{(0)}^{\circ j}$.

We therefore seek more information on the relations in the Hopf ring.

12.1 The First Higher-Order Relation

We need the Hopf ring version for $P(n)$ of Bendersky's lemma [Be86, Theorem 6.2], which immediately implies Lemma 8.7.

Lemma 12.1 *For $q \ge n$, we have in $P(n)_*(\underline{P(n)}_{g(n,q-1)+1})$ the reduction formula*

$$e_{g(n,q)-1} \circ [v_q] \equiv v_q e_{g(n,q-1)+1} \mod I_q P(n)_*(\underline{P(n)}_{g(n,q-1)+1}). \tag{12.2}$$

[*If $p = 2$, this is almost superseded by*

$$e_{g(n,q)-2} \circ [v_q] \equiv v_q e_{g(n,q-1)} + F(e_{g(n,q-1)-1} \circ a_{(n-1)}) \mod I_q, \text{ for } q \ge n+1. \tag{12.3}$$**]**

Proof We establish (12.2) by induction on q. For $q = n$, it follows immediately from (10.6). For $q > n$, we return to the definition of the relation $(\mathcal{R}_q)$. We expand $[p](x) = \sum_K \lambda(K) v^K x^{d(K)}$, summing over multi-indices K, with coefficients $\lambda(K) \in \mathbb{F}_p$ and exponents $d(K)$; then if we write $b(x) = 1_2 + \overline{b}(x)$, (10.5) becomes

$$1_2 + \overline{b}\Big(\sum_K \lambda(K) v^K x^{d(K)}\Big) = \mathop{\ast}_K \{1_2 + \overline{b}(x)^{\circ d(K)} \circ [v^K]\}^{*\lambda(K)}. \tag{12.4}$$

We apply the suspension $e_h \circ -$, where $h = g(n, q-1) - 1$, which kills 1_2 and most $*$-products and thus drastically simplifies (12.4) to

$$e_h \circ \overline{b}\Big(\sum_K \lambda(K) v^K x^{d(K)}\Big) = \sum_K \lambda(K) e_h \circ \overline{b}(x)^{\circ d(K)} \circ [v^K].$$

We take the coefficients of x^{p^q} and work mod I_q. On the left, by (6.10), the only surviving term in $[p](x)$ is $v_q x^{p^q}$, giving $e_h \circ v_q b_{(0)}$, the right side of (12.2). On the right, $e_h \circ [v_k] \equiv 0$ for all $k < q$, by induction on q, since $h = g(n, q-1) - 1 \geq g(n,k) - 1$. This leaves only $e_h \circ b_{(0)}^{\circ p^q} \circ [v_q]$, as required.

[For (12.3), we take $h = g(n, q-1) - 2$ instead. We still have enough e's to kill $[v_k]$ for any $k < q-1$, but not $[v_{q-1}]$. By (6.8), the only terms of interest in $[2](x)$ are $v_{q-1}x^{2^{q-1}}$ and $v_q x^{2^q}$. Instead of (12.2), we find

$$e_h \circ v_q b_{(0)} \equiv e_h \circ b_{(1)}^{\circ 2^{q-1}} \circ [v_{q-1}] + e_h \circ b_{(0)}^{\circ 2^q} \circ [v_q].$$

The first and third terms appear as the second and first terms in (12.3).

The second term is not allowable; but if $q = n+1$, we use (11.7) to write it as

$$b_{(0)}^{\circ N} \circ b_{(1)}^{\circ 2^n} \circ [v_n] \equiv F(e \circ a_{(n-1)} \circ b_{(0)}^{\circ N}).$$

If $q > n+1$, we have $e_h \circ [v_{q-1}] \equiv F(e_{g(n,q-2)-1} \circ a_{(n-1)})$ by induction. Then

$$e_h \circ b_{(1)}^{\circ 2^{q-1}} \circ [v_{q-1}] \equiv b_{(1)}^{\circ 2^{q-1}} \circ F(e_{g(n,q-2)-1} \circ a_{(n-1)}) = F(b_{(0)}^{\circ 2^{q-1}} \circ e_{g(n,q-2)-1} \circ a_{(n-1)}),$$

as required, with the help of (11.5).**]** ∎

12.2 Proofs for Section 3

Now we can finish the proofs of two lemmas.

Proof of Lemma 3.4 For (i), by (12.1),

$$r_* \Sigma^k(v_n v) = \Sigma^m \langle r, e_{k+q+2N} \circ [v_n v] \rangle = \Sigma^m \langle r, e_{k+q+2N} \circ [v_n] \circ [v] \rangle.$$

Since $k + q > 0$, we can use (10.6) to rewrite this as

$$r_* \Sigma^k(v_n v) = \Sigma^m \langle r, v_n e_{k+q} \circ [v] \rangle = v_n r_* \Sigma^k v.$$

Part (ii) is similar, with (12.2) in place of (10.6). ∎

Lemma 12.2 *Let $r\colon \underline{P(n)}_k \to \underline{P(n)}_k$ be any map, where $k > g(n, m-1)$, and suppose that on homotopy, $r_*\colon \Sigma^k P(n)_* \to \Sigma^k P(n)_*$ is given on the bottom class by $r_*\Sigma^k 1 = \lambda\Sigma^k 1$, where $\lambda \in \mathbb{F}_p$. Then on any monomial v^K in the elements v_n, v_{n+1}, …, v_m, r_* has the form*

$$r_*\Sigma^k v^K = \lambda\Sigma^k v^K + \sum_{L>K} c_L \Sigma^k v^L \tag{12.5}$$

with coefficients $c_L \in \mathbb{F}_p$, where we order the multi-indices $L = (l_n, l_{n+1}, \dots)$ lexicographically (as in Section 8).

Proof We use induction on the length of v^K, starting from $K = 0$. If (12.5) holds for $\Sigma^k v^K$, Lemma 3.4(ii) gives

$$r_*\Sigma^k(v_q v^K) \equiv \lambda\Sigma^k(v_q v^K) + \sum_{L>K} c_L \Sigma^k(v_q v^L) \mod I_q.$$

If we assume (as we may) that v^K contains no factors v_t with $t < q$, all monomials in I_q will be larger lexicographically than $v_q v^K$, and we have the result for $v_q v^K$. ∎

Proof of Lemma 3.2 Take any map $f\colon \underline{P(n,m)}_k \to \underline{P(n,m)}_k$, where $g(n, m-1) < k \le g(n, m)$. Suppose $f_*\Sigma^k 1 = \lambda\Sigma^k 1$. We apply Lemma 12.2 to the composite

$$\underline{P(n)}_k \xrightarrow{\rho(m)} \underline{P(n,m)}_k \xrightarrow{f} \underline{P(n,m)}_k \xrightarrow{\overline{\theta(m)}} \underline{P(n)}_k$$

to deduce that

$$\overline{\theta(m)}_* f_* \Sigma^k v^K = \lambda\Sigma^k v^K + \sum_{L>K} c_L \Sigma^k v^L$$

for any monomial v^K in the generators v_n, v_{n+1}, …, v_m. We apply $\rho(m)_*$, to see that $f_*\Sigma^k v^K$ has the same form (possibly with some terms v^L deleted). It is now clear that if $\lambda \neq 0$, f_* is an isomorphism and f is a homotopy equivalence. ∎

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