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# On the Hopf ring for $ER(n)$

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**Abstract**

Kriz and Hu construct a real Johnson–Wilson spectrum, $ER(n)$, which is $2^{n+2}(2^n-1)$ periodic. $ER(1)$ is just $KO_{(2)}$. We do two things in this paper. First, we compute the homology of the $2^n-1$ spaces $\underline{ER(n)}_{2^{n+2}k}$ in the Omega spectrum for $ER(n)$. It turns out the double of these Hopf algebras gives the homology Hopf algebras for the even spaces for $E(n)$. As a byproduct of this we get the homology of the zeroth spaces for the Omega spectrum for real complex cobordism and real Brown–Peterson cohomology. The second result is to compute the homology Hopf ring for all 48 spaces in the Omega spectrum for $ER(2)$. This turns out to be generated by very few elements.


*Keywords:* Hopf ring; Omega spectra; Johnson–Wilson spectra; Real spectra

## 1. Introduction

Kriz and Hu construct [5] a real Johnson–Wilson spectrum, $ER(n)$, which is $2^{n+2}(2^n-1)$ periodic and has a ring spectrum map $ER(n)\to E(n)$, where $E(n)$ is the standard Johnson–Wilson $2(2^n-1)$ periodic spectrum for the prime 2, [8]. In [10] it was shown that the fiber of this map was just another copy of $ER(n)$, i.e. we have the fibration

$$\Sigma^{\lambda(n)}ER(n)\to ER(n)\to E(n),$$

where $\lambda(n)=2^{2n+1}-2^{n+2}+1$. The $n=1$ example is just $KO_{(2)}\to KU_{(2)}$. The $n=2$ case, $ER(2)$, is equivalent to an uncompleted version of the fixed point spectrum of a finite group acting on the Hopkins–Miller spectrum $E_2$. Rezk–Mahowald have also constructed a model for $ER(2)$, known as $TMF(3)$.

The homology of the Omega spectrum for the $E(n)$ is well known; [4,7]. We are concerned with the homology of the Omega spectrum for $ER(n)$. Of course the homology for $n=1$ is well known. We have two main results. First, we compute the homology of the spaces indexed by multiples of $2^{n+2}$. There are $2^n-1$ of these spaces. These have a nice description in terms of the homology of the even spaces in the Omega spectrum for $E(n)$. Second, we compute this homology for all 48 spaces in the Omega spectrum for the $n=2$ case.

Let $\Phi$ be the doubling functor on our Hopf algebra, i.e. $(\Phi B)_i=B_{i/2}$. Recall that the grading for the spaces for $\underline{E(n)}_*$ is over $\mathbb{Z}/(2(2^n-1))$ but the grading for the spaces $\underline{ER(n)}_*$ is over $\mathbb{Z}/(2^{n+2}(2^n-1))$ and that there is

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a standard reduction map $\mathbb{Z}/(2^{n+2}(2^n-1)) \to \mathbb{Z}/(2(2^n-1))$. Note that $-2^{n+2}(2^{n-1}-1)k$ goes to $2k$ under this map.

**Theorem 1.1.** *There is an isomorphism of Hopf rings*

$$\Phi H_* \underline{ER(n)}_{-2^{n+2}(2^{n-1}-1)*} \simeq H_* \underline{E(n)}_{2*}.$$

*The Hopf ring map*

$$H_* \underline{ER(n)}_{-2^{n+2}(2^{n-1}-1)*} \to H_* \underline{E(n)}_{2*}$$

*is a surjection and takes $x$ to $V(\Phi(x))$ where $V$ is the Vershiebung.*

This result is very informative since we know all about the homology $H_*\underline{E(n)}_{2*}$. It is polynomial and there is an explicit basis for the generators. In the case $n=1$ this recovers the homology of $Z \times BO$ from that of $Z \times BU$.

We use this as our starting point for our computation of $H_*\underline{ER(2)}_*$. When $n=2$ we begin by knowing the homology of the three spaces in $H_*\underline{ER(2)}_{-16*}$. Our technique is straightforward, we just deloop and use the bar spectral sequence to compute $H_*\underline{ER(2)}_{-16*+i}$ as $i$ runs from 1 to 15. One could conceivably do this for higher $n$ but the number of steps would be $2^{n+2}-1$. A more compact form of the answer is probably necessary before more progress is made. However, it is fairly straightforward to get started. In particular, it is easy to get $H_*\underline{ER(n)}_{-2^{n+2}(2^{n-1}-1)*+k}$ where $k=\pm 1$ and $\pm 2$. You can use Cotor to loop down as well as Tor to deloop.

Motivation for the $n=2$ case varied with time. To begin with we were struck by the fact that 5 out of 8 of the spaces in the real Bott periodicity case for $KO$ have no torsion. This is the $ER(1)$ case for us. With $ER(2)$ we had 48 spaces and we thought there might be quite a few with no torsion and since we knew the homotopy we thought they might be interesting examples to study. By the time we figured out that all of the spaces had torsion we were too deeply involved to back out. In addition, the connection of $ER(2)$ to $TMF(3)$ seemed interesting enough to keep us going. Our initial interest really came from our computation of the Morava $K$-theory of the spaces in the Omega spectrum for connective real $K$-theory, [9]. We were looking for other similar things to try out our techniques on when we became enamored with the $ER(n)$.

In addition to the sub-Hopf ring of $H_*\underline{ER(n)}_*$ we have given above we also have another known sub-Hopf ring, namely, the zero degree homology, which is just the group-ring, (really the ring-ring), on the homotopy of the spectrum. The homotopy of an Omega spectrum determines the zeroth homology group of the entire spectrum. If $y$ is in the $k$th homotopy group then $y$ is also an element of the zeroth homotopy group of the $-k$ space in the Omega spectrum. The Hurevicz homomorphism then gives us an element, $[y]$, in the zeroth homology of the $-k$ space. So, we have $H_0\underline{ER(n)}_* \simeq \mathbb{Z}/(2)[ER(n)^*]$.

The homotopy of $E(n)$ is $\mathbb{Z}_{(2)}[v_1, v_2, \ldots, v_n, v_n^{-1}]$ and the degree of $v_i$ is $2(2^i-1)$. However, we set the element $[v_n]$ equal to $[1]$ to grade our spaces over $\mathbb{Z}/(2(2^n-1))$. $H_0\underline{E(n)}_*$ is then the group ring on $\mathbb{Z}_{(2)}[v_1, v_2, \ldots, v_{n-1}]$ accompanied by the caution that because of the grading we have an infinite number of elements in each even space. We will identify the periodicity element for $ER(n)$ with $[1]$ as well. This simplifies notation significantly by giving us just a finite number of spaces to consider. The spaces $\underline{E(n)}_{2(2^n-1)i+k}$ are the same for all $i$ because of the invertibility of $[v_n]$ so we need only keep one of them around. If we do not invert $[v_n]$ by setting it equal to $[1]$ then the space we would be studying is not the product of all of these spaces but the colimit with respect to $j$ of the product of $\underline{E(n)}_{2(2^n-1)i+k}$ for $-j<i<j$. We need to mention a couple of other elements. There is the element $e \in H_1\underline{ER(n)}_1$ which is just the suspension of the homotopy identity, $[1] \in H_0\underline{ER(n)}_0$. We have elements coming from real projective space. Our generators there are $\beta_{(i)} \in H_{2^i}\underline{ER(n)}_{-2^{n+2}(2^{n-1}-1)}$. We have two products in our Hopf ring, the circle product which comes from the ring structure (we suppress the circle in our notation) and the star product $(*)$ coming from the Hopf algebra structure. The Hopf ring for $\underline{E(n)}_{2*}$ is generated by $\mathbb{Z}/(2)[E(n)^*]$ and the corresponding elements from complex projective space $b_{(i)} \in H_{2^{i+1}}\underline{E(n)}_2$. There are few relations (which come from the formal group law). The general Hopf ring reference is [12]. Note that we have $\beta_{(i+1)}$ maps to $b_{(i)}$.

For the $n=2$ case we can give the generators of the homotopy without describing the whole thing. There is an $x$ in degree 17, $\alpha_i$ in degree $12i$ for $i=1$, 2, and 3, $w$ in degree 8, and $\alpha$ in degree 32. The $[\alpha]$ and $[w]$ map to $[v_1]$ in $H_0\underline{ER(2)}_*$ and the $[\alpha_i]$ all map to $[2] \in H_0\underline{E(2)}_*$. The element $[x]$ goes to zero.

We have (recall our homology is always mod 2):

**Theorem 1.2.** *The Hopf ring $H_*\underline{ER(2)}_*$ is generated by the element $e \in H_1\underline{ER(2)}_1$ and the two sub-Hopf rings $H_*\underline{ER(2)}_{-16*}$ and $\mathbb{Z}/(2)[ER(2)^*] \simeq H_0\underline{ER(2)}_*$. There are only two types of relations that have to be introduced. First, we need to know what happens to the suspensions of the homotopy generators*:

$$e^8 = e^8[1] = [\alpha_2]\beta_{(3)} + \beta_{(2)}^2[w] + \left(\beta_{(1)}^2\right)^{*2}[w] + \left(\left(\beta_{(0)}^2\right)^{*2} * \beta_{(1)}^2\right)[w],$$
$$e[x] = \beta_{(0)}, \qquad e^4[\alpha] = [\alpha_1]\beta_{(2)}, \qquad e^4[w] = [\alpha_3]\beta_{(2)} + \left[x^4\right]\beta_{(1)}^2,$$
$$e^2[\alpha_1] = \left[x^2\right]\beta_{(1)}[w], \qquad e^2[\alpha_2] = \left[x^6\right]\beta_{(1)}, \qquad e^2[\alpha_3] = \left[x^2\right]\beta_{(1)}[\alpha].$$

*The second type of relation we need to know is a few of the squares of what we might call the fundamental class*:

$$e^{*2} = e\beta_{(0)}[\alpha],$$
$$\left(e^2\right)^{*2} = e^2\beta_{(1)}[\alpha], \qquad \left(e^3\right)^{*2} = e^3\beta_{(0)}^3,$$
$$\left(e^4\right)^{*2} = e^4\beta_{(0)}^2\beta_{(1)} + [\alpha_1]\beta_{(2)}^2, \qquad \left(e^5\right)^{*2} = e^5\beta_{(0)}\beta_{(1)}^2,$$
$$\left(e^6\right)^{*2} = e^6\beta_{(1)}^3 + \left[x^2\right]\beta_{(2)}^3[w], \qquad \left(e^7\right)^{*2} = 0.$$

The obvious conjecture to make is that something similar happens with $ER(n)$, i.e., that $H_*\underline{ER(n)}_*$ is generated by $e$ and the two known sub-Hopf rings $H_*\underline{ER(n)}_{-2^{n+2}(2^{n-1}-1)*}$ and $\mathbb{Z}/(2)[ER(n)^*]$ and that the relations all come from the suspensions of the homotopy generators and the squares of $e^i$ for $0 < i < 2^{n+1}$. This certainly works for the $n = 1$ case. Another view of this is that the entire Hopf ring is generated by the elements, $e$, $\beta_{(i)}$ and the homotopy in the form of the zero degree homology $\mathbb{Z}/(2)[ER(n)^*]$.

Our first result is actually a corollary of a much more general result. Our work begins with a real spectrum, $\mathbb{E}$, as defined in [5]. In particular, $\mathbb{E}$ is given by a collection of pointed $\mathbb{Z}/(2)$-spaces $\mathbb{E}_V$ indexed by the representation ring $RO(\mathbb{Z}/(2))$ of the group $\mathbb{Z}/(2)$. Recall that $RO(\mathbb{Z}/(2)) = \mathbb{Z} \oplus \mathbb{Z}\alpha$, where $\alpha$ is the sign representation. Moreover, we require that the spaces $\mathbb{E}_V$ be compatible in the following sense:

Given a representation $U$, and a pointed $\mathbb{Z}/(2)$-space $X$, let $\Omega^U X$ denote the space $\mathrm{Map}_*(S^U, X)$, where $S^U$ is the one-point compactification of $U$. The space $\Omega^U X$ has an induced diagonal action of the group $\mathbb{Z}/(2)$. For the spectrum $\mathbb{E}$, we require the existence of a family of equivariant homeomorphisms $\alpha_{U,V} : \Omega^U \mathbb{E}_{U\oplus V} \to \mathbb{E}_V$, that satisfy obvious compatibility.

A multiplicative real spectrum $\mathbb{E}$ is one that admits a multiplication preserving the real structure (see [5]).

Complex cobordism, the Brown–Peterson spectrum, $BP\langle n\rangle$, the Johnson–Wilson spectra, $\mathbb{E}(n)$, all have an appropriate $\mathbb{Z}/(2)$ action coming from complex conjugation which gives them associated real spectra, [5]. These spectra are in fact multiplicative real spectra which are homotopy commutative.

We will use the notation $\mathbb{ER}_V$ to denote the homotopy fixed points of the $\mathbb{Z}/(2)$-action on $\mathbb{E}_V$. Notice that for a fixed $V \in RO(\mathbb{Z}/(2))$, the collection of spaces $\{\mathbb{ER}_{n+V}, n \in \mathbb{Z}\}$ form a spectrum in the usual sense. We shall abuse notation and refer to the spectra $\{\mathbb{ER}_{n+V}, n \in \mathbb{Z}\}$ and $\{\mathbb{E}_{n+V}, n \in \mathbb{Z}\}$ as the spectra $\mathbb{ER}_V$ and $\mathbb{E}_V$ respectively. The Omega spectrum for $ER(n)$ is given by the spaces $\underline{ER(n)}_k = \mathbb{ER}(n)_{k+0\alpha}$. The underlying space $\mathbb{E}_{i+j\alpha}$ does not depend on the $\mathbb{Z}/(2)$ action. In the cases for the Omega spectra of interest to us we have $\mathbb{E}_{i+j\alpha} = \underline{E}_{i+j}$.

All of our homology groups are with $\mathbb{Z}/(2)$ coefficients.

**Definition 1.3.** A $\mathbb{Z}/(2)$-space $X$ is said to be projective if

(i) $H_*(X)$ is of finite type.
(ii) $X$ is homeomorphic to $\bigvee_I (\mathbb{C}P^\infty)^{\wedge k_I}$ for some weakly increasing sequence of integers $k_I$, with the $\mathbb{Z}/(2)$ action given by complex conjugation.

**Definition 1.4.** An $H$-space $\mathbb{E}_V$ is said to have the projective property if there exists a projective space $X$, along with a $\mathbb{Z}/(2)$-equivariant map $f : X \to \mathbb{E}_V$, such that $H_*(\mathbb{E}_V)$ is generated as an algebra by the image of $f$.

Such spaces are not rare because many spaces have homology generated by the image of elements coming from complex projective space. This definition covers such spaces even if they must use another product of the elements to get the generators such as in the case of the Hopf ring product. Our examples all come up in this way. In particular, our examples include $\mathbb{MU}_{k(1+\alpha)}$, $\mathbb{BP}_{k(1+\alpha)}$, and $\mathbb{BP}\langle n\rangle_{k(1+\alpha)}$ where this last is only for $k \leqslant 2^{n+1}-1$. Only these diagonal spaces have the correct $\mathbb{Z}/(2)$ action. Keeping these spaces in mind we have:

**Theorem 1.5.** *Let $\mathbb{E}_V$ have the projective property, then there is an isomorphism of Hopf algebras $H_*(\mathbb{E}_V) = \Phi H_*(\mathbb{ER}_V)$. Moreover, the natural map $H_*(\mathbb{ER}_V) \to H_*(\mathbb{E}_V)$ is given by taking $x$ to $V(\Phi(x))$.*

This theorem immediately gives us the homologies $H_*\mathbb{MUR}_{k(1+\alpha)}$ and $H_*\mathbb{BP}_{k(1+\alpha)}$. In particular, when $k=0$ we get the homology of the zeroth spaces for the Omega spectra for $MUR$ and $BPR$ respectively. This is because in [12] it is shown that the Hopf rings for $MU$ and $BP$ are generated by the zeroth homology and elements coming from complex projective space. In [13] (and later in [1,2]) it is shown that the spaces for $BP\langle n\rangle$ split off of those for $BP$ when $k \leqslant 2(2^{n+1}-1)$. Consequently the result holds for them too.

The theorem does not apply directly to the spaces we care most about, the $\mathbb{ER}(n)_V$, but the diagonal spaces here are the limit of the diagonal spaces for the $\mathbb{BPR}\langle n\rangle$ case and the same result holds.

To get from the diagonal spaces to the Omega spectrum spaces we are interested in requires a theorem from [10] giving an equivalence

$$\Sigma^{\lambda(n)}\mathbb{ER}(n)_V \to \mathbb{ER}(n)_{V-\alpha}.$$

Translating this into our situation the map

$$\mathbb{ER}(n)_{k(1+\alpha)} \to \mathbb{E}(n)_{k(1+\alpha)}$$

becomes

$$\underline{ER(n)}_{-2^{n+2}(2^{n-1}-1)k} \to \underline{E(n)}_{2k}.$$

Since $H_*\mathbb{E}(n)_{k+k\alpha} = H_*\underline{E(n)}_{2k}$, we have $[v_n] \in H_0\underline{E(n)}_{-2(2^n-1)}$ which is $H_0\mathbb{E}(n)_{-(2^n-1)(1+\alpha)}$. Along this diagonal, these homotopy elements are fixed under the action and so this gives rise to an element in $H_0\mathbb{ER}(n)_{-(2^n-1)(1+\alpha)}$ which is $H_0\underline{ER(n)}_{(2^n-1)2^{n+2}(2^{n-1}-1)}$. This element is just a power of our periodicity element which we have set equal to [1]. (In the case $n=2$ it is exactly our periodicity element.) For $n=2$ we have $[v_1^i] \in H_0\underline{E(2)}_{-2i} = H_0\mathbb{E}(2)_{-i-i\alpha}$ which gives rise to $[\alpha^i] \in H_0\mathbb{ER}(2)_{-i-i\alpha} = H_0\underline{ER(2)}_{16i}$. This is why we like to view the homotopy of $ER(2)$ from the point of view of $\alpha$. Apologies for the double use of $\alpha$ in our notation. Recalling that our indexing is over $\mathbb{Z}/(48)$, the $-k$ space is also the $48-k$ space.

In the next section we give the details of the description of the Hopf ring for $H_*\underline{ER(2)}_*$. After that we describe the homotopy of $ER(2)$. We then review Lannes's theory, prove the main theorem, review the homology for $E(2)$ and the bar spectral sequence. After that we do the computation for the Hopf ring for $ER(2)$. Finally, we have a brief appendix dealing with the $ER(1) = KO_{(2)}$ case.

## 2. Detailed results

**Proposition 2.1.** *The homotopy of $ER(2)$, $\pi_*ER(2)$, graded over $\mathbb{Z}/(48)$ is generated by elements, $x$, $w$, $\alpha$, $\alpha_1$, $\alpha_2$, and $\alpha_3$ of degrees* 17, 8, 32, 12, 24, *and* 36 *respectively. The relations are given by*:

$$0 = 2x = x^7 = x^3w = x^3\alpha = x\alpha_i, \qquad w\alpha_2 = 2\alpha, \qquad \alpha^2 = w^2,$$
$$\alpha_1^2 = 2\alpha_2, \qquad \alpha_2^2 = 4, \qquad \alpha_3^2 = 2\alpha_2,$$
$$\alpha_1\alpha_2 = 2\alpha_3, \qquad \alpha_1\alpha_3 = 4, \qquad \alpha_2\alpha_3 = 2\alpha_1,$$
$$\alpha\alpha_1 = \alpha_3 w, \qquad \alpha\alpha_2 = 2w, \qquad \alpha\alpha_3 = \alpha_1 w.$$

*As a module over $\mathbb{Z}_{(2)}[\alpha]$, the homotopy can be described as having generators*:

$$1, \quad w, \quad \alpha_1, \quad \alpha_3, \quad \text{and} \quad \alpha_2$$

*with one relation*:

$$\alpha\alpha_2 = 2w,$$

*copies of* $\mathbb{Z}/(2)[\alpha]$ *on generators*

$$x, \quad x^2, \quad xw, \quad x^2w,$$

*and copies of* $\mathbb{Z}/(2)$ *on*

$$x^3, \quad x^4, \quad x^5, \quad x^6.$$

Recall that we set $[v_2] = [1]$ in our Hopf ring for $H_*\underline{E(2)}_{2*}$. This is generated by the zero degree elements $\mathbb{Z}/(2)[E(2)^*] = \mathbb{Z}/(2)[\mathbb{Z}[v_1^i]]$ and the elements $b_{2^i} = b_{(i)} \in H_{2^{i+1}}\underline{E(2)}_2$ that come from the complex projective space elements $b_j$. We have corresponding elements $\beta_{2^i} = \beta_{(i)} \in H_{2^i}\underline{ER(2)}_{-16} = H_{2^i}\underline{ER(2)}_{1+\alpha}$. The $\beta_{(i)}$ all come from the real projective space elements $\beta_{2^i}$.

We need some definitions.

**Definition 2.2.** Let $J = (j_0, j_1, \ldots)$ have $j_i \geqslant 0$ with only a finite number not equal to zero. We define

$$\beta^J = \beta_{(0)}^{j_0}\beta_{(1)}^{j_1}\beta_{(2)}^{j_2}\cdots.$$

Recall that in a Hopf ring we have two products, the circle product coming from the ring structure and the star product coming from the Hopf algebra structure. We suppress the circle from our notation so the above products are circle products. We define $\beta^J[\alpha^i]$ to be *allowable* if all $j_k < 2$ when $i > 0$ and $J \neq 2\Delta_{i_1} + 4\Delta_{i_2} + J'$, $i_1 \leqslant i_2$, when $i = 0$. This follows [12] (having set $[v_2] = [1]$).

**Definition 2.3.** We let $m(J) = k$ where $k$ is the smallest number with $j_k > 0$.

**Definition 2.4.** We need to establish a systematic abuse of notation. If $z$ is the generator $1 \in \mathbb{Z}_{(2)}$ we have the group ring $\mathbb{Z}/(2)[\mathbb{Z}_{(2)}]$. We will always refer to this as $P[[z]]$, as if it were a polynomial generator. It is not, but this convention allows us to write the zero degree homology with the higher degree homology without always pointing out the exception. Tor of this is just an exterior algebra, just as it is with a polynomial algebra. Likewise, when we say "exterior" we mean the group ring $\mathbb{Z}/(2)[\mathbb{Z}/(2)]$ related to a $\mathbb{Z}/(2)$ element $z$ and we write it $E[[z]]$. Using this (abused) notation we have $H_0\underline{E(2)}_{2*} \simeq P[[v_1^i]]$ and $H_0\underline{ER(2)}_{-16*} \simeq P[[\alpha^i]]$.

**Remark 2.5.** It is important to note that our elements $\beta_{(i)}$ are in the zeroth component of $\underline{ER(2)}_{-16}$. This is an important distinction for people used to working with $K$-theory where these elements are traditionally in the component of [1]. Our elements ultimate derive from the complex orientation $CP^\infty \to \underline{MU}_2$ where there is only the zero component because this space is connected. The elements $\beta_{(i)}$ are created from these by Theorem 1.1 in the introduction and so must also be in the zeroth component there. There are subtle differences between the use of these elements and those in the component of [1] when you study $KO$. To see these in detail see [3] and our appendix on $KO$.

For $E(2)$, an infinite number of copies of $BU$ split off of the zeroth space. We believe something similar happens for $ER(2)$ and $BO$, certainly it happens with homology. The exact nature of these splittings could be quite interesting.

For $n = 2$ we list the homology of the spaces. Our answers are sometimes a little complicated and we have to stack up our terms. When we do that it is understood that we mean the tensor product of these algebras.

$$H_*\underline{ER(2)}_{-16*} \simeq P\big[\beta^J\big[\alpha^i\big]\big] \quad \beta^J\big[\alpha^i\big] \text{ allowable,} \tag{2.6}$$

$$H_*\underline{ER(2)}_{-16*+1} \simeq P\big[e\beta^J\big[\alpha^i\big]\big] \quad \text{with } \beta^{J+\Delta_0}\big[\alpha^i\big] \text{ allowable,} \tag{2.7}$$

$$H_*\underline{ER(2)}_{-16*+2} \simeq \begin{cases} P[e^2\beta^J[\alpha^i]] & j_0 = 0,\ \beta^{J+\Delta_1}[\alpha^i] \text{ allowable,} \\ P[e^2\beta^J] & 0 < j_0,\ J + 2\Delta_0 \text{ allowable,} \end{cases} \tag{2.8}$$

$$H_*\underline{ER(2)}_{-16*+3} \simeq \begin{cases} E[e^3\beta^J[\alpha^i]] & j_0 = 0,\ \begin{matrix}\beta^{J+\Delta_1}[\alpha^i] \text{ allowable and} \\ \beta^{J+2\Delta_1}[\alpha^i] \text{ non-allowable,}\end{matrix} \\ P[e^3\beta^J] & \beta^{J+3\Delta_0} \text{ allowable,} \end{cases} \tag{2.9}$$

$$H_*\underline{ER(2)}_{-16*+4} \simeq \begin{cases} P[[\alpha_1]\beta^J[\alpha^i]] & j_0 = j_1 = 0,\ \beta^J[\alpha^i] \text{ allowable}, \\ P[e^4\beta^J] & \beta^{J+4\Delta_0} \text{ allowable}, \\ P[e^4\beta^{J+2\Delta_0}] & j_0 = j_1 = 0,\ \beta^{J+2\Delta_0} \text{ allowable}, \end{cases} \tag{2.10}$$

$$H_*\underline{ER(2)}_{-16*+5} \simeq \begin{cases} E[e[\alpha_1]\beta^J[\alpha^i]] & j_0 = j_1 = 0,\ \beta^J[\alpha^i] \text{ allowable}, \\ E[e^5\beta^{J+2\Delta_0}] & j_0 = j_1 = 0,\ \beta^{J+2\Delta_0} \text{ allowable}, \\ P[e^5\beta^J] & \beta^{J+4\Delta_0} \text{ allowable},\ J \neq J' + \Delta_0 + 2\Delta_1, \end{cases} \tag{2.11}$$

$$H_*\underline{ER(2)}_{-16*+6} \simeq \begin{cases} E[[x^2]\beta^J[\alpha^i][w]], & j_0 = 0,\ j_1 < 2,\ \beta^J[\alpha^i] \text{ allowable}, \\ E[e^6\beta^{J+\Delta_0}], & j_0 = 0,\ \beta^{J+4\Delta_1} \text{ allowable}, \\ P[e^6\beta^J] & j_0 = 0,\ \beta^{J+3\Delta_1} \text{ allowable}, \end{cases} \tag{2.12}$$

$$H_*\underline{ER(2)}_{-16*+7} \simeq \begin{cases} E[[x]\beta^J[\alpha^i][w]] & j_0 \text{ and } j_1 < 2,\ \beta^J[\alpha^i] \text{ allowable}, \\ E[e^7\beta^J] & j_0 = 0,\ \beta^{J+3\Delta_1} \text{ allowable}, \end{cases} \tag{2.13}$$

$$H_*\underline{ER(2)}_{-16*+8} \simeq \begin{cases} P[\beta^J[\alpha^i][w]] & \begin{array}{l} \beta^J[\alpha^i] \text{ allowable}, \\ \text{if } m(J) = 1 \text{ then } j_1 \leqslant 2, \text{ and if } j_1 = 2 \text{ then } j_2 \leqslant 2, \\ \text{if } m(J) = 0 \text{ then } j_0 \leqslant 2 \text{ and } j_1 < 2, \end{array} \\ P[[\alpha_2]\beta^J] & j_0 = j_1 = j_2 = 0,\ \beta^{J+2\Delta_{m(J)}} \text{ allowable}, \end{cases} \tag{2.14}$$

$$H_*\underline{ER(2)}_{-16*+9} \simeq \begin{cases} P[e\beta^J[\alpha^i][w]] & \begin{array}{l} \beta^J[\alpha^i] \text{ allowable}, \\ j_0 = j_1 = 0, \text{ or} \\ j_0 = 0,\ j_1 = 1, \text{ or} \\ j_0 = 1,\ j_1 < 2,\ \beta^{J+\Delta_0}[\alpha^i] \text{ allowable}, \end{array} \\ E[e\beta^J[w]] & \begin{array}{l} j_0 = 2,\ j_1 < 2,\ \beta^J \text{ allowable, or} \\ j_0 = 0,\ j_1 = 2,\ j_2 \leqslant 2,\ \beta^J \text{ allowable}, \end{array} \\ E[e[\alpha_2]\beta^J] & j_0 = j_1 = j_2 = 0,\ \beta^{J+2\Delta_{m(J)}} \text{ allowable}, \end{cases} \tag{2.15}$$

$$H_*\underline{ER(2)}_{-16*+10} \simeq \begin{cases} E[e^2\beta^J[w]] & j_0 = 1,\ j_1 < 2,\ \beta^{J+\Delta_0} \text{ allowable}, \\ E[[x^6]\beta^J] & j_0 = 0,\ \beta^{J+2\Delta_{\min}} \text{ allowable}, \\ P[e^2\beta^J[\alpha^i][w]] & j_0 = 0,\ j_1 < 2,\ \beta^{J+\Delta_1}[\alpha^i] \text{ allowable}, \end{cases} \tag{2.16}$$

$$H_*\underline{ER(2)}_{-16*+11} \simeq \begin{cases} E[[x^5]\beta^J] & \beta^{J+4\Delta_0} \text{ allowable}, \\ E[e^3\beta^J[\alpha^i][w]] & j_0 = 0,\ j_1 < 2,\ \beta^{J+\Delta_1}[\alpha^i] \text{ allowable}, \end{cases} \tag{2.17}$$

$$H_*\underline{ER(2)}_{-16*+12} \simeq \begin{cases} E[[x^4]\beta^J] & \beta^{J+3\Delta_0} \text{ allowable}, \\ P[[\alpha_3]\beta^J[\alpha^i]] & j_0 = 0,\ j_1 = 0,\ \beta^J[\alpha^i] \text{ allowable}, \end{cases} \tag{2.18}$$

$$H_*\underline{ER(2)}_{-16*+13} \simeq \begin{cases} E[[x^3]\beta^J] & \beta^{J+2\Delta_0} \text{ allowable}, \\ E[e[\alpha_3]\beta^J[\alpha^i]] & j_0 = 0,\ j_1 = 0,\ \beta^J[\alpha^i] \text{ allowable}, \end{cases} \tag{2.19}$$

$$H_*\underline{ER(2)}_{-16*+14} \simeq E\big[\big[x^2\big]\beta^J\big[\alpha^i\big]\big] \quad \beta^{J+\Delta_0}\big[\alpha^i\big] \text{ allowable}, \tag{2.20}$$

$$H_*\underline{ER(2)}_{-16*+15} \simeq E\big[[x]\beta^J\big[\alpha^i\big]\big] \quad \beta^J[\alpha^i] \text{ allowable}. \tag{2.21}$$

Notice that we have polynomial for $-16*$, $-16*+1$, $-16*+2$, $-16*+4$, $-16*+8$ and cofree for $-16*$, $-16*-1$, $-16*-2$, $-16*-4$, and $-16*-8$. This is interesting, especially in light of the corresponding facts for $KO$ where we have polynomial for 0, 1, 2, 4, and cofree for 0, $-1$, $-2$, and $-4$.

## 3. Homotopy

We describe the homotopy of the spectrum $ER(2)$.

From [10] we have the homotopy of $\mathbb{ER}(2)$ is

$$\mathbb{Z}_{(2)}\big[2\sigma^{\ell 2}, v_1\sigma^{\ell 2^2}, a, v_2^{\pm 1}, \sigma^{\pm 2^3}\big]/I, \quad \ell \in \mathbb{Z}$$

where $I$ is the ideal generated by the relations:

$$2a, \quad v_1 a^3, \quad v_2 a^7,$$
$$v_1\sigma^{\ell 2^2}.2\sigma^{s2^2} = 2v_1\sigma^{(\ell+s)2^2}.$$

The bidegrees of the generators are given by

$$|a| = -\alpha, \qquad \left|v_k\sigma^{\ell 2^{k+1}}\right| = \left(2^k - 1\right)(1+\alpha) + \ell 2^{k+1}(\alpha - 1).$$

We see that monomials without $a$ are represented as

$$2^{k_0} v_1^{k_1} v_2^{k_2} \sigma^{\ell 2^i}$$

where $k_2$ and $\ell$ are integers and the others are $\geqslant 0$. Also, for the lowest $j$ with $k_j > 0$ then $i > j$.

The bidegree is

$$k_1(1+\alpha) + 3k_2(1+\alpha) + \ell 2^i(\alpha - 1)$$

so if the $\alpha$ degree is zero (that's what we are after) then we must have $\ell 2^i = -k_1 - 3k_2$.

We define some elements in here with $\alpha$ degree equal to zero. We have

$$w = v_1 v_2 \sigma^{-4}, \qquad \alpha = v_1 v_2^5 \sigma^{-16},$$
$$\alpha_1 = 2v_2^2\sigma^{-6}, \qquad \alpha_2 = 2v_2^4\sigma^{-12}, \qquad \alpha_3 = 2v_2^6\sigma^{-18}.$$

The real degrees are 8, 32, 12, 24, and 36 respectively. Apologies for the double use of the notation $\alpha$. We also have the periodicity element $v_2^8\sigma^{-24}$ which we are setting equal to 1 so that our powers of $v_2$ can always be represented between 0 and 7.

If $k_1 > 1$ then our general monomial is decomposable as

$$2^{k_0} v_1^{k_1} v_2^{k_2} \sigma^{-k_1-3k_2} = \alpha 2^{k_0} v_1^{k_1-1} v_2^{k_2-5} \sigma^{-k_1-3k_2+16}.$$

If $k_0 > 0$ then 2 divides $-k_1 - 3k_2$ and so 2 still divides $-k_1 - 3k_2 + 16$. If $k_0 = 0$ then 4 divides $-k_1 - 3k_2$ and so 4 still divides $-k_1 - 3k_2 + 16$. So, for a generator we must have $k_1 < 2$.

If $k_0 > 1$ then our general monomial is decomposable as

$$2^{k_0} v_1^{k_1} v_2^{k_2} \sigma^{-k_1-3k_2} = \alpha_1 2^{k_0-1} v_1^{k_1} v_2^{k_2-2} \sigma^{-k_1-3k_2+6}$$

$k_0 - 1 > 0$. 2 divides $-k_1 - 3k_2$ and so 2 still divides $-k_1 - 3k_2 + 6$. So, for a generator we must have $k_0 < 2$.

If $k_0 = 1$ and $k_1 = 1$ then our general monomial is decomposable as

$$2v_1 v_2^{k_2} \sigma^{-1-3k_2} = \alpha 2 v_2^{k_2-5} \sigma^{-1-3k_2+16}$$

since we must have that 2 divides $-1 - 3k_2$ and 2 still divides $-1 - 3k_2 + 16$. So, for a generator we must have $k_0 k_1 = 0$ and, as before, $k_0 < 2$ and $k_1 < 2$.

If $k_0 = 0 = k_1$ we can look for generators of the form $v_2^{k_2}\sigma^{-3k_2}$. We must have 8 divides $-3k_2$, so 8 must divide $k_2$, i.e. it must be a power of the periodicity element which we have set equal to 1.

If $k_0 = 0$ and $k_1 = 1$ we can look for generators of the form $v_1 v_2^{k_2}\sigma^{-1-3k_2}$. We must have 4 divides $-1 - 3k_2$, so, since $k_2$ runs between 0 and 7 by periodicity, $k_2$ must be either 1 or 5. These correspond to the elements $w$ and $\alpha$ defined above.

If $k_0 = 1$ and $k_1 = 0$ we can look for generators of the form $2v_2^{k_2}\sigma^{-3k_2}$. We must have 2 divides $-3k_2$, so, since $k_2$ runs between 0 and 7 by periodicity, $k_2$ can be 0, 2, 4, or 6. 0 is just 2 times the unit. 2, 4, and 6, correspond, respectively, to $\alpha_1$, $\alpha_2$, and $\alpha_3$ defined above.

We now need to consider elements with $a$, which has degree $-\alpha$. If we have a monomial

$$a^j 2^{k_0} v_1^{k_1} v_2^{k_2} \sigma^{\ell 2^i}$$

with zero $\alpha$ degree then we can show, just as above, that we can either divide by $\alpha$ (the other $\alpha$) or $\alpha_1$ unless $k_0 < 2$ and $k_1 < 2$ and $k_0 k_1 = 0$. Since $2a = 0$, if $j > 0$ then $k_0 = 0$. If $k_1 = 1$, then the real part of the degree is still $-1 - 3k_2$ and our monomial is divisible by the element $\alpha$ which gets rid of the $v_1$. So, to be a generator, we must have $k_0 = 0 = k_1$. So, our candidates are

$$a^j v_2^{k_2} \sigma^{\ell 2^i}$$

with $j < 8$ and $i \geqslant 3$. The bidegree here is $-j\alpha + 3k_2(\alpha + 1) + 8\ell(\alpha - 1)$. We need $-j + 3k_2 + 8\ell$ to be zero (the $\alpha$ degree). In particular, we need 8 to divide $3k_2 - j$. This happens when $j = 1$ and $k_2 = 3$ and we get the special element of real degree 17:

$$x = a v_2^3 \sigma^{-8}.$$

For $j = 1$ and $k_2$ ranging between 0 and 7 this is the only possibility. For each $0 < j < 8$ there is only one solution and it corresponds to $x^j$.

We need the image of these elements in the homotopy of $E(2)$. We have the following:

$$w \to v_1 v_2, \qquad \alpha \to v_2^5 v_1,$$
$$\alpha_1 \to 2v_2^2, \qquad \alpha_2 \to 2v_2^4, \qquad \alpha_3 \to 2v_2^6.$$

This is a ring homomorphism and the only generator in the kernel of the map $\pi_* ER(2) \to \pi_* E(2)$ is the element $x$.

There is another element which we hide. $ER(2)$ is a 48 periodic theory and the periodicity element maps to $v_2^8$. In our description of the homotopy we set $v_2$ in $\pi_* E(2)$ equal to 1 and the periodicity element in $\pi_{48} ER(2)$ equal to 1.

We can now read off the relations in homotopy and we have collected the results in Proposition 2.1.

To complete our description we need the maps on homotopy in the fibration

$$\Sigma^{17} ER(2) \to ER(2) \to E(2).$$

We have already done the second map. The first is just multiplication by $x$. We need the map on the homotopy for the boundary, $E(2) \to \Sigma^{18} ER(2)$. This is given as:

$$v_2 \to \alpha_3, \qquad v_2^2 \to x^2 w, \qquad v_2^3 \to 2,$$
$$v_2^4 \to x^6, \qquad v_2^5 \to \alpha_1, \qquad v_2^6 \to x^2\alpha, \qquad v_2^7 \to \alpha_2$$

but this is only part of the story. If multiplication by $\alpha^k$ is nonzero on the right then you can multiply by $v_2^5 v_1$ on the left to get more of the map. This works for all but the $v_2^4$.

## 4. Lannes theory

We fix the prime as $p = 2$ throughout. All (co)homology groups in this section will be assumed with 2-primary coefficients. Let $\mathcal{U}$ denote the category of unstable modules over the Steenrod algebra, and let $\mathcal{K}$ denote the category of unstable algebras over the Steenrod algebra. We will use $\mathbb{Z}/2\mathcal{K}$ to denote the category under the object $H^*(B\mathbb{Z}/2) \in \mathcal{K}$. Hence, an object in $\mathbb{Z}/2\mathcal{K}$ is a morphism $H^*(B\mathbb{Z}/2) \to M$ in $\mathcal{K}$, and a morphism in $\mathbb{Z}/2\mathcal{K}$ is a commutative triangle in $\mathcal{K}$. We will use the notation $\mathbb{Z}/2\mathcal{U}$ to denote the category of objects in $\mathcal{U}$ with a compatible $H^*(B\mathbb{Z}/2)$ action.

The functor $G$ from $\mathcal{U}$ to $\mathbb{Z}/2\mathcal{U}$ given by $G(M) = M \otimes H^*(B\mathbb{Z}/2)$, has a left adjoint $F : \mathbb{Z}/2\mathcal{U} \to \mathcal{U}$. The functor $F$ enjoys two important properties.

(P1) The functor $F : \mathbb{Z}/2\mathcal{U} \to \mathcal{U}$ is exact.
(P2) The functor $F$ commutes with tensor products. More precisely, there is a natural isomorphism $F(M_1 \otimes_{H^*(B\mathbb{Z}/2)} M_2) \to F(M_1) \otimes F(M_2)$, for modules $M_1$ and $M_2$ in $\mathbb{Z}/2\mathcal{U}$.

If $M \in \mathbb{Z}/2\mathcal{U}$ descends from an element in $\mathbb{Z}/2\mathcal{K}$, then $F(M)$ has the canonical structure of an unstable algebra over the Steenrod algebra. Hence $F$ lifts to a functor $F : \mathbb{Z}/2\mathcal{K} \to \mathcal{K}$. This lift is in fact the left adjoint of the functor $G$ seen as a functor from $\mathcal{K}$ to $\mathbb{Z}/2\mathcal{K}$.

**Remark 4.1.** In [11] Section 4 the functor $F$ is denoted as *Fix*.

The functor $F:\mathbb{Z}/2\mathcal{K}\to\mathcal{K}$ can be interpreted geometrically as follows. Let $X$ be a space with a $\mathbb{Z}/2$ action. Let $X_{h\mathbb{Z}/2}$ denote the homotopy orbit space $X_{h\mathbb{Z}/2}=E\mathbb{Z}/2\times_{\mathbb{Z}/2}X$. Let $H^*_{\mathbb{Z}/2}(X)$ denote the cohomology of $X_{h\mathbb{Z}/2}$. The natural map $X_{h\mathbb{Z}/2}\to B\mathbb{Z}/2$ makes $H^*_{\mathbb{Z}/2}(X)$ into an element of $\mathbb{Z}/2\mathcal{K}$. Let $X^{h\mathbb{Z}/2}$ denote the homotopy fixed point space $X^{h\mathbb{Z}/2}=\mathrm{Map}^{\mathbb{Z}/2}(E\mathbb{Z}/2,X)$. The map

$$E\mathbb{Z}/2\times X^{h\mathbb{Z}/2}\to E\mathbb{Z}/2\times X,\qquad (e,f)\mapsto\big(e,f(e)\big)$$

descends to a map $j:B\mathbb{Z}/2\times X^{h\mathbb{Z}/2}\to X_{h\mathbb{Z}/2}$. Taking the adjoint of $j$ in cohomology yields a map $k:FH^*_{\mathbb{Z}/2}(X)\to H^*(X^{h\mathbb{Z}/2})$ in the category $\mathcal{K}$. The theorem of Lannes gives conditions for the map $k$ to be an isomorphism.

**Theorem 4.2.** *(See [11, (4.7)].) Let $X$ be a $\mathbb{Z}/2$-CW complex of finite dimension, then the natural map $k:FH^*_{\mathbb{Z}/2}(X)\to H^*(X^{h\mathbb{Z}/2})$ is an isomorphism in $\mathcal{K}$.*

Another theorem of Lannes we will need in the sequel is:

**Theorem 4.3.** *(See [11, (4.9.3)].) Let $X$ be a 2-complete space. Assume that $FH^*_{\mathbb{Z}/2}(X)$ is of finite type and free in degrees less than 3 (see below), then the natural map $k:FH^*_{\mathbb{Z}/2}(X)\to H^*(X^{h\mathbb{Z}/2})$ is an isomorphism in $\mathcal{K}$.*

An algebra $A\in\mathcal{K}$ is said to be free in degrees less than 3 if the multiplication map in homogeneous degree one: $(A^1\otimes A^1)_{\Sigma_2}\to A^2$, is injective.

We may use the properties of the functor $F$ to prove the following

**Theorem 4.4.** *Let $X$ be a connected space with a $\mathbb{Z}/2$ action. Assume that there exists a connected space $Z$ and a map $Z\to X^{h\mathbb{Z}/2}$ such that the composite*

$$FH^*_{\mathbb{Z}/2}(X)\to H^*\big(X^{h\mathbb{Z}/2}\big)\to H^*(Z)$$

*is an isomorphism, then the following composite map is an isomorphism*

$$FH^*_{\mathbb{Z}/2}(\Omega\Sigma X)\to H^*\big((\Omega\Sigma X)^{h\mathbb{Z}/2}\big)\to H^*(\Omega\Sigma Z).$$

**Proof.** The functor $J$ from connected spaces to spaces given by $J(X)=\Omega\Sigma X$ has a natural filtration known as the James filtration $J^n$. The associated graded pieces of the James filtration are given by $Q^nJ(X)=X^{\wedge n}$. Hence we get a convergent spectral sequence $EJ(X)$ in the category $\mathcal{U}$ converging to $\tilde{H}^*(J(X))$, with $E_1J(X)^{(p,*)}=\tilde{H}^*(X)^{\otimes p}$ (in fact, the Snaith splitting implies that the spectral collapses). Equivariantly, one gets a spectral sequence $EJ_{\mathbb{Z}/2}(X)$ in the category $\mathbb{Z}/2\mathcal{U}$ converging to $\tilde{H}^*_{\mathbb{Z}/2}(J(X))$, with $E_1J_{\mathbb{Z}/2}(X)^{(p,*)}=\tilde{H}^*_{\mathbb{Z}/2}(X)^{\otimes p}$. Since $F$ is an exact functor and commutes with tensor products, applying $F$ to $EJ_{\mathbb{Z}/2}(X)$ yields another spectral sequence in $\mathcal{U}$ converging to $F\tilde{H}^*_{\mathbb{Z}/2}(J(X))$ with an isomorphism $FE_1J_{\mathbb{Z}/2}(X)^{(p,*)}=F\tilde{H}^*_{\mathbb{Z}/2}(X)^{\otimes p}\to\tilde{H}^*(Z)^{\otimes p}=E_1J(Z)^{(p,*)}$. The proof follows once we observe that this isomorphism is induced by the composite

$$FH^*_{\mathbb{Z}/2}(\Omega\Sigma X)\to H^*\big((\Omega\Sigma X)^{h\mathbb{Z}/2}\big)\to H^*(\Omega\Sigma Z).\qquad\square$$

## 5. (Co)homology of $\mathbb{ER}_V$

We use the definitions of projective and projective property in the introduction.

**Claim 5.1.** *Let $X$ be a projective space. Then the following map is an isomorphism*

$$FH^*_{\mathbb{Z}/2}(X)\to H^*\big(X^{h\mathbb{Z}/2}\big)\to H^*\big(X^{\mathbb{Z}/2}\big).$$

**Proof.** For a projective space $X$, it is clear that $X$ has the structure of a CW complex with cells of the form $\mathbb{C}^n$, and the $\mathbb{Z}/2$-action given by complex conjugation. Therefore, $X^{\mathbb{Z}/2}$ (the honest fixed points) has the structure of a CW complex with a cell in dimension $n$ for each cell $\mathbb{C}^n$ of $X$.

The cellular filtration of $X$ induces a filtration of $H^*_{\mathbb{Z}/2}(X)$. Now it is easy to check that the Serre spectral sequence for the fibration $X \to X_{h\mathbb{Z}/2} \to B\mathbb{Z}/2$ collapses, and hence the associated graded quotients of the filtration of $H^*_{\mathbb{Z}/2}(X)$ have the form $\bigoplus_k H^*_{\mathbb{Z}/2}(S^{n(1+\alpha)})$, where $S^{n(1+\alpha)}$ is the one point compactification of $\mathbb{C}^n$. By the exactness of $F$, it follows that $FH^*_{Z/2}(X)$ has a filtration, where the associated graded has the form $\bigoplus_k FH^*_{\mathbb{Z}/2}(S^{n(1+\alpha)})$. By Theorem 4.2, the latter is $\bigoplus_k H^*(S^n)$. It is straightforward to verify that this identification is given by the map

$$FH^*_{\mathbb{Z}/2}(X) \to H^*\big(X^{h\mathbb{Z}/2}\big) \to H^*\big(X^{\mathbb{Z}/2}\big). \qquad \square$$

This theorem, along with 4.4 immediately implies

**Corollary 5.2.** *If $X$ is a projective space, then the composite map*

$$FH^*_{\mathbb{Z}/2}(\Omega\Sigma X) \to H^*\big((\Omega\Sigma X)^{h\mathbb{Z}/2}\big) \to H^*(\Omega\Sigma Z)$$

*is an isomorphism, where $Z = X^{\mathbb{Z}/2}$ (the honest fixed points). Moreover, $H^*(\Omega\Sigma X) = \Phi H^*(\Omega\Sigma Z)$, and the natural map $H^*(\Omega\Sigma X) \to H^*(\Omega\Sigma Z)$ is given by the Frobenius, i.e. $\Phi(x)$ goes to $F(x) = x^{*2}$.*

**Proof.** Only the last part needs explanation. Recall that the doubling functor $\Phi : \mathcal{K} \to \mathcal{K}$ defined by $(\Phi A)^i = A^{i/2}$, $\Phi(Sq^k) = Sq^{2k}$, with the algebra structure on $\Phi A$ induced by $A$. There is a natural transformation between $\Phi$ and the identity functor given by the Frobenius $F(x) = x^2 = Sq^{|x|}x$.

Now if $X = \bigvee_I (\mathbb{C}P^\infty)^{\wedge k_I}$, it follows that $X^{\mathbb{Z}/2} = \bigvee_I (\mathbb{R}P^\infty)^{\wedge k_I}$. Notice that $H^*(\mathbb{C}P^\infty) = \Phi H^*(\mathbb{R}P^\infty)$, and the natural map $H^*(\mathbb{C}P^\infty) \to H^*(\mathbb{R}P^\infty)$ is given by the Frobenius. The final statement of the corollary now follows using a simple argument with the Snaith splitting. $\square$

We now come to the proof of the main theorem (Theorem 1.5) stated in the introduction.

**Proof.** Since our techniques are better suited for cohomology, we shall prove the theorem for cohomology.

We need to show that there is an isomorphism of Hopf algebras $H^*(\mathbb{E}_V) = \Phi H^*(\mathbb{ER}_V)$, and that the natural map $H^*(\mathbb{E}_V) \to H^*(\mathbb{ER}_V)$ is given by the Frobenius.

The projective property of $\mathbb{E}_V$ means that there is a $\mathbb{Z}/2$-equivariant map $f : X \to \mathbb{E}_V$ from a projective space $X$ into $\mathbb{E}_V$. The map $f$ extends to an equivariant map (also called $f$), $f : \Omega\Sigma X \to \mathbb{E}_V$. The projective property implies that $f^* : H^*(\mathbb{E}_V) \to H^*(\Omega\Sigma X)$ is injective. We now need to digress by first proving the following claim:

**Claim 5.3.** *The Serre spectral sequence for the following fibration collapses*

$$\Omega\Sigma X \to (\Omega\Sigma X)_{h\mathbb{Z}/2} \to B\mathbb{Z}/2.$$

**Proof.** To show that this spectral sequence collapses, it is sufficient to know the Poincaré series of $H^*_{\mathbb{Z}/2}(\Omega\Sigma X)$. It is easy to see that the Serre spectral sequence for the fibration $X \to X_{h\mathbb{Z}/2} \to B\mathbb{Z}/2$ collapses. It follows that the Serre spectral sequence for the fibration $X^{\wedge n} \to (X^{\wedge n})_{h\mathbb{Z}/2} \to B\mathbb{Z}/2$ also collapses. This gives us the Poincaré series for $H^*_{\mathbb{Z}/2}(X^{\wedge n})$. Now recall the functor $J(X) = \Omega\Sigma X$. The Snaith splitting provides a stable splitting of this functor into summands given by $Q^n J(X) = X^{\wedge n}$. It follows that $H^*_{\mathbb{Z}/2}(J(X))$ has a splitting in the category $\mathbb{Z}/2\mathcal{U}$ into the summands $H^*_{\mathbb{Z}/2}(X^{\wedge n})$. This splitting, along with the previous observation, gives us the Poincaré series for $H^*_{\mathbb{Z}/2}(\Omega\Sigma X)$ from which the claim follows easily. $\square$

Returning to the proof of the theorem, the above claim says that the spectral sequence for the fibration $\mathbb{E}_V \to (\mathbb{E}_V)_{h\mathbb{Z}/2} \to B\mathbb{Z}/2$ also collapses. Consequently, the map $H^*_{\mathbb{Z}/2}(\mathbb{E}_V) \to H^*_{\mathbb{Z}/2}(\Omega\Sigma X)$ is injective. Using the exactness of $F$, and 5.2, we notice that $FH^*_{\mathbb{Z}/2}(\mathbb{E}_V) \to FH^*_{\mathbb{Z}/2}(\Omega\Sigma X) = H^*(\Omega\Sigma X^{\mathbb{Z}/2})$ is injective. It is clear that $H^*(\Omega\Sigma X^{\mathbb{Z}/2})$ is free in degrees less than 3, and hence so is $FH^*_{\mathbb{Z}/2}(\mathbb{E}_V)$. Therefore, by Theorem 4.3, the natural map $FH^*_{\mathbb{Z}/2}(\mathbb{E}_V) \to H^*(\mathbb{ER}_V)$ is an isomorphism (we have used the fact that the restriction map $H^*(\hat{\mathbb{ER}}_V) \to H^*(\mathbb{ER}_V)$ is an isomorphism, where $\hat{\mathbb{ER}}_V$ denotes the homotopy fixed points of the $\mathbb{Z}/2$ action on the 2-completion of $\mathbb{E}_V$).

We would now like to investigate the algebra $FH^*_{\mathbb{Z}/2}(\mathbb{E}_V)$ further. Using the James filtration, we see that the space $\Omega\Sigma X$ is a CW complex with cells of the form $\mathbb{C}^n$, with the $\mathbb{Z}/2$ action given by complex conjugation. The skeletal filtration of $H^*_{\mathbb{Z}/2}(\Omega\Sigma X)$ induces a filtration of $H^*_{\mathbb{Z}/2}(\mathbb{E}_V)$, via the inclusion $H^*_{\mathbb{Z}/2}(\mathbb{E}_V) \to H^*_{\mathbb{Z}/2}(\Omega\Sigma X)$. The associated graded quotients of this filtration of $H^*_{\mathbb{Z}/2}(\mathbb{E}_V)$ have the form $\bigoplus_k H^*_{\mathbb{Z}/2}(S^{n(1+\alpha)})$. By exactness of $F$ and 4.2, it follows that $FH^*_{\mathbb{Z}/2}(\mathbb{E}_V)$ has a filtration, where the associated graded quotients have the form $\bigoplus_k H^*(S^n)$. Hence, for every basis element of $H^i(\mathbb{E}_V)$, there is a basis element of $FH^*_{\mathbb{Z}/2}(\mathbb{E}_V)^{i/2} = H^{i/2}(\mathbb{ER}_V)$. Now consider the commutative diagram:

$$\begin{tikzcd} H^*(\mathbb{E}_V) \arrow[r] \arrow[d] & H^*(\Omega\Sigma X) \arrow[d] \\ H^*(\mathbb{ER}_V) \arrow[r] & H^*(\Omega\Sigma X^{\mathbb{Z}/2}) \end{tikzcd}$$

Since all maps are injective, the proof of the theorem follows using 5.2. □

**Remark 5.4.** Note that the spectrum $\mathbb{E}(n)$ does not have the projective property. However, the above theorem still remains true if $V = k(1+\alpha)$. To see this, notice that $\mathbb{ER}(n)_V = \varinjlim_m \mathbb{BPR}\langle n\rangle_{V-m(2^n-1)(1+\alpha)}$ with the maps given by powers of $v_n$. This fact follows from the computation of $\mathbb{ER}(n)_*$ given in [10], along with the computation of $\mathbb{BPR}\langle n\rangle_*$ given in [6]. Now, one uses the projective property of $\mathbb{BP}\langle n\rangle_{k(1+\alpha)}$ when $k \leqslant 2^{n+1}-1$ to establish the theorem for $\mathbb{ER}(n)$.

## 6. The old Hopf ring

Ref. [12] describes the relations in the Hopf ring for the homology of the even spaces in the Omega spectrum for $BP$. By [4] and [7] we know these relations also determine the homology of $\underline{ER(n)}_*$. We start off with elements $b_i \in H_{2i}\underline{E(n)}_2$ coming from complex projective space. We have special elements $b_{(i)} = b_{2^i}$ and $\beta_{(i)} = \beta_{2^i}$ where the $\beta_i \in H_i\underline{ER(n)}_{-2^{n+2}(2^{n-1}-1)}$ come from real projective space by way of Theorem 1.1. Let $b(x) = \sum_i b_i x^i$ and $\beta(x) = \sum_i \beta_i x^i$. The relation we need is:

**Theorem 6.1.** *(See [12].) At $p = 2$. In $H_*\underline{E(2)}_2$ we have*

$$[0_2] = \bigl(b(x)^{*2}\bigr) *_{[F]} [v_1]b(x)^2 *_{[F]} [v_2]b(x)^4.$$

*And so in $H_*\underline{ER(2)}_{-16}$.*

$$[0_{-16}] = \bigl(\beta(x)^{*2}\bigr) *_{[F]} [\alpha]\beta(x)^2 *_{[F]} \beta(x)^4.$$

The formal group law for the first equation uses our $[v_1]$ and $[v_2]$ where we set the $[v_2] = [1]$. When we translate this to the second formula we replace $[v_1]$ with $[\alpha]$ and $[v_2]$ with $[1]$.

Whatever we say about the $b_{(i)}$ has a corresponding fact about the $\beta_{(i)}$. We tend to state only those facts we need for the $\beta_{(i)}$ but may use the corresponding fact for the $b_{(i)}$ if necessary.

We need a number of elementary facts about Hopf rings as well as corollaries of this relation. The following lemma collects some useful elementary Hopf ring information where, as usual, we suppress the circle in the notation for the circle product.

**Lemma 6.2.**

$$V(\beta_{(i+1)}) = \beta_{(i)},$$
$$V(wy) = V(w)V(y),$$
$$\bigl(zV(y)\bigr)^{*2} = (z^{*2})y.$$

*If $y$ is a primitive (such as $e$ or $\beta_0$) and $w$ and $z$ have positive degrees then*

$$y(z * w) = 0.$$

*For z positive degree we have*

$$\big([x]V(z)\big)^{*2} = 0.$$

**Proof.** The only thing that is not a basic Hopf ring fact is the last one involving our $[x]$. This follows by

$$\big([x]V(z)\big)^{*2} = [x]^{*2}z = [x+x]z = [2x]z = [0_{-17}]z = 0. \qquad \square$$

From the relation above we get some facts we need:

**Lemma 6.3.** *We have relations in $H_*\underline{ER(2)}_{-16}$ coming from corresponding relations in $H_*\underline{E(2)}_2$.*

$$\beta^2_{(0)}[\alpha] = \beta^{*2}_{(0)},$$
$$\beta^2_{(1)}[\alpha] = \beta^{*2}_{(1)} + \beta^{*4}_{(0)} + \beta^4_{(0)}.$$

*Modulo star products*

$$\beta^2_{(2)}[\alpha] = \beta^4_{(1)}.$$

*Precisely,*

$$\beta^2_{(2)}[\alpha] = \beta^4_{(1)} + \beta^{*4}_{(0)} * \beta^4_{(0)} + \beta^{*4}_{(0)} * \beta^{*2}_{(1)} + \beta^{*2}_{(1)} * \beta^4_{(0)} + \beta^{*4}_{(1)} + \beta^{*2}_{(2)} + \big(\beta^4_{(0)}\big)^{*2}$$

*and*

$$\beta^2_{(2)}\big[\alpha^2\big] = \beta^4_{(0)}\beta^2_{(1)} + \big(\beta^2_{(0)}\beta_{(1)}\big)^{*2} + \big(\beta^{*4}_{(0)} * \beta^{*2}_{(1)}\big)[\alpha] + \beta^{*4}_{(1)}[\alpha] + \beta^{*2}_{(2)}[\alpha].$$

*Modulo star products and $[\alpha]$ we have*

$$\beta^4_{(n)} = 0.$$

**Proof.** These follow from reading off the coefficient of $x^2$, $x^4$, $x^8$, and $x^{2^{n+2}}$ in the relation respectively. We need the formula for $\beta_{(2)}[\alpha^2]$ which we derive from that for $\beta_{(2)}[\alpha]$.

$$\beta^2_{(2)}\big[\alpha^2\big] = \big(\beta^4_{(1)} + \beta^{*4}_{(0)} * \beta^4_{(0)} + \beta^{*4}_{(0)} * \beta^{*2}_{(1)} + \beta^{*2}_{(1)} * \beta^4_{(0)} + \beta^{*4}_{(1)} + \beta^{*2}_{(2)} + \big(\beta^4_{(0)}\big)^{*2}\big)[\alpha].$$

Recalling that $\beta^4_{(0)}[\alpha] = 0$ this gives

$$\begin{aligned}
\beta^2_{(2)}\big[\alpha^2\big] &= \beta^4_{(1)}[\alpha] + \big(\beta^{*4}_{(0)} * \beta^{*2}_{(1)}\big)[\alpha] + \beta^{*4}_{(1)}[\alpha] + \beta^{*2}_{(2)}[\alpha] \\
&= \beta^2_{(1)}\big(\beta^4_{(0)} + \beta^{*4}_{(0)} + \beta^{*2}_{(1)}\big) + \big(\beta^{*4}_{(0)} * \beta^{*2}_{(1)}\big)[\alpha] + \beta^{*4}_{(1)}[\alpha] + \beta^{*2}_{(2)}[\alpha] \\
&= \beta^4_{(0)}\beta^2_{(1)} + \big(\beta^2_{(0)}\beta_{(1)}\big)^{*2} + \big(\beta^{*4}_{(0)} * \beta^{*2}_{(1)}\big)[\alpha] + \beta^{*4}_{(1)}[\alpha] + \beta^{*2}_{(2)}[\alpha]. \qquad \square
\end{aligned}$$

## 7. The bar spectral sequence

We use the bar spectral sequence of Hopf algebras (all our spaces are infinite loop spaces)

$$\mathrm{Tor}^{H_*\Omega X}_{*,*}\big(\mathbb{Z}/(2), \mathbb{Z}/(2)\big) \implies H_*X.$$

All of our homologies are either polynomial or exterior algebras or a mix of the two. There are no truncated polynomial algebras and everything is bicommutative. When Tor is computed we get exterior algebras with generators in the first filtration from the polynomial algebras and divided power algebras with the primitive in the first filtration from the exterior part.

The first differential found must start on a generator in filtration higher than 2 but the only such generators are the $\gamma_{2^i}$ from the divided powers and these are all even degree. The target must be primitive and now we see it must also be in odd degrees and so must be in the first filtration. Thus to show the spectral sequence collapses all that is necessary is to show that the odd degree elements in the first filtration are not zero.

Elements in the first filtration are all primitives and so they are either primitive generators or they are squares. If we calculate that the square of one element in the first filtration, say $x$, is another, say $y$, and they are both primitives for divided powers, then by the commutativity of the Frobenius and the Vershiebung we also get $\gamma_{2^i}(x)^{*2} = \gamma_{2^k}(y)$ modulo lower filtrations. This is generally all we have to do to solve our extension problems.

When it comes to naming elements, which is quite important for us, we usually have a name for the primitives because they are suspensions of previously named elements. Suspension is just circle product with $e$. Naming the $\gamma_{2^i}(x)$ is a different matter though. However, we know the iterated Vershiebung takes $\gamma_{2^i}(x)$ to $x$. If we have a nice enough name for $x$ so that, using the Hopf ring structure, we can construct an element $y$ such that $V^i(y) = x$ then we can replace $\gamma_{2^i}(x)$ with $y$.

## 8. $H_*\underline{ER(2)}_{-16*}$

This Hopf ring, $H_*\underline{ER(2)}_{-16*}$ is given by Theorem 1.1:

$$H_*\underline{ER(2)}_{-16*} \simeq P\big[\beta^J\big[\alpha^i\big]\big],$$

the polynomial algebra on $\beta^J[\alpha^i]$ allowable. The "polynomial" algebra $H_0\underline{ER(2)}_{-16*}$ is $P[[\alpha^i]]$, which is our notation for the group ring over $\mathbb{Z}/(2)$ on $\mathbb{Z}_{(2)}[\alpha]$. This fits our notation nicely.

Before we move on from the zero space we need a couple of relations. The degree 1 part, $H_1\underline{ER(2)}_{-16*}$, is generated by $\beta_{(0)}[\alpha^i]$, $i \geqslant 0$. On the other hand, $H_0\underline{ER(2)}_{-16*-1}$ is, in our notation, $E[[x][\alpha^i]]$. When we use the bar spectral sequence to go from $H_*\underline{ER(2)}_{-16*-1}$ to $H_*\underline{ER(2)}_{-16*}$ we must suspend the elements $[x][\alpha^i]$ to $e[x][\alpha^i]$, but the only elements in this degree for these spaces are the $\beta_{(0)}[\alpha^i]$. So, there must be an element that suspends to $\beta_0$. If, in the spectral sequence that computed the homotopy of $ER(2)$ we had found an $x'$, our special degree 17 element, then the element that suspends to $\beta_0$ can be written $\sum_i a_i x'\alpha^{3i}$ with $a_0 = 1$. Replace $x'$ with this element and call it $x$. It has all the right properties and, in addition, we have

**Lemma 8.1.**

$$e[x] = \beta_{(0)} \in H_1\underline{ER(2)}_{-16}.$$

We have already proven the relation $\beta_{(0)}^{*2} = \beta_{(0)}^2[\alpha]$. There is yet another relation, which, unfortunately, is known not to be true on the nose. However, the approximation has been a guiding light for us.

$$\beta_{(n)}^{*2} \sim \beta_{(n)}^2[\alpha] + \beta_{(n-1)}^4 \in H_{2^n+1}\underline{ER(2)}_{-16}. \tag{8.2}$$

If $j_0 = 0$ we can define $s^{-1}J = (j_1, j_2, \ldots)$. Similarly, $sJ = (0, j_0, j_1, \ldots)$. We have our map: $H_*\underline{ER(2)}_{-16*} \to H_*\underline{E(2)}_{2*}$, given as:

$$\beta^J\big[\alpha^i\big] \to 0 \quad \text{if } j_0 > 0 \quad \text{and} \quad \beta^J\big[\alpha^i\big] \to b^{s^{-1}J}\big[v_1^i\big] \quad \text{if } j_0 = 0. \tag{8.3}$$

In the language of Theorem 1.1 this is $\Phi\beta^J[\alpha^i] = b^J[v_1^i]$ and the map takes $\beta^J[\alpha^i]$ to $V\Phi\beta^J[\alpha^i] = Vb^J[v_1^i] = b^{s^{-1}J}[v_1^i]$.

One last thing that should be pointed out is that if $\beta^J[\alpha^i]$ is allowable, then $\beta^{J+\Delta_0}[\alpha^i]$ is non-zero even when non-allowable. When it is non-allowable it is approximately a square of either an allowable element or another non-allowable element of the same sort. These elements can be found (or approximated) quite easily. If $i > 0$ and $J$ is as above, then $j_0 = 1$ and we have (exactly)

$$\beta^{J+\Delta_0}\big[\alpha^i\big] = \big(\beta_{(0)}^2[\alpha]\big)\beta^{J-\Delta_0}\big[\alpha^{i-1}\big] = \big(\beta_{(0)}^{*2}\big)\beta^{J-\Delta_0}\big[\alpha^{i-1}\big] = \big(\beta_{(0)}\beta^{s^{-1}(J-\Delta_0)}\big[\alpha^{i-1}\big]\big)^{*2}. \tag{8.4}$$

If $i = 0$ with $J$ as above, then we write $J + \Delta_0 = 2\Delta_0 + 4\Delta_k + J'$ and we have, approximately,

$$\beta^{J+\Delta_0} \sim \big(\beta_{(0)}\beta_{(k)}^2\beta^{s^{-1}(J')}\big)^{*2}. \tag{8.5}$$

This last comes about because

$$\big(\beta_{(0)}\beta_{(k)}^2\big)^{*2} = \big(\beta_{(0)}^{*2}\big)\beta_{(k+1)}^2 = \beta_{(0)}^2[\alpha]\beta_{(k+1)}^2 \sim \beta_{(0)}^2\beta_{(k)}^4. \tag{8.6}$$

This is just an approximation and it was our guide through many of the preliminary computations for this paper. However, to actually prove things we need a much more rigorous statement.

**Lemma 8.7.** *If $\beta^J[\alpha^i]$ is allowable and $\beta^{J+\Delta_0}[\alpha^i]$ is not allowable then $\beta^{J+\Delta_0}[\alpha^i]$ is a non-zero square.*

**Proof.** This is non-zero because the corresponding element in $H_*\underline{E(2)}_{2*}$ is non-zero from [12]. It is primitive because of the $\beta_{(0)}$. If $i>0$ then above we computed precisely what element this is the square of. Our problem only comes when $i=0$. To be non-allowable we can write it as $\beta^{2\Delta_0+4\Delta_k+J'}$. By Lemma 6.3 we have $\beta^4_{(k)}$ is zero modulo star products and $[\alpha]$. $\beta_{(0)}$ kills the star product and the $[\alpha]$ combined with $\beta^2_{(0)}$ give $\beta^{*2}_{(0)}$ and this leads us to the whole thing being squared. □

There are a couple of relations proven elsewhere in the paper but since they occur in these spaces we collect them here.

**Lemma 8.8.**

$$e^4[\alpha_3]=(\beta^2_{(0)})^{*2}\in H_4\underline{ER(2)}_{16},$$
$$e^8[w]=\big(\beta^2_{(0)}\beta_{(1)}\big)^{*2}+\beta^4_{(0)}\beta^2_{(1)}\in H_8\underline{ER(2)}_0.$$

## 9. $H_*\underline{ER(2)}_{-16*+1}$

$\underline{ER(2)}_{-16*+1}$ has trivial zero degree homology because $\pi_{16*-1}ER(2)=0$. The $E^2$ term of the bar spectral sequence from $H_*\underline{ER(2)}_{-16*}$ to $H_*\underline{ER(2)}_{-16*+1}$ is $\mathrm{Tor}^{H_*\underline{ER(2)}_{-16*}}(\mathbb{Z}/(2),\mathbb{Z}/(2))$. Since $H_*\underline{ER(2)}_{-16*}$ is polynomial this is just

$$E\big[e\beta^J\big[\alpha^i\big]\big],\quad \beta^J\big[\alpha^i\big]\text{ allowable}$$

where $E$ denotes the exterior algebra. From this we know the spectral sequences collapses (all the generators are in filtration 1). We consider the map $[x]:\underline{ER(2)}_{-16*+17}\to\underline{ER(2)}_{-16*}$ taking $e\beta^J[\alpha^i]$ to $\beta^{J+\Delta_0}[\alpha^i]$ because of the relation $e[x]=\beta_{(0)}$. This map is an injection and is isomorphic to the sub-Hopf algebra generated by the primitives (anything with a $\beta_{(0)}$ in it) and so is polynomial and we can read off a basis for the generators to get our result. If we look just at the element $e$,

$$[x]\big(e^{*2}\big)=\big([x]e\big)^{*2}=\beta^{*2}_{(0)}=\beta^2_{(0)}[\alpha]=[x]e\beta_{(0)}[\alpha].$$

From this and our injection we get the important relation:

**Lemma 9.1.**

$$e^{*2}=e\beta_{(0)}[\alpha]\in H_2\underline{ER(2)}_1.$$

## 10. $H_*\underline{ER(2)}_{-16*+2}$

$\underline{ER(2)}_{-16*+2}$ has trivial zero degree homology because $\pi_{16*-2}ER(2)=0$. Since $H_*\underline{ER(2)}_{-16*+1}$ is polynomial our Tor is:

$$E\big[e^2\beta^J\big[\alpha^i\big]\big]\qquad \beta^{J+\Delta_0}\big[\alpha^i\big]\text{ allowable.}$$

All generators are in the first filtration so the spectral sequence collapses. Multiplying these elements by $[x]$ we have $e^2\beta^J[\alpha^i]$ goes to $e\beta^{J+\Delta_0}[\alpha^i]$ and we see that $H_*\underline{ER(2)}_{-16*+2}\to H_*\underline{ER(2)}_{-16*+2-17}$ injects. Consequently we have a polynomial algebra and our only problem is to find the generators. If we look just at the element $e^2$, we know that $e^2[x]=e\beta_{(0)}$ and that

$$[x]\big(\big(e^2\big)^{*2}\big)=\big([x]e^2\big)^{*2}=\big(e\beta_{(0)}\big)^{*2}=e^{*2}\beta_{(1)}=e\beta_{(0)}[\alpha]\beta_{(1)}=[x]e^2\beta_{(1)}[\alpha].$$

From this and our injection we get the important relation, the first in our lemma:

**Lemma 10.1.**

$$\left(e^2\right)^{*2} = e^2\beta_{(1)}[\alpha] \in H_4\underline{ER(2)}_{-14},$$
$$\left(e^2\beta_{(0)}\right)^{*2} = e^2\beta_0^4 \in H_6\underline{ER(2)}_{-14}.$$

**Proof.** We have already proven the first equation. The second is done as follows:

$$\left(e^2\beta_{(0)}\right)^{*2} = \left(e^2\right)^{*2}\beta_1 = \left(e^2\right)\beta_{(1)}^2[\alpha] = e^2\beta_0^4. \qquad \square$$

There are two cases to consider, first, with $j_0 > 0$ and second, with $j_0 = 0$. If we have a $\beta_{(0)}$ then we have no $[\alpha]$ because $\beta^{J+\Delta_0}[\alpha^i]$ is allowable. If $j_0 > 0$ then we have that $j_k < 4$ for $k > 0$ ($j_0 < 5$).

$$\left(e^2\beta^J\right)^{*2} = \left(e^2\beta_{(0)}\right)^{*2}\beta^{s(J-\Delta_0)} = e^2\beta_{(0)}^4\beta^{s(J-\Delta_0)}.$$

Note that all terms of $s(J-\Delta_0)$ are less than 4. So, $4\Delta_0 + s(J-\Delta_0) + \Delta_0$ is allowable. Furthermore, we see that of the elements with $j_0 > 0$, if $j_0 = 4$ and $j_k < 4$ ($k > 0$), then $e^2\beta^J$ is a square. We conclude that we have a polynomial algebra:

$$P\left[e^2\beta^J\right] \quad j_0 > 0,\ \beta^{J+2\Delta_0} \text{ allowable.}$$

Now we need to contend with the $j_0 = 0$ terms. We use the map $\underline{ER(2)}_{-16*+2} \to \underline{E(2)}_{2*}$. On these terms our map in homology injects taking $e^2\beta^J[\alpha^i]$ to $b_{(0)}b^{s^{-1}J}[v_1^i]$ where we know that $b^{s^{-1}J}$ is allowable. ($e^2$ goes to $b_{(0)}$.) This image is precisely the sub-Hopf algebra of $H_*\underline{E(2)}_{2*}$ generated by the primitives. Pulling this knowledge back to $H_*\underline{ER(2)}_{-16*+2}$ this part is $P[e^2\beta^J[\alpha^i]]$ with $\beta^{J+\Delta_1}[\alpha^i]$ allowable and with $j_0 = 0$.

If we collect our terms we have finished the computation.

## 11. $H_*\underline{ER(2)}_{-16*+3}$

$\underline{ER(2)}_{-16*+3}$ has trivial zero degree homology because $\pi_{16*-3}ER(2) = 0$. Since $H_*\underline{ER(2)}_{-16*+2}$ is polynomial Tor is:

$$E\left[e^3\beta^J\left[\alpha^i\right]\right] \quad j_0 = 0,\ \beta^{J+\Delta_1}[\alpha^i] \text{ allowable,}$$
$$E\left[e^3\beta^J\right] \qquad 0 < j_0,\ \beta^{J+2\Delta_0} \text{ allowable.}$$

All generators are in the first filtration so the spectral sequence collapses. The map given by $[x]$ is no long injective but is still informative. We have:

$$[x]\left(\left(e^3\right)^{*2}\right) = \left([x]e^3\right)^{*2} = \left(e^2\beta_{(0)}\right)^{*2} = \left(e^2\right)^{*2}\beta_{(1)} = e^2\beta_{(1)}^2[\alpha] = e^2\beta_{(0)}^4 = [x]e^3\beta_{(0)}^3.$$

The element $e^3\beta_{(0)}^3$ is the only element in the first filtration of the third space and so we get our relation:

**Lemma 11.1.**

$$\left(e^3\right)^{*2} = e^3\beta_{(0)}^3 \in H_*\underline{ER(2)}_3.$$

Calculating,

$$\left(e^3\beta^J\left[\alpha^i\right]\right)^{*2} = \left(e^3\right)^{*2}\beta^{sJ}\left[\alpha^i\right] = e^3\beta_{(0)}^3\beta^{sJ}\left[\alpha^i\right].$$

If $i > 0$ then $\beta_{(0)}^3[\alpha] = \beta_{(0)}\beta_{(0)}^{*2} = 0$.

For $i = 0$ in the first part of Tor we consider the elements with $\beta^{J+2\Delta_1}$ not allowable (recall $j_0 = 0$ here). In this case there is some $\beta_{(n)}^4$ and we know that this is zero modulo star products and $[\alpha]$ and we know that the $\beta_{(0)}^3$ above kills all star products and $[\alpha]$ so these squares are trivial and give rise to exterior generators also.

Still with $i = 0$ we can combine what is left of the two parts of Tor into $e^3\beta^J$ with $\beta^{J+2\Delta_0}$ allowable (when $j_0 = 0$ this is the same condition as $\beta^{J+2\Delta_1}$ allowable). We see that the square is of the same form (see the calculation above) and we have a polynomial algebra with generators given by $e^3\beta^J$ with $\beta^{J+3\Delta_0}$ allowable. This concludes our proof.

## 12. $H_*\underline{ER(2)}_{-16*+4}$

Since $\pi_{16*-4}ER(2)$ is $\mathbb{Z}_{(2)}$ free on $\alpha_1\alpha^i$, $H_0\underline{ER(2)}_{-16*+4}$ is what we call "polynomial" (it is really the group-ring on this homotopy), which we denote by $P[[\alpha_1][\alpha^i]]$. Tor is:

$$\begin{array}{lll} \Gamma[e^4\beta^J[\alpha^i]] & j_0=0, & \beta^{J+\Delta_1}[\alpha^i] \text{ allowable, and} \\ & & \beta^{J+2\Delta_1}[\alpha^i] \text{ non-allowable,} \\ E[e^4\beta^J] & & \beta^{J+3\Delta_0} \text{ allowable.} \end{array}$$

Any differentials must hit odd degree elements in the first filtration. These are the $e^4\beta^J$ with $j_0=1$. However, these are all non-zero because if you multiply them by $[x]$ you get $e^3\beta^J$ with $j_0=2$ and this is non-zero. So, our spectral sequence collapses.

We need to identify $[\alpha_1]\beta_{(2)}$. The only possibility is:

$$[\alpha_1]\beta_{(2)}=\sum_{i\geqslant 0}a_ie^4[\alpha^{3i+1}].$$

Map this to $H_*\underline{E(2)}_2$. It goes to:

$$[2]b_{(1)}=2_*b_{(1)}=b_{(0)}^{*2}=\sum_{i\geqslant 0}a_ib_{(0)}^2[v_1^{3i+1}]=\sum_{i\geqslant 0}a_i(b_{(0)}^{*2})[v_1^{3i}].$$

Consequently $a_i=0$, $i>0$ and $a_0=1$.

**Lemma 12.1.**

$$\begin{aligned} &[\alpha_1]\beta_{(2)}=e^4[\alpha]\in H_4\underline{ER(2)}_{20}, \\ &[\alpha_1]\beta_{(1)}=0\in H_2\underline{ER(2)}_{20}, \\ &[\alpha_1]\beta_{(0)}=0\in H_1\underline{ER(2)}_{20}, \\ &(e^4)^{*2}=e^4\beta_{(0)}^2\beta_{(1)}+[\alpha_1]\beta_{(2)}^2\in H_8\underline{ER(2)}_4, \\ &e^5\beta_{(0)}^2\beta_{(1)}=e^5\beta_{(2)}[\alpha]=e\beta_{(2)}^2[\alpha_1]\in H_7\underline{ER(2)}_5. \end{aligned}$$

**Proof.** We have just proven the first equation. The next two follow immediately by applying the Vershiebung. For the fourth one we have:

$$[x](e^4)^{*2}=([x]e^4)^{*2}=(e^3\beta_{(0)})^{*2}=(e^3)^{*2}\beta_{(1)}=e^3\beta_{(0)}^3\beta_{(1)}=[x]e^4\beta_{(0)}^2\beta_{(1)},$$

so, modulo the kernel of $[x]$, we have $(e^4)^{*2}=e^4\beta_{(0)}^2\beta_{(1)}$. Note that $(e^4)^{*2}$ must be in the first filtration of the spectral sequence. So, we really have:

$$(e^4)^{*2}=e^4\beta_{(0)}^2\beta_{(1)}+\sum_{i\geqslant 0}a_ie^4\beta_{(2)}[\alpha^{3i+1}]$$

with $a_i\in\mathbb{Z}/(2)$. All of these extra terms are in the kernel of $[x]$ so we can not use that approach to solve this problem. Instead, we map to $H_*\underline{E(2)}_4$ where this equation goes to

$$(b_{(0)}^2)^{*2}=\sum_{i\geqslant 0}a_ib_{(0)}^2b_{(1)}[v_1^{3i+1}]$$

which is

$$(b_{(0)}^2)^{*2}=a_0(b_{(0)}^2)^{*2}+\sum_{i>0}a_i(b_{(0)}^2)^{*2}[v_1^{3i}]=a_0(b_{(0)}^2)^{*2}+\sum_{i>0}a_ib_{(0)}^{*4}[v_1^{3i-1}].$$

That first term on the right went to zero but all of the other terms are non-zero. However, the only one that gives the correct relation has $a_i=0$, $i>0$ and $a_0=1$. We get

$$(e^4)^{*2}=e^4\beta_{(0)}^2\beta_{(1)}+e^4\beta_{(2)}[\alpha]\in H_8\underline{ER(2)}_4.$$

We have already evaluated $e^4[\alpha]$ so this is our fourth equation. Multiply by $e$ to get the final equation. $\square$

We have extensions to solve. We first map to $H_*\underline{E(2)}_{2*}$. The terms in the exterior part of Tor with $j_0 > 0$ go to zero and the other terms in the exterior part go to $b^{s^{-1}J+2\Delta_0}$, all of which are allowable and thus primitive generators. We will only concern ourselves with the primitive part of the divided power part of Tor for now. These primitives map to $b^{s^{-1}J+2\Delta_0}[v_1^i]$ where this is non-allowable but $b^{s^{-1}J+\Delta_0}[v_1^i]$ is allowable. This hits all of the non-allowable primitives. Primitives in $H_*\underline{E(2)}_{2*}$ are all either primitive generators or $2^i$ powers of primitive generators. We show that all of our terms are decomposables and so none of them are generators and they give all of the primitive decomposables. Consequently we see that the image of the sub-algebra generated by the primitives of the divided power part of Tor is exactly the polynomial algebra on the squares of the allowable primitive elements. To see these are all decomposable, if $i > 0$ we have

$$b_{(0)}^2[v_1]y = \big(b_{(0)}^{*2}\big)y = \big(b_{(0)}V(y)\big)^{*2}.$$

If $i = 0$ but our term is not allowable then we must have that it is $b_{(0)}^2 b_{(k)}^4 y$. We already know, from 6.3, that $b_{(k)}^4$ is zero modulo star products and $[v_1]$, so again it is decomposable.

If $i > 0$ our conditions say that $j_1 = 0$ and we have $e^4[\alpha] = [\alpha_1]\beta_{(2)}$. This substitution accounts for all $[\alpha_1]\beta^J[\alpha^{i-1}]$ with $j_0 = j_1 = 0$ and $j_2 > 0$ which are allowable and non-allowable if you multiply by $[\alpha]$, that is to say that $j_k < 2$ unless $k = 2$ in which case it is either 1 or 2. The image of this set is given by all $2^i$, $i > 0$, powers of generators $b^J[v_1^i]$ with $j_0 > 0$ and all $j_k < 2$.

All that is left of the primitives of the divided power algebra for the first part of Tor are those $e^4\beta^J$ with $j_0 = 0$, $\beta^{J+\Delta_1}$ allowable and $\beta^{J+2\Delta_1}$ non-allowable. This requires some $j_k > 3$. Note that there are only a finite number of these elements in each degree. This set maps to the $2^i$, $i > 0$, powers of the remaining primitive generators, i.e., $b^J$ allowable with $J = \Delta_0 + 2\Delta_k + J'$. We have other elements that accomplish the same task. Take $[\alpha_1]\beta^J$ with $\beta^J$ allowable, $j_0 = 0 = j_1$ and $J = \Delta_2 + 2\Delta_k + J'$. We can replace our primitives with these elements. Our conclusion is that the sub-algebra generated by the primitives of the divided power part of our Tor is just $P[[\alpha_1]\beta^J[\alpha^i]]$ with $\beta^J[\alpha^i]$ allowable, $j_2 > 0$, and $j_0 = 0 = j_1$. The rest of the divided power algebra follows by eliminating the condition that $j_2 > 0$.

We now need to study the exterior part of our Tor. We have $e^4\beta^J$ with $J + 3\Delta_0$ allowable. To square this we have:

$$\big(e^4\beta^J\big)^{*2} = \big(e^4\big)^{*2}\beta^{sJ} = \big(e^4\beta_{(0)}^2\beta_{(1)} + e^4\beta_{(2)}[\alpha]\big)\beta^{sJ} = \big(e^4\beta_{(0)}^2\beta_{(1)} + [\alpha_1]\beta_{(2)}^2\big)\beta^{sJ}.$$

If $j_0 > 0$ this second term is zero since $[\alpha_1]\beta_{(1)} = 0$. So this second term is contained in the divided power part that we have already dealt with. Since it was polynomial by itself we can work modulo that and get rid of this second term. From this point of view our square is $e^4\beta_{(0)}^2\beta_{(1)}\beta^{sJ}$. If you multiply this by $\beta_{(0)}^3$ it is still allowable so this square is one of our terms. So, if $J = 2\Delta_0 + \Delta_1 + J'$ then it is a square. Our preferred way to write this is $P[e^4\beta^J]$ with $J + 4\Delta_0$ allowable and $P[e^4\beta^{J+2\Delta_0}]$ with $J + 2\Delta_0$ allowable and $j_0 = 0 = j_1$.

Combining what we have, we get our result.

## 13. $H_*\underline{ER(2)}_{-16*+5}$

The zero degree homology here is trivial. Tor is:

$$\begin{array}{ll} E\big[e[\alpha_1]\beta^J\big[\alpha^i\big]\big] & j_0 = 0 = j_1,\ \beta^J\big[\alpha^i\big] \text{ allowable}, \\ E\big[e^5\beta^J\big] & \beta^{J+4\Delta_0} \text{ allowable}, \\ E\big[e^5\beta^{J+2\Delta_0}\big] & \beta^{J+2\Delta_0} \text{ allowable},\ j_0 = 0 = j_1. \end{array}$$

As all the generators are in the first filtration the spectral sequence collapses.

We need to evaluate $(e^5)^{*2}$. Since $e^5\beta_{(0)}\beta_{(1)}^2$ is the only degree 10 element in the fifth space we have the relation:

$$\big(e^5\big)^{*2} = ae^5\beta_{(0)}\beta_{(1)}^2 \in H_{10}\underline{ER(2)}_5$$

where $a$ is 0 or 1. If $(e^5)^{*2}$ is non-zero then $a$ must be 1. It is easy to see that this is non-zero by multiplying by $[x]$ to get $(e^4\beta_{(0)})^{*2}$ which we know is non-zero. We have

**Lemma 13.1.**

$$\big(e^5\big)^{*2} = e^5\beta_{(0)}\beta_{(1)}^2 \in H_{10}\underline{ER(2)}_5,$$
$$e^5\beta_{(0)}^2\beta_{(1)} = e^5\beta_{(2)}[\alpha] = e\beta_{(2)}^2[\alpha_1] \in H_9\underline{ER(2)}_5.$$

We can now move on to solving extension problems. Taking the first term of our Tor and squaring we have

$$\big(e[\alpha_1]\beta^J\big[\alpha^i\big]\big)^{*2} = e^{*2}[\alpha_1]\beta^{sJ}\big[\alpha^i\big] = e\beta_{(0)}[\alpha_1]\beta^{sJ}\big[\alpha^{i+1}\big] = 0$$

because we have $\beta_{(0)}[\alpha_1] = 0$.

Looking at the third term and squaring we have

$$\begin{aligned}\big(e^5\beta^{J+2\Delta_0}\big)^{*2} &= \big(e^5\big)^{*2}\beta^{sJ+2\Delta_1} = e^5\beta_{(0)}\beta_{(1)}^2\beta^{sJ+2\Delta_1} = e^5\beta_{(0)}\beta_{(1)}^4\beta^{sJ}\\ &= e^5\beta_{(0)}\beta_{(2)}^2[\alpha]\beta^{sJ} = e^5[\alpha]\beta_{(0)}\beta_{(2)}^2\beta^{sJ} = e\beta_{(2)}[\alpha_1]\beta_{(0)}\beta_{(2)}^2\beta^{sJ} = 0\end{aligned}$$

again because we have $[\alpha_1]\beta_{(0)} = 0$. We have used 6.3 here.

All that is left of our Tor is the middle term: $E[e^5\beta^J]$, with $J + 4\Delta_0$ allowable. Squaring this term gives:

$$\big(e^5\beta^J\big)^{*2} = \big(e^5\big)^{*2}\beta^{sJ} = e^5\beta_{(0)}\beta_{(1)}^2\beta^{sJ}.$$

This is certainly of the same form so we get:

$$P\big[e^5\beta^J\big] \quad \beta^{J+4\Delta_0} \text{ allowable, } J \neq J' + \Delta_0 + 2\Delta_1.$$

We now have our final answer.

## 14. $H_*\underline{ER(2)}_{-16*+6}$

The degree zero homology of $\underline{ER(2)}_{-16*+6}$ is given by the group ring on the homotopy, which is $\mathbb{Z}/(2)$ free on $x^2\alpha^i w$. Our notation for this would be as an exterior algebra, i.e. $H_0\underline{ER(2)}_{-16*+6} \simeq E[[x^2][\alpha^i][w]]$. Computing our Tor we have:

$$\begin{array}{ll}\Gamma\big[e^2[\alpha_1]\beta^J\big[\alpha^i\big]\big] & j_0 = j_1 = 0,\ \beta^J\big[\alpha^i\big] \text{ allowable},\\ \Gamma\big[e^6\beta^{J+2\Delta_0}\big] & j_0 = j_1 = 0,\ \beta^{J+2\Delta_0} \text{ allowable},\\ E\big[e^6\beta^J\big] & J \neq J' + \Delta_0 + 2\Delta_1,\ \beta^{J+4\Delta_0} \text{ allowable}.\end{array}$$

Differentials must hit odd degree elements in the first filtration. The only such elements are in the last term of Tor where $j_0$ could be 1. If $j_1 = 0$ then this maps by $[x]$ to a non-zero element so we need only be concerned with elements with $j_0 = 1 = j_1$. Multiply such an element by $e^2\beta_{(0)}[w]$. Below we show that $e^6\beta_{(0)}^2 = [x^2]\beta_{(2)}^2[w]$ which is in a degree lower than this so our problems with collapsing will not interfere with that proof. Let $j_0 = j_1 = 0$. We have

$$\begin{aligned}e^6\beta_{(0)}\beta_{(1)}\beta^J e^2\beta_{(0)}[w] &= e^2\big[x^2\big]\beta_{(1)}\beta_{(2)}^2\big[w^2\big]\beta^J = \beta_{(0)}^2\beta_{(1)}\beta_{(2)}^2\big[\alpha^2\big]\beta^J\\ &= \beta_{(0)}^{*2}\beta_{(1)}\beta_{(2)}^2[\alpha]\beta^J = \big(\beta_{(0)}^2\beta_{(1)}^2[\alpha]\beta^{s^{-1}J}\big)^{*2} = \big(\beta_{(0)}^{*2}\beta_{(1)}^2\beta^{s^{-1}J}\big)^{*2} = \big(\beta_{(0)}^3\beta^{s^{-2}J}\big)^{*4}.\end{aligned}$$

Since all $j_k < 4$ we see that this is non-zero so our original element could not be hit by a differential and our spectral sequence collapses.

Before we proceed we must make some identifications.

**Lemma 14.1.**

$$\big[x^2\big]\beta_{(1)}[w] = e^2[\alpha_1] \in H_2\underline{ER(2)}_{38},$$
$$\big[x^2\big]\beta_{(1)}^2[w] = 0 \in H_4\underline{ER(2)}_{22},$$
$$e^6\beta_{(0)}^2 = \big[x^2\big]\beta_{(2)}^2[w] \in H_8\underline{ER(2)}_{22},$$

$$e^8\beta_{(0)}^2[w] = \left(\beta_{(0)}^5\right)^{*2} \in H_{10}\underline{ER(2)}_{16},$$
$$\left(e^6\right)^{*2} = e^6\beta_{(1)}^3 + \left[x^2\right]\beta_{(2)}^3[w] \in H_{12}\underline{ER(2)}_6,$$
$$e^7\beta_{(1)}^3 = [x]\beta_{(0)}\beta_{(2)}^3[w] \in H_{13}\underline{ER(2)}_7.$$

**Proof.** We need to first identify $[x^2]\beta_{(1)}[w]$. We have already named all the degree two elements in this space. So

$$\left[x^2\right]\beta_{(1)}[w] = \sum_{i\geqslant 0} a_i e^2[\alpha_1]\left[\alpha^{3i}\right]$$

where $a_i \in \mathbb{Z}/(2)$. We use a number of known relations: $e[x] = \beta_{(0)}$, $w^2 = \alpha^2$, $\alpha_1 w = \alpha_3\alpha$, $e^4[\alpha] = \beta_{(2)}[\alpha_1]$, $\alpha_1\alpha_3 = 4$, $\beta_{(0)}^2[\alpha] = \beta_{(0)}^{*2}$, and $[4]\beta_{(2)} = \beta_{(0)}^{*4}$. Circle multiply both sides with the element $e^2[w]$.

$$\beta_{(0)}^2\beta_{(1)}\left[\alpha^2\right] = \sum_{i\geqslant 0} a_i e^4[\alpha_3][\alpha]\left[\alpha^{3i}\right],$$
$$\left(\beta_{(0)}^{*2}\right)\beta_{(1)}[\alpha] = \sum_{i\geqslant 0} a_i \beta_{(2)}[\alpha_1][\alpha_3]\left[\alpha^{3i}\right],$$
$$\left(\beta_{(0)}^2\right)^{*2}[\alpha] = \sum_{i\geqslant 0} a_i \beta_{(2)}[4]\left[\alpha^{3i}\right],$$
$$\beta_{(0)}^{*4} = \sum_{i\geqslant 0} a_i\left(\beta_{(0)}^{*4}\right)\left[\alpha^{3i}\right].$$

This relation is in the $-16$ space and we already know all about it. All of the terms are non-zero and independent, so to have an equation we must have $a_i = 0$, $i > 0$, and $a_0 = 1$. Multiply this relation by $\beta_{(1)}$ and use $[\alpha_1]\beta_{(1)} = 0$ to get the second one.

The next element we need to identify is $[x^2]\beta_{(2)}^2[w]$. Because of the previous equation this is primitive and so:

$$\left[x^2\right]\beta_{(2)}^2[w] = ce^6\beta_{(0)}^2.$$

Multiply both sides by $e^2[w]$ and use the same relations we have already used:

$$\beta_{(0)}^2\beta_{(2)}^2\left[\alpha^2\right] = ce^8\beta_{(0)}^2[w],$$
$$\left(\beta_{(0)}\beta_{(1)}^2\right)^{*2}[\alpha] = ce^8\beta_{(0)}^2[w],$$
$$\left(\beta_{(0)}^5\right)^{*2} = ce^8\beta_{(0)}^2[w].$$

So $c = 1$ because the left-hand side is non-zero.

Next we need to evaluate $(e^6)^{*2}$. We start with our usual multiplication by $[x]$:

$$[x]\left(e^6\right)^{*2} = \left(e^5\beta_{(0)}\right)^{*2} = \left(e^5\right)^{*2}\beta_{(1)} = e^5\beta_{(0)}\beta_{(1)}^2\beta_{(1)} = [x]e^6\beta_{(1)}^3.$$

So we have $(e^6)^{*2} = e^6\beta_{(1)}^3$ modulo the kernel of $[x]$. The only other element in this space and degree in filtration 1 of our spectral sequence is $e^6\beta_{(0)}^2\beta_{(2)} = [x^2]\beta_{(2)}^3[w]$. We have the relation: $(e^6)^{*2} = e^6\beta_{(1)}^3 + c[x^2]\beta_{(2)}^3[w]$ with $c \in \mathbb{Z}/(2)$. If we multiply the left-hand side by $e$ it is zero so if we multiply the term $e^6\beta_{(1)}^3$ by $e$ and it is not zero then $c$ must be non-zero. If we multiply this term by $e^2\beta_{(1)}[w]$ we have, using most of our previous relations:

$$\begin{aligned} e^8\beta_{(1)}^4[w] &= e^8\beta_{(2)}^2[\alpha][w] = e^4\beta_{(2)}^2e^4[\alpha][w] = e^4\beta_{(2)}^2\beta_{(2)}[\alpha_1][w] \\ &= e^2\beta_{(2)}^3e^2[\alpha_1][w] = e^2\beta_{(2)}^3\left[x^2\right]\beta_{(1)}[w][w] = e^2\left[x^2\right]\beta_{(2)}^3\beta_{(1)}\left[\alpha^2\right] = \beta_{(0)}^2\beta_{(1)}\beta_{(2)}^3\left[\alpha^2\right] \\ &= \left(\beta_{(0)}^{*2}\right)\beta_{(1)}\beta_{(2)}^3[\alpha] = \left(\left(\beta_{(0)}^2\right)^{*2}\right)\beta_{(2)}^3[\alpha] = \left(\beta_{(0)}^{*4}\right)\beta_{(2)}^3 = \left(\beta_{(0)}^4\right)^{*4} \neq 0. \end{aligned}$$

Thus we have our complete relation.

Multiply by $e$ to get the last equation. □

We can replace $\Gamma[e^2[\alpha_1]\beta^J[\alpha^i]]$, $j_0 = j_1 = 0$, $\beta^J[\alpha^i]$ allowable with $E[[x^2]\beta^{s^j(J+\Delta_1)}[\alpha^i][w]]$, $j_0 = j_1 = 0$, $j \geqslant 0$, $\beta^J[\alpha^i]$ allowable. Note that 6.2 says that this is exterior. Likewise for the second term, $\Gamma[e^6\beta^{J+2\Delta_0}]$, $j_0 = j_1 = 0$, $\beta^{J+2\Delta_0}$ allowable, which can be replaced with $E[[x^2]\beta^{s^j(J+2\Delta_2)}[w]]$, $j_0 = j_1 = 0$, $j \geqslant 0$, $\beta^{J+2\Delta_0}$ allowable. Combining both terms we get

$$E\big[\big[x^2\big]\beta^J\big[\alpha^i\big][w]\big], \quad j_0 = 0,\ j_1 < 2,\ \beta^J\big[\alpha^i\big] \text{ allowable.}$$

We look now at the third term in our Tor: $E[e^6\beta^J]$, $\beta^{J+4\Delta_0}$, allowable, $J \neq J' + \Delta_0 + 2\Delta_1$. If $j_0 = 1$ ($j_0 < 2$ always) we have

$$\begin{aligned}
\big(e^6\beta_{(0)}\big)^{*2} &= \big(e^6\big)^{*2}\beta_{(1)} = e^6\beta_{(1)}^4 + \big[x^2\big]\beta_{(1)}\beta_{(2)}^3[w] = e^6\beta_{(2)}^2[\alpha] + \big[x^2\big]\beta_{(1)}\beta_{(2)}^3[w] \\
&= e^2\beta_{(2)}^2 e^4[\alpha] + \big[x^2\big]\beta_{(1)}\beta_{(2)}^3[w] = e^2\beta_{(2)}^3[\alpha_1] + \big[x^2\big]\beta_{(1)}\beta_{(2)}^3[w] \\
&= \beta_{(2)}^3 e^2[\alpha_1] + \big[x^2\big]\beta_{(1)}\beta_{(2)}^3[w] = \big[x^2\big]\beta_{(1)}\beta_{(2)}^3[w] + \big[x^2\big]\beta_{(1)}\beta_{(2)}^3[w] = 0.
\end{aligned}$$

So we get $E[e^6\beta^{J+\Delta_0}]$, $j_0 = 0$, $\beta^{J+4\Delta_1}$ allowable.

We are left with $E[e^6\beta^J]$, $j_0 = 0$, $\beta^{J+4\Delta_0}$ allowable. We have

$$\big(e^6\beta^J\big)^{*2} = \big(e^6\big)^{*2}\beta^{sJ} = \big(e^6\beta_{(1)}^3 + \big[x^2\big]\beta_{(2)}^3[w]\big)\beta^{sJ}.$$

We can ignore the part with the $[x]$ in it because it is exterior. We get $e^6\beta^{sJ+3\Delta_1}$. This has the property that there is no $\beta_{(0)}$ and if you add $4\Delta_0$ to it then it is allowable so it is still in our $E[e^6\beta^J]$. Anything in here with a $j_1 = 3$ (it cannot be $\geqslant 3$ because $\beta_{(0)}^4\beta_{(1)}^4$ is not allowable) is such a square so we have, after solving these extensions, $P[e^6\beta^J]$, $j_0 = 0$, $\beta^{J+3\Delta_1}$ allowable.

Collecting our terms we are done.

## 15. $H_*\underline{ER(2)}_{-16*+7}$

The zero degree homology is:

$$E\big[[x]\big[\alpha^i\big][w]\big].$$

It will be subsumed in our notation for the final answer when we are finished.

Tor is:

$$\begin{array}{ll}
\Gamma\big[[x]\beta^{J+\Delta_0}\big[\alpha^i\big][w]\big], & j_0 = 0,\ j_1 < 2,\ \beta^J\big[\alpha^i\big] \text{ allowable,} \\
\Gamma\big[e^7\beta^{J+\Delta_0}\big], & j_0 = 0,\ \beta^{J+4\Delta_1} \text{ allowable,} \\
E\big[e^7\beta^J\big] & j_0 = 0,\ \beta^{J+3\Delta_1} \text{ allowable.}
\end{array}$$

The differential must, as usual, hit an odd degree element in the first filtration. The primitives of the third term here map to the exterior algebra $H_*\underline{E(2)}_{2*+1}$; $e^7\beta^J$ goes to $eb^{s^{-1}J+3\Delta_0}$. Since these are all non-trivial, they cannot be hit by differentials. The second term of Tor has no odd degree elements. All that remains is the first term of Tor and all of the first filtration elements there are odd degree. We split this into a couple of parts. If $j_k < 2$ for all $k$ then we multiply by $e[w]$ to get

$$[x]\beta^{J+\Delta_0}\big[\alpha^i\big][w]e[w] = \beta^{J+2\Delta_0}\big[\alpha^{i+2}\big]$$

which is non-zero (even though a $2^j$ power) and so this could not be hit by a differential. If some $j_k > 1$ then we have $i = 0$ and we consider two cases. First we do the $j_1 = 1$ case. Again, we multiply by $e[w]$ to get:

$$[x]\beta^{J+\Delta_0}[w]e[w] = \beta^{J+2\Delta_0}\big[\alpha^2\big] = \big(\beta^{s^{-1}J+\Delta_0}\big)^{*2}[\alpha].$$

Since $j_1 = 1$ we have this is

$$\big(\beta^{s^{-2}(J-\Delta_1)+\Delta_0}\big)^{*4}$$

which is non-zero even if it is non-allowable since $\beta^{s^{-2}(J-\Delta_1)}$ is clearly allowable. If $j_1 = 0$ then multiply by $e\beta_{(1)}[w]$ and the same argument works. The spectral sequence collapses because all of the potential targets of the differentials are non-zero elements.

**Lemma 15.1.**

$$[x]\beta_{(2)}^2[w] = e^7\beta_{(0)} \in H_8\underline{ER(2)}_{39},$$
$$\left(e^7\right)^{*2} = 0 \in H_{14}\underline{ER(2)}_7,$$
$$e^7\beta_{(1)}^3 = [x]\beta_{(0)}\beta_{(2)}^3[w] \in H_{13}\underline{ER(2)}_7.$$

**Proof.** The elements of degree 8 in the first term of Tor are given by $[x]\beta_{(3)}[\alpha^i][w]$. We want to identify $[x]\beta_{(2)}^2[w]$. We have:

$$[x]\beta_{(2)}^2[w] = ce^7\beta_{(0)} + \sum_i a_i[x]\beta_{(3)}\left[\alpha^{3i+2}\right][w].$$

Multiply by $e\beta_{(0)}[w]$ to get

$$\beta_{(0)}^2\beta_{(2)}^2\left[\alpha^2\right] = ce^8\beta_{(0)}^2[w] + \sum_i a_i\beta_{(0)}^2\beta_{(3)}\left[\alpha^{3i+4}\right],$$
$$\left(\beta_{(0)}^{*2}\right)\beta_{(2)}^2[\alpha] = ce^2\left[x^2\right]\beta_{(2)}^2\left[\alpha^2\right] + \sum_i a_i\left(\beta_{(0)}^{*2}\right)\beta_{(3)}\left[\alpha^{3i+3}\right],$$
$$\left(\beta_{(0)}\beta_{(1)}^2\right)^{*2}[\alpha] = c\beta_{(0)}^2\beta_{(2)}^2\left[\alpha^2\right] + \sum_i a_i\left(\beta_{(0)}\beta_{(2)}\right)^{*2}\left[\alpha^{3i+3}\right],$$
$$\left(\beta_{(0)}^5\right)^{*2} = c\left(\beta_{(0)}^5\right)^{*2} + \sum_i a_i\left(\beta_{(0)}\beta_{(2)}\right)^{*2}\left[\alpha^{3i+3}\right].$$

From this we see that $c = 1$ and $a_i = 0$. We used $e^6\beta_{(0)}^2 = [x^2]\beta_{(2)}^2[w]$.

Next we need $(e^7)^{*2} = 0$. There are no 14 degree elements in the first filtration. The last one has already been proven. □

We have now identified everything and everything is exterior. A careful study of the two Gamma parts allows us to combine them to get the stated homology.

## 16. $H_*\underline{ER(2)}_{-16*+8}$

The zero degree homology is:

$$P\left[[\alpha_2]\right] \otimes P\left[\left[\alpha^i\right][w]\right] \quad i \geqslant 0.$$

Tor is:

$$\Gamma\left[\beta^{J+\Delta_0}\left[\alpha^i\right][w]\right] \quad j_0 \text{ and } j_1 < 2,\ \beta^J\left[\alpha^i\right] \text{ allowable},$$
$$\Gamma\left[e^8\beta^J\right] \quad j_0 = 0,\ \beta^{J+3\Delta_1} \text{ allowable}.$$

This is the most interesting space but also the most tedious to compute with. The spectral sequence will be seen to collapse and so we see it is cofree as a coalgebra. It is also polynomial but this requires some work to see. The generators for the first term in Tor are easy to identify but to get nice ones we can work with for the second term is significantly more complicated than anything we have run into before. First, of course, we have to show collapse of the spectral sequence. The second term is all even degree so there are no differentials hitting first filtration odd degree elements. The odd degree first filtration elements which are potential targets of differentials in the first term are those with $j_0 = 0$. The way we show these are all non-zero is very much like what we did for the $-16* + 7$ spaces. If $j_k < 2$ then this is easy to deal with by multiplying by $[w]$. If some $j_k > 1$ then $k > 1$ and again we can multiply by $[w]$ and, if it helps, we can also multiply by $\beta_{(0)}$ and/or $\beta_{(1)}$ to make the computations work nicely. In the end, the spectral sequence collapses.

We need a series of relations. Some of them are ahead of our time but we need them now and the information we need from spaces we have not yet studied does not depend on the information here.

**Lemma 16.1.**

(1) $e^2[\alpha_3] = [x^2]\beta_{(1)}[\alpha] \in H_2\underline{ER(2)}_{14}$,
(2) $e^4[\alpha_3] = (\beta^2_{(0)})^{*2} \in H_4\underline{ER(2)}_{16}$,
(3) $e^4[w] = [\alpha_3]\beta_{(2)} + [x^4]\beta^2_{(1)} \in H_4\underline{ER(2)}_{-4}$,
(4) $e^8[w] = (\beta^2_{(0)}\beta_{(1)})^{*2} + \beta^4_{(0)}\beta^2_{(1)} \in H_8\underline{ER(2)}_0$,
(5) $e^8 = e^8[1] = [\alpha_2]\beta_{(3)} + \beta^2_{(2)}[w] + (\beta^2_{(1)})^{*2}[w] + ((\beta^2_{(0)})^{*2} * \beta^2_{(1)})[w] \in H_8\underline{ER(2)}_8$,
(6) $[\alpha_2]\beta_{(2)} = \beta^2_{(1)}[w] + (\beta^2_{(0)})^{*2}[w] \in H_4\underline{ER(2)}_8$,
(7) $[\alpha_2]\beta_{(1)} = \beta^2_{(0)}[w] \in H_2\underline{ER(2)}_8$,
(8) $[\alpha_2]\beta_{(0)} = 0 \in H_1\underline{ER(2)}_8$,
(9) $\beta^3_{(2)}[w] = e^8\beta_{(2)} + \beta^2_{(1)}\beta_{(3)}[w] \in H_{12}\underline{ER(2)}_{-8}$,
(10) $\beta^3_{(1)}[w] = \beta^2_{(0)}\beta_{(2)}[w] \in H_6\underline{ER(2)}_{-8}$,
(11) $\beta^3_{(0)}[w] = 0 \in H_3\underline{ER(2)}_{-8}$,
(12) $\beta_{(0)}\beta^2_{(1)}[w] = 0 \in H_5\underline{ER(2)}_{-8}$,
(13) $(e^8)^{*2} = e^8\beta^2_{(1)}\beta_{(2)} + \beta^2_{(0)}\beta_{(1)}\beta_{(2)}\beta_{(3)}[w] \in H_{16}\underline{ER(2)}_8$,
(14) $[\alpha_3]\beta_{(1)} = [x^4]\beta^2_{(0)} \in H_2\underline{ER(2)}_{-4}$,
(15) $[\alpha_3]\beta_{(0)} = 0 \in H_1\underline{ER(2)}_{-4}$.

**Proof.** When we get to the 14th space we will see that

$$\left[x^2\right]\beta_{(1)}[\alpha] = \sum_i a_i e^2[\alpha_3]\left[\alpha^{3i}\right].$$

Multiply this by $e^2$. The left-hand side is

$$e^2\left[x^2\right]\beta_{(1)}[\alpha] = \beta^2_{(0)}\beta_{(1)}[\alpha] = \left(\beta^2_{(0)}\right)^{*2}.$$

Recall that we already know that $e^4[\alpha] = [\alpha_1]\beta_{(2)}$. The right-hand side is

$$\sum_i a_i e^4[\alpha_3]\left[\alpha^{3i}\right] = a_0 e^4[\alpha_3] + \sum_{i>0} a_i[\alpha_3]\left[\alpha^{3i-1}\right]\beta_{(2)}[\alpha_1].$$

Since $\alpha_1\alpha_3 = 4$ we have this is

$$a_0 e^4[\alpha_3] + \sum_{i>0} a_i\left[\alpha^{3i-1}\right]\beta^{*4}_{(0)}.$$

This forces $a_0 = 1$. Multiply by $[\alpha]$ and use $[\alpha][\alpha_3] = [\alpha_1][w]$ to get

$$\beta^{*4}_{(0)} = e^4[\alpha_1][w] + \sum_{i>0} a_i\left[\alpha^{3i}\right]\beta^{*4}_{(0)}.$$

We use $e^2[\alpha_1] = [x^2]\beta_{(1)}[w]$ with this to get

$$\beta^{*4}_{(0)} = e^2\left[x^2\right]\beta_{(1)}\left[\alpha^2\right] + \sum_{i>0} a_i\left[\alpha^{3i}\right]\beta^{*4}_{(0)}.$$

This is:

$$\beta^{*4}_{(0)} = \beta^{*4}_{(0)} + \sum_{i>0} a_i\left[\alpha^{3i}\right]\beta^{*4}_{(0)}.$$

We are left with $a_i = 0$ for $i > 0$.

The second equation follows immediately from the first.

The next equation happens in space 44 but in a low degree. We will see there that we must have

$$[\alpha_3]\beta_{(2)} = \sum_i a_i e^4\left[\alpha^{3i}\right][w] + b\left[x^4\right]\beta^2_{(1)}.$$

Multiply by $[\alpha_1]$ to get

$$[4]\beta_{(2)} = \sum_i a_i e^2 e^2 [\alpha_1]\big[\alpha^{3i}\big][w] + 0,$$

$$\beta_{(0)}^{*4} = \sum_i a_i e^2 \big[x^2\big]\beta_{(1)}[w]\big[\alpha^{3i}\big][w] + 0,$$

$$\beta_{(0)}^{*4} = \sum_i a_i \beta_{(0)}^2 \beta_{(1)}\big[\alpha^{3i+2}\big] = \sum_i a_i \beta_{(0)}^{*4}\big[\alpha^{3i}\big].$$

From this we have $a_i = 0$ for $i > 0$ and $a_0 = 1$. We are left with

$$[\alpha_3]\beta_{(2)} = e^4[w] + b\big[x^4\big]\beta_{(1)}^2.$$

Now multiply by $e^4$ to get

$$e^8[w] + be^4\big[x^4\big]\beta_{(1)}^2 = e^4[\alpha_3]\beta_{(2)} = e^2\big[x^2\big]\beta_{(1)}[\alpha]\beta_{(2)} = \beta_{(0)}^2\beta_{(1)}[\alpha]\beta_{(2)} = \big(\beta_{(0)}^2\beta_{(1)}\big)^{*2}.$$

We now have

$$e^8[w] = b\beta_{(0)}^4\beta_{(1)}^2 + \big(\beta_{(0)}^2\beta_{(1)}\big)^{*2}.$$

Multiply both sides by $\beta_{(0)}^2$ to get

$$\begin{aligned} e^8\beta_{(0)}^2[w] &= b\beta_{(0)}^6\beta_{(1)}^2 + \big(\beta_{(0)}^2\beta_{(1)}\big)^{*2}\beta_{(0)}^2 \\ &= b\big(\beta_{(0)}^3\big)^{*2}\beta_{(1)}^2 + 0 = b\big(\beta_{(0)}^5\big)^{*2}. \end{aligned}$$

We can use a fact from the $-16 * +6$ spaces where we know $b = 1$.

The fourth equation follows directly from the previous two.

We have names for all of our elements up through degree 8. It is a fairly simple calculation (observation really) to see that if we multiply the elements in the 8th space by $[w]$ to get into the zeroth space we have an injection up through this degree. Because of this injection, to prove the formula for $e^8$ it is enough to check that it holds when we map by $[w]$. We have our formula for $e^8[w]$ already so we only need to check that the right-hand side is the same:

$$\begin{aligned} &[\alpha_2][w]\beta_{(3)} + \beta_{(2)}^2\big[\alpha^2\big] + \big(\beta_{(1)}^2\big)^{*2}\big[\alpha^2\big] + \big(\big(\beta_{(0)}^2\big)^{*2} * \beta_{(1)}^2\big)\big[\alpha^2\big] \\ &\quad = [2\alpha]\beta_{(3)} + \beta_{(0)}^4\beta_{(1)}^2 + \big(\beta_{(0)}^2\beta_{(1)}\big)^{*2} \\ &\qquad + \big(\beta_{(0)}[\alpha]\big)^{*4} * \big(\beta_{(1)}[\alpha]\big)^{*2} + \big(\beta_{(1)}[\alpha]\big)^{*4} + \big(\beta_{(2)}[\alpha]\big)^{*2} \\ &\qquad + \big(\beta_{(1)}^{*2} + \beta_{(0)}^4 + \beta_{(0)}^{*4}\big)^{*2}[\alpha] + \big(\beta_{(0)}^{*4} * \big(\beta_{(1)}^{*2} + \beta_{(0)}^4 + \beta_{(0)}^{*4}\big)\big)[\alpha] \\ &\quad = [\alpha]\beta_{(2)}^{*2} + \beta_{(0)}^4\beta_{(1)}^2 + \big(\beta_{(0)}^2\beta_{(1)}\big)^{*2} \\ &\qquad + \big(\beta_{(0)}[\alpha]\big)^{*4} * \big(\beta_{(1)}[\alpha]\big)^{*2} + \big(\beta_{(1)}[\alpha]\big)^{*4} + \big(\beta_{(2)}[\alpha]\big)^{*2} \\ &\qquad + \big(\beta_{(1)}^{*4} + (\beta_{(0)}^4)^{*2} + \beta_{(0)}^{*8}\big)[\alpha] + \big(\beta_{(0)}^{*4} * \beta_{(1)}^{*2} + \beta_{(0)}^{*4} * \beta_{(0)}^4 + \beta_{(0)}^{*8}\big)[\alpha] \\ &\quad = [\alpha]\beta_{(2)}^{*2} + \beta_{(0)}^4\beta_{(1)}^2 + \big(\beta_{(0)}^2\beta_{(1)}\big)^{*2} \\ &\qquad + \big(\beta_{(2)}[\alpha]\big)^{*2} + \big(\big(\beta_{(0)}^4\big)^{*2} + \beta_{(0)}^{*8}\big)[\alpha] + \big(\beta_{(0)}^{*4} * \beta_{(0)}^4 + \beta_{(0)}^{*8}\big)[\alpha] \\ &\quad = \beta_{(0)}^4\beta_{(1)}^2 + \big(\beta_{(0)}^2\beta_{(1)}\big)^{*2}. \end{aligned}$$

And so we are done with (5).

(6), (7), and (8) follow from (5) by iterating the Vershiebung. For (9) we multiply $\beta_{(2)}$ by the equation for $e^8$.

$$e^8\beta_{(2)} = [\alpha_2]\beta_{(3)}\beta_{(2)} + \beta_{(2)}^3[w] + \beta_{(2)}\big(\beta_{(1)}^2\big)^{*2}[w] + \beta_{(2)}\big(\big(\beta_{(0)}^2\big)^{*2} * \beta_{(1)}^2\big)[w].$$

Substitute our equation for $[\alpha_2]\beta_{(2)}$ to get

$$= \beta_{(3)}\big(\beta_{(1)}^2[w] + \big(\beta_{(0)}^2\big)^{*2}[w]\big) + \beta_{(2)}^3[w] + \beta_{(2)}\big(\beta_{(1)}^2\big)^{*2}[w] + \beta_{(2)}\big(\big(\beta_{(0)}^2\big)^{*2} * \beta_{(1)}^2\big)[w]$$
$$= \beta_{(3)}\beta_{(1)}^2[w] + \big(\beta_{(0)}^2\beta_{(2)}\big)^{*2}[w] + \beta_{(2)}^3[w] + \big(\beta_{(1)}^3\big)^{*2}[w] + \big(\big(\beta_{(0)}^3\big)^{*2} * \beta_{(1)}^3\big)[w].$$

We used the Hopf ring distributive law for the last term. Applying the Vershiebung twice to this we get (11). Applying just once and using (11) we get (10). Substituting Eqs. (10) and (11) into the above we get:

$$= \beta_{(3)}\beta_{(1)}^2[w] + \big(\beta_{(0)}^2\beta_{(2)}\big)^{*2}[w] + \beta_{(2)}^3[w] + \big(\beta_{(0)}^2\beta_{(2)}\big)^{*2}[w].$$

This gives us (9).

Multiply (6) by $\beta_{(0)}$ and use (8) to get (12).

The equation for $(e^8)^{*2}$ is not "new" in that it can be derived from our equation for $e^8$. Just plug in our equation for $e^8$ on both sides and evaluate. It is roughly the same length of calculation. We do not do it that way though. $(e^8)^{*2}$ must be in the first filtration of the 8th space so:

$$\big(e^8\big)^{*2} = be^8\beta_{(1)}^2\beta_{(2)} + \sum_i a_i\beta_{(0)}^2\beta_{(1)}\beta_{(2)}\beta_{(3)}\big[\alpha^{3i}\big][w].$$

Multiplying by $[w]$ we get

$$\big(e^8[w]\big)^{*2} = be^8[w]\beta_{(1)}^2\beta_{(2)} + \sum_i a_i\beta_{(0)}^2\beta_{(1)}\beta_{(2)}\beta_{(3)}\big[\alpha^{3i+2}\big].$$

The left side is easy to evaluate so we have:

$$\big(\big(\beta_{(0)}^2\beta_{(1)}\big)^{*2} + \beta_{(0)}^4\beta_{(1)}^2\big)^{*2} = b\big(\big(\beta_{(0)}^2\beta_{(1)}\big)^{*2} + \beta_{(0)}^4\beta_{(1)}^2\big)\beta_{(1)}^2\beta_{(2)} + \sum_i a_i\big(\beta_{(0)}^2\beta_{(1)}\big)^{*4}\big[\alpha^{3i}\big],$$
$$\big(\beta_{(0)}^2\beta_{(1)}\big)^{*4} + \big(\beta_{(0)}^4\beta_{(1)}^2\big)^{*2} = b\big(\beta_{(0)}^4\beta_{(1)}^2\big)^{*2} + b\beta_{(0)}^4\beta_{(1)}^4\beta_{(2)} + a_0\big(\beta_{(0)}^2\beta_{(1)}\big)^{*4} + \sum_{i>0} a_i\beta_{(0)}^{*16}\big[\alpha^{3i-2}\big],$$
$$\big(\beta_{(0)}^2\beta_{(1)}\big)^{*4} + \big(\beta_{(0)}^4\beta_{(1)}^2\big)^{*2} = b\big(\beta_{(0)}^4\beta_{(1)}^2\big)^{*2} + 0 + a_0\big(\beta_{(0)}^2\beta_{(1)}\big)^{*4} + \sum_{i>0} a_i\beta_{(0)}^{*16}\big[\alpha^{3i-2}\big].$$

We get $a_i = 0$ for $i > 0$ and $a_0 = 1 = b$.

Eqs. (14) and (15) are just the iterated Vershiebung of (3). □

We have several steps left. First we want to show that this is polynomial. Second we will find generators and then finally we will find good names for the generators. So, we now show that this homology is polynomial.

We look first at the primitives of the second part of our Tor. They are given by $e^8\beta^J$, $j_0 = 0$, $\beta^{J+3\Delta_1}$ allowable. If we map these elements to $H_*\underline{E(2)}_*$ they go to $b^{s^{-1}J+4\Delta_0}$. Studying the sub-algebra generated by these primitives is fairly easy because we know that $(b_{(0)}^3)^{*2} = b_{(0)}^6$, so

$$\big(b^{s^{-1}J+4\Delta_0}\big)^{*2} = \big(b_{(0)}^3\big)^{*2}b^{J+\Delta_1} = b^{J+6\Delta_0+\Delta_1}$$

which is the image of $e^8\beta^{sJ+2\Delta_1+\Delta_2}$ which is one of our elements. We see that the sub-algebra generated by the primitives of the second part of our Tor is just a polynomial algebra on $e^8\beta^J$ with $j_0 = 0$, $\beta^{J+3\Delta_1}$ allowable such that if $j_1 = 2$ then $j_2 = 0$. We would like to show that this splits off as algebras. The only possible problem is that something in the first part of our Tor squares to something in this. If that were the case it would have to be true in the image in the homology of $\underline{E(2)}_*$. However, most of our polynomial generators from the second part of Tor are generators after we map them to the homology of $\underline{E(2)}_*$ except for those coming from $j_1 = 2$ with $j_2 = 0$. The image of these has a $b_{(0)}^6$ in them and are squares but we know that the element squaring to it must have a $b_{(0)}^3$ in it and the image of the first part of our Tor never has such an element in it. We conclude that the second part of our Tor is polynomial and splits off, as algebras, from the first part.

We need to show that the first part of our Tor is polynomial. Again we map it to $H_*\underline{E(2)}_{2*}$. The primitives are all in the kernel because they all have a $\beta_{(0)}$ in them. We need to show that nothing else is in the kernel and this is not so obvious. We need only look at the elements that are primitive after we have taken out the sub-algebra generated by the primitives, i.e. $\beta^{J+\Delta_1}[\alpha^i][w]$ with $\beta^J[\alpha^i]$ allowable and $j_0 = 0$ and $j_1$ and $j_2$ are less than 2. These map to

$b^{s^{-1}J+\Delta_0}[v_1^{i+1}]$. This would be very easy to show except for that $i+1$, that is, $b^{s^{-1}J+\Delta_0}[v_1^i]$ certainly is all non-zero. We just have to show that multiplying by $[v_1]$ on this is injective. To simplify a bit let us work with $b^{J+\Delta_0}[v_1^i]$ with $j_0$ and $j_1$ both less than 2. First, if $j_0=1$ then we get

$$b^{J+\Delta_0}\left[v_1^{i+1}\right]=b_{(0)}^2[v_1]b^{J-\Delta_0}\left[v_1^i\right]=b_{(0)}^{*2}b^{J-\Delta_0}\left[v_1^i\right]=\left(b^{s^{-1}(J-\Delta_0)+\Delta_0}\left[v_1^i\right]\right)^{*2}.$$

This is non-zero. If we multiply this by $b_{(0)}$ we get zero. So, to show the rest (those with $j_0=0$) are non-zero and independent we can multiply not just by our necessary $[v_1]$ but also by $b_{(0)}$. The same argument works as just given above.

We have shown that the kernel of the map of the first part of Tor is generated by the primitives and the image must be polynomial since it is a sub-algebra of a polynomial algebra. That means that if we mod out by the sub-algebra generated by the primitives we have a polynomial algebra. In particular, the sub-algebra of that quotient algebra generated by its primitives must be polynomial but if we lift that polynomial sub-algebra back to the whole of our first part of Tor we see that the Vershiebung takes it isomorphically to the sub-algebra generated by the primitives there. Thus the whole of the first part of Tor is polynomial. We have now shown that $H_*\underline{ER(2)}_{-16*+8}$ is polynomial.

Now we need to find names for the generators and then we need to find good names.

We will only look at the primitives first. We have already found generators for the second part of our Tor with $e^8$ in them. Much of the first part of Tor is easy to do. If all of the $j_k<2$ then

$$\left(\beta^{J+\Delta_0}\left[\alpha^i\right][w]\right)^{*2}=\left(\beta_{(0)}^{*2}\right)\beta^{sJ}\left[\alpha^i\right][w]=\left(\beta_{(0)}^2\right)\beta^{sJ}\left[\alpha^{i+1}\right][w]$$

which is also an element of the same form as those of the first term in Tor, so this square is non-zero. Also, if $j_0=1$ and $i>0$ then the element in the first part of Tor is a square of another one in the first part of the Tor. So, for the $j_k<2$ case the generators of a polynomial algebra are those with $j_0=0$ or with $i=0$ and $j_0=1$. This takes care of all elements with $i>0$. There are only a finite number of elements left in each degree with some $j_k>1$. The remaining elements of the form we have as primitives which come from generators in $H_*\underline{ER(2)}_{-16*}$ by way of the map $[w]$ are all $\beta^J$ allowable with $0<j_0\leqslant 2$, $j_1<2$, and some $j_k>1$. To show that these are actually our generators it is enough to show that all of the other elements, $\beta^{J+\Delta_0}[w]$, $\beta^J$ allowable, $\beta^{J+\Delta_0}$ not allowable, $j_0$ and $j_1$ less than 2 and some $j_k>1$, are decomposable and there are just the right number of them to be the $2^i$ powers of our generators. To get non-allowable we must have $j_0=1$ and some $j_k\geqslant 4$. We know that $\beta_{(k)}^4$ is zero modulo star products and $[\alpha]$. The star product gives decomposable and the $[\alpha]$ combined with the $\beta_{(0)}^2$ we have gives decomposable. All that remains now is a counting argument showing that for every one of our elements there is a corresponding decomposable in twice the degree. We follow [12]. The approximate square of $\beta^{\Delta_0+2\Delta_k+J}$ is $\beta^{2\Delta_0+4\Delta_k+sJ}$. There is alternative proof that this is polynomial once we have reduced it to this finite set in each degree. We do show that certain elements are decomposable and the same counting argument just given shows that if the numbers are to work out right then all the squares have to be non-zero and in the same filtration. There just aren't enough elements in higher filtration (because their degrees are higher) to square to all the things in lower filtration.

We now have nice generators for the sub-algebra generated by the primitives. This is:

$$\begin{array}{ll} P\left[\beta^J\left[\alpha^i\right][w]\right] & \beta^J\left[\alpha^i\right] \text{ allowable},\ 0<j_0\leqslant 2,\ j_1<2,\\ P\left[e^8\beta^J\right] & \beta^{J+3\Delta_1} \text{ allowable},\ j_0=0,\ \text{ if } j_1=2 \text{ then } j_2=0. \end{array}$$

The generators for the first part of this are all in the image of the iterated Vershiebung and so it is easy to find nice names for all of the rest of the polynomial generators for the first part of Tor. The second part though is a different story. Although the primitives have nice enough names they are not in the image of the Vershiebung as they stand. In order to give nice names to all of the generators it is necessary to find equivalent elements to these primitive generators which are in the image of the iterated Vershiebung.

This replacement of generators takes place in several steps. Since we are dealing only with generators we can work modulo decomposables which means we can simplify the formula for $e^8$ to $e^8=[\alpha_2]\beta_{(3)}+\beta_{(2)}^2[w]$. We can also work modulo everything in the first part of our Tor which is:

$$P\left[\beta^J\left[\alpha^i\right][w]\right]\quad \beta^J\left[\alpha^i\right] \text{ allowable},\ j_{m(J)}\leqslant 2,\ j_{m(J)+1}<2.$$

First we consider the case of $e^8\beta^{J+2\Delta_1}[w]$ when $j_0=j_1=j_2=0$. We substitute for $e^8$ and use our relation $[\alpha_2]\beta_{(1)}=\beta_{(0)}^2[w]$ to get:

$$\begin{aligned} e^8\beta^{J+2\Delta_1} &= \beta^J\big(e^8\beta_{(1)}^2\big) = \beta^J\big(\beta_{(1)}^2\beta_{(2)}^2 + \beta_{(0)}^2\beta_{(1)}\beta_{(3)}\big)[w] \\ &= \beta^{J+2\Delta_1+2\Delta_2}[w] + \beta^{J+2\Delta_0+\Delta_1+\Delta_3}[w]. \end{aligned}$$

The second term is in the first part of Tor so we can ignore it. The first term is new so we can use it for our replacement generator.

Looking at the second term when we have a $\beta_{(1)}$ we use the same relation to get ($j_0 = 0 = j_1$)

$$e^8\beta^{J+\Delta_1} = \beta^{J+\Delta_1+2\Delta_2}[w] + \beta^{J+2\Delta_0+\Delta_3}[w].$$

Again the second term fits into the first part of Tor and the first term is all $\beta^J[w]$ with $\beta^J$ allowable and $j_0 = 0$, $j_1 = 1$, $j_2 \geqslant 2$. We now have these as generators.

We now move on to the case where $j_0 = j_1 = 0$. We start with $j_2 > 0$. It must be less than 4, so in the following we have $j_2 \leqslant 2$. We use the same technique as above (working modulo star products).

$$e^8\beta^{J+\Delta_2} = \beta^{J+3\Delta_2}[w] + [\alpha_2]\beta^{J+\Delta_2+\Delta_3} = \beta^{J+3\Delta_2}[w] + \beta^{J+2\Delta_1+\Delta_3}[w].$$

The last term is in the first part of Tor if $j_2 < 2$ and has already been used as a generator if $j_2 = 2$. There is one little complication here. If $j_3 = 3$ and $j_2 = 2$ then this is not something we have seen before. However, it has $\beta_{(1)}^2\beta_{(3)}^4[w]$ in it. We know that the fourth power is trivial modulo decomposables and $[\alpha]$ and we know that, modulo decomposables, $\beta_{(1)}^2[\alpha] = \beta_{(0)}^4$ and that $\beta_{(0)}^3[w] = 0$ so in this case our term is decomposable. So, we can use the first term as our replacement generator. It is $\beta^J[w]$ with $\beta^J$ allowable and $j_0 = j_1 = 0$ and $j_2 > 2$.

We move to the case of $e^8\beta^J$ where $j_0 = j_1 = j_2 = 0$. Our allowable condition tells us that $j_k < 4$. We have

$$e^8\beta^J = \big([\alpha_2]\beta_{(3)} + \beta_{(2)}^2[w]\big)\beta^J = [\alpha_2]\beta^{J+\Delta_3} + \beta^{J+2\Delta_2}[w].$$

When $j_3 = 0$, 1, or 2, we have seen the second term before so we can use the first as a generator. If, however, $j_3 = 3$, then we have not seen the second term before and we would like it for our generator but must deal with the first term before we can. It has a $[\alpha_2]\beta_{(3)}^4$ in it. All we need to do is show that this is zero modulo decomposables. The fourth power is always trivial modulo decomposables and $[\alpha]$. We need only worry about the $[\alpha]$ part but $[\alpha_2][\alpha] = [2w]$ and $[2]$ times anything is decomposable.

Collecting all our terms and simplifying the answer we are done.

## 17. $H_*\underline{ER(2)}_{-16*+9}$

Tor is:

$$\begin{array}{ll} & \beta^J\big[\alpha^i\big] \text{ allowable,} \\ E\big[e\beta^J\big[\alpha^i\big][w]\big] & \text{if } m(J) = 1 \text{ then } j_1 \leqslant 2, \text{ and if } j_1 = 2 \text{ then } j_2 \leqslant 2, \\ & \text{if } m(J) = 0 \text{ then } j_0 \leqslant 2, \text{ and } j_1 < 2, \\ & \\ E\big[e[\alpha_2]\beta^J\big] & j_0 = j_1 = j_2 = 0,\ \beta^{J+2\Delta_{m(J)}} \text{ allowable.} \end{array}$$

The generators are all in filtration 1 so the spectral sequence collapses.

We recall the formula $e^{*2} = e\beta_{(0)}[\alpha]$. In addition, we need $\beta_{(0)}^3[w] = 0$ and $\beta_{(0)}\beta_{(1)}^2[w] = 0$. We must check a number of cases. If $j_k < 2$

$$\big(e\beta^J\big[\alpha^i\big][w]\big)^{*2} = e\beta_{(0)}\beta^{sJ}\big[\alpha^{i+1}\big][w]$$

so if $e\beta_{(0)}[\alpha]$ is present then we have a square. From this we get $P[e\beta^J[\alpha^i][w]]$, $j_k < 2$, and if $i > 0$ then $j_0 = 0$.

If our $e\beta^J[\alpha^i][w]$ is $e\beta^{J'+2\Delta_k}[w]$ (with $j_q < 2$ for $q < k$) then a direct calculation shows its square is zero if $k = 0$ or 1. For $k > 1$ there are only a finite number of elements and as in many previous calculations, any element with $e\beta_{(0)}\beta_{(k)}^4$ in it is decomposable and there are just enough of them to give all $2^i$ powers of our allowable elements using the usual counting technique of approximating the square as:

$$\big(e\beta^{J+2\Delta_k}[w]\big)^{*2} \sim e\beta_{(0)}\beta_{(k)}^4\beta^{sJ}[w].$$

As for the terms with $[\alpha_2]$, we have

$$\big(e[\alpha_2]\big)^{*2} = e\beta_{(0)}[\alpha][\alpha_2] = e\beta_{(0)}[2w] = 0$$

and so they are all exterior generators.

Our description is as stated.

## 18. $H_*\underline{ER(2)}_{-16*+10}$

The degree zero homology is:

$$E[x^6].$$

Tor is:

$$\begin{array}{ll} & \beta^J[\alpha^i] \text{ allowable,} \\ & j_0 = j_1 = 0, \text{ or} \\ E[e^2\beta^J[\alpha^i][w]] & j_0 = 0,\ j_1 = 1, \text{ or} \\ & j_0 = 1,\ j_1 < 2,\ \beta^{J+\Delta_0}[\alpha^i] \text{ allowable,} \\ \Gamma[e^2\beta^J[w]] & j_0 = 2,\ j_1 < 2,\ \beta^J \text{ allowable, or} \\ & j_0 = 0,\ j_1 = 2,\ j_2 \leqslant 2,\ \beta^J \text{ allowable,} \\ \Gamma[e^2[\alpha_2]\beta^J] & j_0 = j_1 = j_2 = 0,\ \beta^{J+2\Delta_{m(J)}} \text{ allowable.} \end{array}$$

To show collapse we must show that the odd degree elements in the first filtration survive. The only such elements are the $e^2\beta^J[\alpha^i][w]$ with $j_0 = 1$, $j_1 < 2$ and $\beta^{J+\Delta_0}$ allowable. This forces $i = 0$. To see these are non-zero all we have to do is multiply by $[x]$ to get $e\beta^{J+\Delta_0}[w]$ in the $-16*+9$ spaces where we see they are all non-zero.

We need some relations.

**Lemma 18.1.**

$$e^2[\alpha_2] = x^6\beta_{(1)} \in H_2\underline{ER(2)}_{26},$$
$$e^2\beta_{(0)}^2[w] = x^6\beta_{(1)}^2 \in H_4\underline{ER(2)}_{10},$$
$$e^2\beta_{(1)}^2[w] = x^6\beta_{(1)}\beta_{(2)} \in H_6\underline{ER(2)}_{10}.$$

**Proof.** The second and third relations follow from the first and the already known relations $[\alpha_2]\beta_{(2)} = \beta_{(1)}^2[w]$ (modulo decomposables) and $[\alpha_2]\beta_{(1)} = \beta_{(0)}^2[w]$. Because we know Tor we know that

$$[x^6]\beta_{(1)} = \sum_i a_i e^2[\alpha^{3i+2}][w] + be^2[\alpha_2].$$

Multiply both sides by $e^6$. The left-hand side is then

$$\beta_{(0)}^6\beta_{(1)} = (\beta_{(0)}^3)^{*2}\beta_{(1)} = (\beta_{(0)}^4)^{*2}.$$

The right-hand side is

$$\begin{aligned}\sum_i a_i e^8[\alpha^{3i+2}][w] + be^8[\alpha_2] &= \sum_i a_i[\alpha^{3i+2}]e^8[w] + be^8[\alpha_2] \\ &= \sum_i a_i[\alpha^{3i+2}]\big((\beta_{(0)}^2\beta_{(1)})^{*2} + \beta_{(0)}^4\beta_{(1)}^2\big) + be^8[\alpha_2] \\ &= \sum_i a_i[\alpha^{3i+1}](\beta_{(0)}^2)^{*4} + 0 + be^8[\alpha_2] = \sum_i a_i[\alpha^{3i}](\beta_{(0)})^{*8} + be^8[\alpha_2].\end{aligned}$$

From this we see that $b \neq 0$. Multiply both sides by $[\alpha]$ now. The left-hand side is zero and so we get:

$$0 = \sum_i a_i[\alpha^{3i+1}](\beta_{(0)})^{*8} + e^8[\alpha_2][\alpha] = \sum_i a_i[\alpha^{3i+1}](\beta_{(0)})^{*8} + e^8[2w] = \sum_i a_i[\alpha^{3i+1}](\beta_{(0)})^{*8}$$

so we see that all $a_i = 0$. $\square$

We can now rewrite our Tor

$$\begin{array}{ll} & \beta^J\big[\alpha^i\big] \text{ allowable,} \\ & j_0 = j_1 = 0, \text{ or} \\ E\big[e^2\beta^J\big[\alpha^i\big][w]\big] & j_0 = 0,\ j_1 = 1, \text{ or} \\ & j_0 = 1,\ j_1 < 2,\ \beta^{J+\Delta_0}\big[\alpha^i\big] \text{ allowable,} \\ \Gamma\big[\big[x^6\big]\beta^{J+2\Delta_1}\big] & j_0 = 0,\ j_1 < 2,\ \beta^{J+2\Delta_1} \text{ allowable, or} \\ \Gamma\big[\big[x^6\big]\beta^{J+\Delta_1+\Delta_2}\big] & j_0 = 0,\ j_1 = 0,\ j_2 \leqslant 2,\ \beta^{J+2\Delta_1} \text{ allowable,} \\ \Gamma\big[\big[x^6\big]\beta^{J+\Delta_1}\big] & j_0 = j_1 = j_2 = 0,\ \beta^{J+2\Delta_1} \text{ allowable.} \end{array}$$

Anything with an $[x]$ in it must be exterior so all of the divided powers are exterior. Solving the extensions for the rest is now routine. Using our formula $(e^2)^{*2} = e^2\beta_{(1)}[\alpha]$ we can compute the squares of all but a finite number of elements in each degree for the first two parts of Tor. Then, using our usual counting techniques we can see that elements with $e^2\beta_{(1)}\beta_{(k)}^4$ are decomposable and we have a polynomial algebra. The third part of Tor has a $\beta_{(0)}$ in it (and $i$ must be zero) and we have $(e^2\beta_{(0)}[w])^{*2} = (e^2)^{*2}\beta_{(1)}[w] = e^2\beta_{(1)}^2[\alpha][w] = [x^6]\beta_{(1)}\beta_{(2)}[\alpha] = 0$ so these are all exterior.

Combining all of this we get our homology.

## 19. $H_*\underline{ER(2)}_{-16*+11}$

The homology in degree zero is:

$$E\big[\big[x^5\big]\big].$$

Tor is:

$$\begin{array}{ll} \Gamma\big[e^3\beta^J[w]\big] & j_0 = 1,\ j_1 < 2,\ \beta^{J+\Delta_0} \text{ allowable,} \\ \Gamma\big[\big[x^5\big]\beta^{J+\Delta_0}\big] & j_0 = 0, \beta^{J+2\Delta_{\min}} \text{ allowable,} \\ E\big[e^3\beta^J\big[\alpha^i\big][w]\big] & j_0 = 0,\ j_1 < 2,\ \beta^{J+\Delta_1}\big[\alpha^i\big] \text{ allowable.} \end{array}$$

To show collapse we need to identify all of the odd degree primitives. There are none in the first part of Tor. All of the primitives in the second and third parts are odd degree though. For those in the second term just multiply by $e^5$ to get $\beta^{J+6\Delta_0} = (\beta^{s^{-1}J+3\Delta_0})^{*2}$ which is non-zero. For the third term with $j_1 = 0$ we multiply by $e$ and then map to $H_*\underline{E(2)}_*$. This maps to $b^{s^{-1}J+2\Delta_0}[v_1^{i+1}]$. This is $(b^{s^{-2}J+\Delta_0}[v_1^i])^{*2}$ which is non-zero. If $j_1 = 1$ then $i = 0$ and we know that all $j_k < 4$ (recall that $j_0 = 0$ and $j_1 = 1$). Multiply this by $e^5$ and use our formula for $e^8[w]$ to get this is $\beta^J(\beta_{(0)}^4\beta_{(1)}^2 + (\beta_{(0)}^2\beta_{(1)})^{*2})$. The first part is $\beta^{J+4\Delta_0+2\Delta_1}$ which is allowable and therefore a non-zero generator. All of our potential differential targets are non-zero so the spectral sequence collapses.

We need one relation:

**Lemma 19.1.**

$$\big[x^5\big]\beta_{(1)}^2 = e^3\beta_{(0)}[w] \in H_4\underline{ER(2)}_{27}.$$

**Proof.** The only element of degree 4 in the same space as $[x^5]\beta_{(1)}^2$ is $e^3\beta_{(0)}[w]$ so if $[x^5]\beta_{(1)}^2$ is non-zero then we must have our equation. To show it is non-zero we multiply it by $e^5$ to get

$$\beta_{(0)}^5\beta_{(1)}^2$$

which is a non-zero basis element. $\square$

Looking first at our exterior term in Tor we have:

$$\big(e^3[w]\big)^{*2} = \big(e^3\big)^{*2}[w] = e^3\beta_{(0)}^3[w] = 0$$

so it stays exterior.

We use our lemma to replace the first $\Gamma$ term in Tor with:

$$\Gamma\big[\big[x^5\big]\beta^{J+2\Delta_1}\big] \quad j_0=0,\, j_1<2,\ \beta^{J+\Delta_0} \text{ allowable.}$$

Since our two $\Gamma$ terms have $[x]$ in them, they are exterior. We can read off our result now but there are different ways of phrasing the same thing. For example, we now have $E[[x^5]\beta^J]$ with $\beta^{J+2\Delta_{m(J)}}$ allowable and if $m(J)=0$ then $j_0=1$. All this really says is that $j_0<2$ and $j_k<4$. Another way to say this is all $E[[x^5]\beta^J]$ with $\beta^{J+4\Delta_0}$ allowable. This concludes our computation.

## 20. $H_*\underline{ER(2)}_{-16*+12}$

The zero degree homology is

$$E\big[\big[x^4\big]\big]\otimes P\big[[\alpha_3]\big[\alpha^i\big]\big].$$

Our Tor is:

$$\begin{aligned}&\Gamma\big[\big[x^4\big]\beta^{J+\Delta_0}\big] && \beta^{J+4\Delta_0} \text{ allowable,}\\ &\Gamma\big[e^4\beta^J\big[\alpha^i\big][w]\big] && j_0=0,\ j_1<2,\ \beta^{J+\Delta_1}\big[\alpha^i\big] \text{ allowable.}\end{aligned}$$

**Lemma 20.1.**

$$\begin{aligned}&e^4[w]=[\alpha_3]\beta_{(2)}+\big[x^4\big]\beta^2_{(1)}\in H_4\underline{ER(2)}_{-4},\\ &[\alpha_3]\beta_{(1)}=\big[x^4\big]\beta^2_{(0)}\in H_2\underline{ER(2)}_{-4},\\ &[\alpha_3]\beta_{(0)}=0\in H_1\underline{ER(2)}_{-4}.\end{aligned}$$

These have already been proven.

We use our relation $e^4[w]=[\alpha_3]\beta_{(2)}+[x^4]\beta^2_{(1)}$ to help us rename the primitives with $e^4$ in them. We replace the $e^4\beta^J[\alpha^i][w]$ with $\beta^J[\alpha^i]([\alpha_3]\beta_{(2)}+[x^4]\beta^2_{(1)})$. When $j_1=0$ either $i>0$ or $i=0$. If $i=0$ then either the second term here is in the first part of Tor or it is zero. If it is not in the first part of Tor it is because $\beta^{J+2\Delta_1}$ is not allowable which means there must be a $\beta^4_{(k)}$ and we know this is zero modulo star products and $[\alpha]$, both of which are killed by $[x^4]$. If $i>0$ then the second term goes away there too because $[x^4][\alpha]=0$. Under all of these circumstances we can keep the first term as a replacement. Namely, we replace these with $[\alpha_3]\beta^{J+\Delta_2}[\alpha^i]$ where $j_0=0=j_1$ and $\beta^J[\alpha^i]$ is allowable.

Our next problem is when $j_1=1$. Our allowable condition tells us that $i=0$. We rewrite our $J$ so that we are working with $\beta^{J+\Delta_1}$ with $j_0=j_1=0$ and all $j_k<4$. We have, using $[\alpha_3]\beta_{(1)}=[x^4]\beta^2_{(0)}$,

$$\beta^{J+\Delta_1}\big([\alpha_3]\beta_{(2)}+\big[x^4\big]\beta^2_{(1)}\big)=\beta^J[\alpha_3]\beta_{(1)}\beta_{(2)}+\beta^J\big[x^4\big]\beta^3_{(1)}=\beta^J\big[x^4\big]\beta^2_{(0)}\beta_{(2)}+\beta^J\big[x^4\big]\beta^3_{(1)}.$$

If the first term here has $\beta^{J+2\Delta_0+\Delta_2}$ allowable then it is already in our first part of Tor. If it is not allowable it is because $j_2=3$ and we have $\beta^4_{(2)}$ which is zero modulo star products and $[\alpha]$ so this term would be zero. In all cases where $j_1=1$ we replace our primitive with the second term above. This is slightly odd but easy enough to write down succinctly.

With our acquired expertise we can now just read off the final answer for our homology. The polynomial part comes from mapping into $H_*\underline{E(2)}_*$.

## 21. $H_*\underline{ER(2)}_{-16*+13}$

The zero degree homology is

$$E\big[\big[x^3\big]\big].$$

Tor is:

$$\begin{aligned}&\Gamma\big[\big[x^3\big]\beta^{J+\Delta_0}\big] && \beta^{J+3\Delta_0} \text{ allowable,}\\ &E\big[e[\alpha_3]\beta^J\big[\alpha^i\big]\big] && j_0=0,\ j_1=0,\ \beta^J\big[\alpha^i\big] \text{ allowable.}\end{aligned}$$

To show collapse just multiply all of the primitives by $e^3$ to see that they are non-zero.

The terms with $[x]$ are exterior and

$$\big(e[\alpha_3]\big)^{*2} = e^{*2}[\alpha_3] = e\beta_{(0)}[\alpha][\alpha_3] = 0$$

because $[\alpha_3]\beta_{(0)} = 0$ so these terms remain exterior also.

## 22. $H_*\underline{ER(2)}_{-16*+14}$

The zero degree homology is

$$E\big[\big[x^2\big]\big[\alpha^i\big]\big].$$

Tor is:

$$\Gamma\big[\big[x^2\big]\beta^{J+\Delta_0}\big] \qquad \beta^{J+2\Delta_0} \text{ allowable},$$
$$\Gamma\big[e^2[\alpha_3]\beta^J\big[\alpha^i\big]\big] \quad j_0 = 0,\ j_1 = 0,\ \beta^J[\alpha^i] \text{ allowable}.$$

All of the odd degree primitives are in the first part of Tor but they are all non-zero. This can be seen just by multiplying by $e^2$.

**Lemma 22.1.**

$$e^2[\alpha_3] = \big[x^2\big]\beta_{(1)}[\alpha] \in H_2\underline{ER(2)}_{14}.$$

This has already been proven. We use it to replace the last part of Tor with

$$\Gamma\big[\big[x^2\big]\beta^{J+\Delta_1}\big[\alpha^{i+1}\big]\big], \quad j_0 = 0,\ j_1 = 0,\ \beta^J\big[\alpha^i\big] \text{ allowable}.$$

$[x]$ is in everything so everything is exterior. However, this is not a satisfactory description of our elements. If $j_k < 2$ for all $k$ then this is fine. In the case where some $j_k \geqslant 2$ (lowest possible $k$ chosen here), then we have $i = 0$ and we would like to replace the $\beta^2_{(k)}[\alpha]$ with $\beta^4_{(k-1)}$. If we can do that then we still have $j_0 = 0$ and if we multiply by $\beta_{(0)}$ it is still allowable but if we multiply by $\beta^2_{(0)}$ it is not allowable. Even though $j_0 = 0$ the replacement on the primitives is still primitive. If you try to apply the Vershiebung then you get a $j_0 > 0$ and so we really have $[x^3]$. Since any $\beta^4_{(k-1)}$ is zero modulo star products (killed by $[x]$) and $[\alpha]$ (killed by $[x^3]$) this is zero. Assuming we can do this then we would get

$$H_*\underline{ER(2)}_{-16*+14} \simeq E\big[\big[x^2\big]\beta^J\big[\alpha^i\big]\big] \quad \beta^{J+\Delta_0}\big[\alpha^i\big] \text{ allowable}.$$

We consider only the primitives. Since $i = 0$ when some $k \geqslant 2$ we only have a finite number of elements in each degree to worry about. They are the $e^2[\alpha_3]\beta^{J+2\Delta_k}$ with $j_0 = 0 = j_1$, $k > 1$, and $\beta^{J+2\Delta_k}$ allowable. We have replaced these with $[x^2]\beta^{J+\Delta_1+2\Delta_k}[\alpha]$ with $j_0 = 0 = j_1$, $k > 1$, and $\beta^{J+\Delta_1+2\Delta_k}$ allowable. We would like to replace this set with $[x^2]\beta^{J+\Delta_1+4\Delta_{k-1}}$ with $j_0 = 0 = j_1$, $k > 1$, and $\beta^{J+\Delta_1+4\Delta_{k-1}}$ allowable. We have already done the counting argument to see these sets are the same size; replacing $\beta^2_{(k)}[\alpha]$ with $\beta^4_{(k-1)}$ gives a one-to-one correspondence. Having made it this far without invoking the algorithm to reduce non-allowables to allowables we will go a bit out of our way to continue this avoidance. We consider these two sets in $H_*\underline{ER(2)}_{-16*}$ without the $[x^2]$. We have, for the first, $\beta^{J+\Delta_1+2\Delta_k}[\alpha]$ with $j_0 = 0 = j_1$, $k > 1$ and $\beta^{J+\Delta_1+2\Delta_k}$ allowable, and, for the second, $\beta^{J+\Delta_1+4\Delta_{k-1}}$ with $j_0 = 0 = j_1$, $k > 1$, and $\beta^{J+\Delta_1+4\Delta_{k-1}}$ allowable. We map both sets to $H_*\underline{E(2)}_*$ and then multiply by $b_{(0)}$. This composite map is an injection of $\beta^J$ allowable with $j_0 = 0$, $j_1 > 0$. The first set's elements go to

$$\begin{aligned} b^{s^{-1}(J+\Delta_1+2\Delta_k)+\Delta_0}[v_1] &= b^{s^{-1}(J+2\Delta_k)+2\Delta_0}[v_1] \\ &= b^{s^{-1}(J+2\Delta_k)}b^2_{(0)}[v_1] = b^{s^{-1}(J+2\Delta_k)}b^{*2}_{(0)} = \big(b^{s^{-2}(J+2\Delta_k)+\Delta_0}\big)^{*2} \end{aligned}$$

where $b^{s^{-2}(J+2\Delta_k)}$ is allowable. The second goes to $b^{s^{-1}(J+\Delta_1+4\Delta_{k-1})+\Delta_0}$ where $b^{s^{-1}(J+\Delta_1+4\Delta_{k-1})}$ is allowable but $b^{s^{-1}(J+\Delta_1+4\Delta_{k-1})+\Delta_0}$ is not allowable. We know that $b^{s^{-1}(J+\Delta_1+4\Delta_{k-1})+\Delta_0} = b^{s^{-1}(J+4\Delta_{k-1})+2\Delta_0}$ is decomposable as we have seen before. ($b^4_{(k)}$ is zero module star products and $[v_1]$. $[v_1]b^2_{(0)}$ is decomposable.) If we square any $b^K$ with

$k_i < 2$ then we never get a $b^2_{(j)}$. Since $b^{s^{-1}(J+4\Delta_{k-1})+2\Delta_0}$ is decomposable (and a primitive which can only be a $2^i$ power of something) it must be a square of things with a $b^2_{(k)}$ in each term (you could have different $k$ for each term). However, this is then the same set we had before. Since we have an injection, the only difference could be elements in the kernel, i.e. with $\beta_{(0)}$. We will show that when you multiply by $[x^2]$ these elements will not affect anything. If all $j_k < 4$ then we are in the first term of Tor. If some $j_k \geqslant 4$ we use $e[x] = \beta_{(0)}$ to see that

$$[x^2]\beta_{(0)}\beta^4_{(k)} = [x^3]e\beta^4_{(k)}.$$

We know that $[x]$ kills decomposables and $[x^3]$ kills $[\alpha]$ so this is trivial. We have succeeded in replacing our generators with what we want. Our final answer is as stated.

## 23. $H_*\underline{ER(2)}_{-16*+15}$

The zero degree homology is

$$E\big[[x][\alpha^i]\big].$$

Tor is

$$\Gamma\big[[x]\beta^{J+\Delta_0}[\alpha^i]\big] \quad \beta^{J+\Delta_0}[\alpha^i] \text{ allowable.}$$

This collapses because all of the primitives are non-zero when you multiply by $e$.

Because of the $[x]$ this is all exterior and our answer is as stated.

## 24. $H_*\underline{ER(2)}_{-16*}$

The zero degree homology is

$$P\big[[\alpha^i]\big].$$

Tor is

$$\Gamma\big[\beta^{J+\Delta_0}[\alpha^i]\big] \quad \beta^{J}[\alpha^i] \text{ allowable.}$$

This is easily recognized as our starting point.

## 25. Appendix on $ER(1) = KO_{(2)}$

Our $ER(1)$ is just $KO_{(2)}$. Since we are interested only in the mod 2 homology we can just look at $KO$ without localizing. The homology of the spaces in this 8-periodic spectrum are, of course, well known. Viewing this calculation from the point of view of Hopf rings is done nicely in [3]. We present our version here for several reasons. First, we differ from their approach in a few ways. They use the traditional map of real projective space to the component of [1] to define their elements used to generate things. However, it really works best if the elements are in the [0] component the way ours naturally arise. Consequently we do not have to worry about their elements at all. We do not have to assume the homology of $\underline{KO}_0$ because it comes out of our general Theorem 1.1 (and this is the only space we get). We also get the 8-fold periodicity for free. We just need to know the homology of $BU$ as a Hopf ring but that is how the computation of the homology of $E(1)$ is done. Thus we give more prominence to the Hopf ring product. They suppress the star in the star product notation and we suppress the circle product notation instead. Because we set the Bott periodicity element in $H_0$ equal to [1] we also do not have to keep track of it, making our space graded over $\mathbb{Z}/(8)$ rather than just 8-periodic.

The second reason for including this result is that the reader who has managed to slog through the computations for $ER(2)$ will find this trivial, but the reader who is having some problems with our proofs for $ER(2)$ might find it instructive to look at this case which is, in principle, familiar, and relatively simple. However, all the techniques are demonstrated here.

The homotopy of $E(1)$ is just $\mathbb{Z}_{(2)}[v_1, v_1^{-1}]$. We set $[v_1] = [1]$ in $H_0$ and as in Theorem 6.1 we get

$$[0_2] = \big(b(x)^{*2}\big) *_{[F]} b(x)^2.$$

The only relation we really need from this is $b_{(0)}^{*2}=b_{(0)}^2$. From [12,4,7] we know that *allowable* here just means $j_k<2$ in $b^J$. Since $E(1)$ is graded over $\mathbb{Z}/(2)$ we only have one even space and we have $H_*\underline{E(1)}_0\simeq P[b^J]$, $j_k<2$. This gives us $H_*\underline{KO}_0\simeq P[\beta^J]$, $j_k<2$. The homotopy of $KO$ can be described adequately after setting the Bott periodicity element equal to 1. Then all we have left is $\eta\in\pi_1KO$ and $\beta\in\pi_4KO$ (apologies for the bad use of $\beta$ in a second way) with relations $0=\eta^3=2\eta=\eta\beta$ and $\beta^2=4$.

We have an analogous theorem to Theorem 1.2:

**Theorem 25.1.** *The Hopf ring $H_*\underline{KO}_*$ is generated by the two sub-Hopf rings $H_*\underline{KO}_0$ and $\mathbb{Z}/(2)[KO^*]\simeq H_0\underline{KO}_*$. There are only two types of relations that have to be introduced.*

$$e^4=e^4[1]=[\beta]\beta_{(2)},\qquad e^2[\beta]=\left[\eta^2\right]\beta_{(1)},\qquad e[\eta]=\beta_{(0)},$$
$$e^{*2}=e\beta_{(0)},\qquad \left(e^2\right)^{*2}=e^2\beta_{(1)},\qquad \left(e^3\right)^{*2}=0.$$

The complete answer is:

**Theorem 25.2.**

$$\begin{array}{ll}
H_*\underline{KO}_0\simeq P\left[\beta^J\right] & j_k<2,\\
H_*\underline{KO}_1\simeq P\left[e\beta^J\right] & j_0=0,\ j_k<2,\\
H_*\underline{KO}_2\simeq P\left[e^2\beta^J\right] & j_0=j_1=0,\ j_k<2,\\
H_*\underline{KO}_3\simeq E\left[e^3\beta^J\right] & j_0=j_1=0,\ j_k<2,\\
H_*\underline{KO}_4\simeq P\left[[\beta]\beta^J\right] & j_0=j_1=0,\ j_k<2,\\
H_*\underline{KO}_5\simeq E\left[e[\beta]\beta^J\right] & j_0=j_1=0,\ j_k<2,\\
H_*\underline{KO}_6\simeq E\left[\left[\eta^2\right]\beta^J\right] & j_0=0,\ j_k<2,\\
H_*\underline{KO}_7\simeq E\left[[\eta]\beta^J\right] & j_k<2.
\end{array}$$

**Proof.** The first theorem follows from our proof of the second theorem. The zero degree element $[\eta]$ must suspend to our $\beta_{(0)}$ giving the relation $e[\eta]=\beta_{(0)}$. $j_k$ will always be less than 2 so we will not mention it anymore. We compute Tor for the first space and get $E[e\beta^J]$. Since the generators are all in filtration 1 it collapses. Mapping by multiplication by $[\eta]$ takes these elements to $\beta^{J+\Delta_0}$. This is injective so $H_*\underline{KO}_1$ must be polynomial. If $j_0=1$ it is a square so we get our answer. It follows from the injection that we get $e^{*2}=e\beta_{(0)}$.

Computing Tor for the second space we get $E[e^2\beta^J]$ where $j_0=0$. Since the generators are all in filtration 1 it collapses. Again, multiplication by $[\eta]$ gives us an injection and so our answer is polynomial. The injection gives $(e^2)^{*2}=e^2\beta_{(1)}$. Our answer follows.

Tor for the third space is $E[e^3\beta^J]$ where $j_0=j_1=0$. Since the generators are all in filtration 1 it collapses. There is no element in degree 6 so $(e^3)^{*2}=0$ and this is the final answer because $(e^3\beta^J)^{*2}=(e^3)^{*2}\beta^{sJ}=0$.

Tor for the fourth space is $\Gamma[e^4\beta^J]$ with $j_0=j_1=0$. This is even degree so it collapses. We need to evaluate $[\beta]\beta_{(2)}$. The only element in degree 4 is $e^4$ so this is either $e^4$ or zero. To show it is non-zero just multiply by $[\beta]$ and use $\beta^2=4$ to get $[\beta][\beta]\beta_{(2)}=[4]\beta_{(2)}=\beta_{(0)}^{*4}\neq 0$. Taking the Vershiebung of this relation we get $[\beta]\beta_{(1)}=[\beta]\beta_{(0)}=0$. In $\Gamma$ replace $e^4$ with $[\beta]\beta_{(2)}$. Note that $V^2$ of $\beta_{(2)}^2$ must be $\beta_{(0)}^2=\beta_{(0)}^{*2}$ so, modulo $\beta_{(0)}$ and $\beta_{(1)}$ we must have the relation $\beta_{(2)}^2=\beta_{(2)}^{*2}$. Looking at just the primitives, i.e. the set $[\beta]\beta^{J+\Delta_2}$ with $j_0=j_1=0$ it is now straightforward to compute that it is polynomial as stated.

Tor for the fifth space is $E[e[\beta]\beta^J]$, $j_0=j_1=0$. This collapses because all of the generators are in filtration 1. It is exterior because there are no elements in even degrees but if we wanted to use the Hopf ring structure we could just take $(e[\beta]\beta^J)^{*2}=[\beta]e\beta_{(0)}\beta^{sJ}=0$ because $[\beta]\beta_{(0)}=0$.

Tor for the sixth space is $\Gamma[e^2[\beta]\beta^J]$, $j_0=j_1=0$. This collapses because it is even degree but it too follows from the Hopf ring structure. We need to show that $[\eta^2]\beta_{(1)}=e^2[\beta]$. Since the right hand element is the only element in degree two in our Tor then either the left side is zero or equal to it. All we need is that it is non-zero. To see this just multiply by $e^2$ to get $\beta_{(0)}^2\beta_{(1)}=\beta_{(0)}^{*2}\beta_{(1)}=(\beta_{(0)}^2)^{*2}=\beta_{(0)}^{*4}\neq 0$. Do the replacement in our Tor. We get $E[[\eta^2]\beta^J]$,

$j_0 = 0$. To show that these are exterior we use the fact that $[\eta]^{*2} = [2\eta] = [0]$ to see $([\eta^2]\beta^J)^{*2} = [\eta][\eta]^{*2}\beta^{sJ} = [\eta][0]\beta^{sJ} = 0$.

Tor for the seventh space is $\Gamma[[\eta]\beta^{J+\Delta_0}]$ where $j_0 = 0$. This collapses because if you suspend the primitives, i.e. multiply by $e$, they are non-zero, i.e. $\beta^{J+2\Delta_0}$ with $j_0 = 0$. This is all exterior again because of the $[\eta]$. This completes the proof. □

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