Unstable splittings related to Brown-Peterson cohomology

J. Michael Boardman and W. Stephen Wilson

Abstract. We give a new and relatively easy proof of the splitting theorem of the second author for the spaces in the Omega spectrum for BP. We then give the first published proofs of our similar theorems for the spectra $P(n)$.

1. Introduction

In [Wil75], unstable splittings were constructed for the spaces in the Omega spectrum for Brown-Peterson cohomology, a cohomology theory with coefficient ring $BP^* \simeq \mathbf{Z}_{(p)}[v_1, v_2, \ldots, v_n, \ldots]$. This was done using the Postnikov decomposition and a multiple induction. The proof, from [Wil73] (again using Postnikov systems), that these spaces had no torsion was essential in this proof. In [RW77], the calculation of the homology of those spaces for BP was done as a Hopf ring, a great improvement, and this was used in [BJW95] to construct an unstable idempotent to get the splittings. Some of the splittings are not as H-spaces and so the full power of non-additive unstable operations was required. The splittings were generalized in [BW] to the spaces in the Omega spectrum for $P(n)$, $n > 0$, a theory with coefficient ring $P(n)^* \simeq BP^*/I_n$ where $I_n = (p, v_1, v_2, \ldots, v_{n-1})$, after the calculation of the Hopf ring for these spaces in [RW96]. This calculation was done with the intent of getting an analogous splitting. The technique is again to construct an unstable idempotent to get the splittings.

Although the technique of constructing unstable idempotents is clearly the proper way to prove these results, it requires an immense amount of technical machinery which cannot be accused of being easily accessible. In fact, this difficulty has led to the present paper being published well before the paper with the first proof in it.

We wish to present a much more direct proof of these splittings which requires none of the machinery of unstable operations. In fact, the proofs could be done quite easily if one could just insert a few short paragraphs into the papers [RW77] and [RW96]. Unfortunately that option is not open to us. If we essentially reproduce those papers to insert what little extra is needed, then the proofs cease to be "easy." On the other hand, if we just create those paragraphs to be inserted,

1991 *Mathematics Subject Classification.* Primary 55N22 ; Secondary 55N20 .

then the result remains obscure. We will try to walk a fine line between these two approaches. Our goal will be to write the necessary insertions in such a way that a rigorous proof has been accomplished when combined with the previous two papers but at the same time discuss the results in enough depth so the reader should be convinced of the result without having to consult the other papers.

First we need to establish some notation and state our results. We let $P(0)$ represent BP so BP is not an exceptional case. In fact it is both easier and quite different from the $n > 0$ case. There are also theories $BP\langle m\rangle$ with homotopy $\mathbf{Z}_{(p)}[v_1, \dots, v_m]$ and $P(n,m)$ with homotopy $BP\langle m\rangle_*/I_n$ ($I_0 = (0)$, so $BP\langle m\rangle = P(0,m)$) which are associative ring spectra by [SY76]. If E is a spectrum we denote the spaces in the Omega spectrum by $\{\underline{E}_k\}$. Define $g(n,m)$, for $n \leq m$ to be $2(p^n + p^{n+1} + \cdots + p^m)$. We will show that for $k \leq g(n,m)$, $\underline{P(n,m)}_k$ splits off of $\underline{P(n)}_k$. Precisely:

Theorem 1.1 ([Wil75], and later [BJW95], for $n = 0$ and also [BW] for $n > 0$)**.** *For $k \leq g(n,m)$ there is an unstable splitting*

$$\underline{P(n)}_k \simeq \underline{P(n,m)}_k \prod_{j>m} \underline{P(n,j)}_{k+2(p^j-1)}.$$

Remark 1.2. Once the bottom piece has been split off the rest of the splitting follows easily. If $k < g(n,m)$ then it follows that this is a splitting of H-spaces.

Remark 1.3. We do not recover the result of [BW] for $n > 0$ and $k = g(n,m)$ that this is still a splitting as H-spaces (only for odd primes). The fact that the homology splits off as Hopf algebras tells us nothing. If $k > g(n,m)$ then it is easy to see from our approach that the homology of the smaller piece no longer splits off and so there is no such homotopy splitting as well. One of the major attractions about our approach is that we don't have to worry in any way about the additivity of our splittings. For the $n = 0$, $k = g(0,m)$ case where the splittings are not additive, this is a major complication in the proof using unstable idempotents. Our approach is indifferent to such matters although it does show the non-additivity of this splitting because we see that the homology splitting cannot be as Hopf algebras.

Remark 1.4. Note that when $n = m$ the small bottom piece of the splitting is just a space in the spectrum for connective Morava K-theory. Furthermore, when $k < 2p^n - 2$, this is a space in the spectrum for periodic Morava K-theory.

Remark 1.5. Another major complication in [BW] for the $n > 0$ case is the prime 2. The theories involved are not homotopy commutative ring spectra and so the machinery for unstable idempotents must be extended and contorted to deal with this special case. These complications have led to significant delays in the publication of this work. One of the benefits of our approach is once again that we do not need to worry about such things. The $p = 2$ proof is identical to the odd prime proof and we need not be concerned whether any of the spectra are commutative ring spectra or not. The $p = 2$ version of the theorem is very important.

A major motivation for the theorem, and even for a second or third proof, is its important applications. First, it is easy to prove a generalization of Quillen's theorem from the splitting.

Theorem 1.6 (For $n = 0$, [Qui71], and, for $n > 0$, also [BW].). *For X a finite complex, $P(n)^*(X)$ is generated by non-negative degree elements.*

Proofs for the $n = 0$ case using the splitting appear in [Wil75] and [BJW95]. In this last case a more general, purely algebraic, version about unstable modules is proven. Strickland has shown that Quillen's proof cannot be generalized to the $n > 0$ case.

Second, all of the results of [RWY98] are unstable, and the only unstable input is this generalized Quillen theorem. Everything depends on it. Thus, we feel it is important to have a relatively simple and accessible proof, especially one with no complications associated with the prime two or the non-additive splittings.

The authors are particularly indebted to the organizers of the 1998 Barcelona Conference on Algebraic Topology and the Centre de Recerca Matemàtica for the conference and terrific work environment during the extended stay of the second author. Thanks are also due to our previous coauthors, David Johnson and Douglas Ravenel, for the work with them which led us to this paper.

2. Approach and simple parts of the proofs

In [RW77] and [RW96] the homology of the spaces in the Omega spectra were calculated by induction on degree using the bar spectral sequence

$$\mathrm{Tor}^{H_*(\underline{P(n)}_*)}(\mathbf{Z}/(p), \mathbf{Z}/(p)) \Rightarrow H_{*+1}(\underline{P(n)}_{*+1}).$$

The H_{n+1} is there to indicate that we gain a degree in this inductive calculation.

What was not noticed in those two papers was that we could easily have simultaneously calculated the homology of the bottom piece which splits off. We would just use the spectral sequence

$$\mathrm{Tor}^{H_*(\underline{P(n,m)}_*)}(\mathbf{Z}/(p), \mathbf{Z}/(p)) \Rightarrow H_{*+1}(\underline{P(n,m)}_{*+1}).$$

There is a map of spectra $P(n) \to P(n,m)$ which induces a map on the spectral sequences. The proof of the calculation of this spectral sequence is identical up to $k = g(n,m)$ with the exception of one slightly modified definition. It is easy to see that the calculation cannot go one step higher.

Once the homology is calculated it is easy to see that the Atiyah-Hirzebruch spectral sequence for the $P(n)$ and $P(n,m)$ homology of these spaces collapses and so they are free over the coefficient rings.

Corollary 2.1. *For $k \leq g(n,m)$ the Atiyah-Hirzebruch spectral sequences*

$$H_*(\underline{P(n,m)}_k; P(n)_*) \text{ and } H_*(\underline{P(n,m)}_k; P(n,m)_*)$$

collapse and $P(n)_*(\underline{P(n,m)}_k)$ is $P(n)_*$ free and $P(n,m)_*(\underline{P(n,m)}_k)$ is $P(n,m)_*$ free.

This in turn allows us to calculate the cohomology theories as the dual and get:

Corollary 2.2. *For $k \leq g(n,m)$, the map*

$$P(n)^*(\underline{P(n,m)}_k) \to P(n,m)^*(\underline{P(n,m)}_k)$$

is surjective.

Proof of Theorem 1.1. To get our splitting we just note that the identity map of $\underline{P(n,m)}_k$, an element of $P(n,m)^k(\underline{P(n,m)}_k)$, has a lift to a map of $\underline{P(n,m)}_k$ to $\underline{P(n)}_k$, an element of $P(n)^k(\underline{P(n,m)}_k)$. That splits off the bottom piece. The other pieces are handled by taking the maps

$$\underline{P(n,j)}_{k+2(p^j-1)} \to \underline{P(n)}_{k+2(p^j-1)} \to \underline{P(n)}_k$$

where the first map is just the bottom piece splitting and the second map comes from the stable map of v_j. Using the H-space structure to add all of these maps up we see we have a homotopy equivalence due to the obvious isomorphism on homotopy groups.

This concludes the proof of the splitting from the calculation of the homology and the collapsing of the Atiyah-Hirzebruch spectral sequence. □

The summary given above constitutes a “proof” that if the splittings exist, then there must be an “easy” proof along the lines just outlined.

We now want to give a proof of the Quillen theorem.

Proof of Theorem 1.6. This proof is exactly the same as the proof of the original Quillen theorem given in [Wil75], but because it is of such major importance, and it is short, we reproduce it here. There are stable cofibrations

$$\Sigma^{2(p^m-1)}P(n,m) \xrightarrow{v_m} P(n,m) \longrightarrow P(n,m-1) \tag{2.1}$$

where $P(n,n-1)$ is the mod p Eilenberg–Mac Lane spectrum. These cofibrations give rise to long exact sequences in cohomology theories. Given a negative degree element $x \in P(n)^*(X)$ where X is a finite complex, we see that there is some $q \geq n$ for which x maps to zero in $P(n,q-1)^*(X)$ because mod p cohomology is zero in negative degrees. Using the above exact squence there is an element $y_q \in P(n,q)^*(X)$ such that $v_q y_q$ is the image of x in $P(n,q)^*(X)$. Because x has negative degree, the element y_q, which is a map of X into the space $\underline{P(n,q)}_{|x|+2(p^q-1)}$, is in the range of the splitting Theorem 1.1 and so the element y_q can be lifted to $P(n)^*(X)$. We now look at the element $x - v_q y_q$. It will go to zero in $P(n,q)^*(X)$ and we can iterate this process. It is a finite process since the splitting Theorem 1.1, combined with the finiteness of X, tells us that for some large m, $P(n)^k(X)$ is the same as $P(n,m)^k(X)$ for a fixed $k = |x|$. At that point we have $x = \sum v_i y_i$, a finite sum. Thus any negative degree element is decomposable in this way and we have proven the generalized Quillen theorem. (For the $n = 0$ case we let $P(0,0)$ be the $\mathbf{Z}_{(p)}$ cohomology and we do not need the mod p cohomology.) □

3. The Homology

Let $H_*(-)$ be the standard mod p homology where p is the prime associated with the spectrum $P(n,m)$. Because $P(n,m)$ is a ring spectrum there are maps

$$\underline{P(n,m)}_i \times \underline{P(n,m)}_j \longrightarrow \underline{P(n,m)}_{i+j}$$

corresponding to cup product, in addition to the loop space product

$$\underline{P(n,m)}_i \times \underline{P(n,m)}_i \longrightarrow \underline{P(n,m)}_i.$$

These induce pairings

$$\circ : H_*(\underline{P(n,m)}_i) \otimes H_*(\underline{P(n,m)}_j) \to H_*(\underline{P(n,m)}_{i+j})$$

and

$$* : H_*(\underline{P(n,m)}_i) \otimes H_*(\underline{P(n,m)}_i) \to H_*(\underline{P(n,m)}_i).$$

Since $H_*(-)$ has a Kunneth isomorphism these pairings satisfy certain identities making $H_*(\underline{P(n,m)}_*)$ into a *Hopf ring*, [RW77], i.e., a ring object in the category of coalgebras.

There are special elements

$$\begin{aligned} e &\in P(n)_1(\underline{P(n,m)}_1), \\ a_{(i)} &\in P(n)_{2p^i}(\underline{P(n,m)}_1) && \text{for } 0 \le i < n, \\ [v_i] &\in P(n)_0(\underline{P(n,m)}_{-2(p^i-1)}) && \text{for } m \ge i \ge n, \quad i > 0, \quad \text{and} \\ b_{(i)} &\in P(n)_{2p^i}(\underline{P(n,m)}_2) && \text{for } i \ge 0, \end{aligned}$$

which have already been defined in [RW96] in $P(n)_*(\underline{P(n)}_*)$ and we get these by just pushing them down using the map from $P(n)$ to $P(n,m)$ to get them first in $P(n)_*(\underline{P(n,m)}_*)$ and then into $P(n,m)_*(\underline{P(n,m)}_*)$. They then push down non-trivially to $H_*(\underline{P(n,m)}_*)$.

A basic property which we need and which comes out of the construction of these elements, (this goes clear back to [Wil84]), is:

Proposition 3.1. *The elements e, $a_{(i)}$, $[v_i]$, and $b_{(i)}$ are permanent cycles in the Atiyah-Hirzebruch spectral sequence for $P(n)_*(-)$ and $P(n,m)_*(-)$.*

Other facts proven about these elements are in [Wil84, Proposition 1.1] and were repeated again in [RW96, Proposition 2.1, p. 1048] and are not repeated again here.

Let

$$e^\varepsilon a^I [v^K] b^J = e^\varepsilon \circ a_{(0)}^{\circ i_0} \circ \cdots \circ a_{(n-1)}^{\circ i_{n-1}} \circ [v_n^{k_n} v_{n+1}^{k_{n+1}} \cdots] \circ b_{(0)}^{\circ j_0} \circ b_{(1)}^{\circ j_1} \cdots$$

where $\varepsilon = 0$ or 1, $i_q = 0$ or 1, $k_q \ge 0$, and $j_q \ge 0$, (K and J finite), and if $n = 0$, $k_0 = 0$.

Definition 3.2. For $n > 0$ we say $e^\varepsilon a^I[v^K]b^J$ is nm**-allowable** if

$$J = p^n\Delta_{d_n} + p^{n+1}\Delta_{d_{n+1}} + \cdots + p^q\Delta_{d_q} + J'$$

where Δ_d has a 1 in the d^{th} place and zeros elsewhere, $d_n \le d_{n+1} \le \cdots \le d_q$ and J' is non-negative implies $k_q = 0$. In other words,

$$[v_q] \circ b^{p^n\Delta_{d_n}+p^{n+1}\Delta_{d_{n+1}}+\cdots+p^q\Delta_{d_q}}$$

does not divide $e^\varepsilon a^I[v^K]b^J$ when $d_n \le d_{n+1} \le \cdots \le d_q$, $q < m$. We will denote the set of such (K, J) by $\mathcal{A}_{nm}$. If we eliminate the reference to m then we have the n-allowable of [RW96]. Because we do not want to use $[v_0]$ in the $n = 0$ case we set $\mathcal{A}_{0m} = \mathcal{A}_{1m}$. There is still a difference between the allowable elements because I is empty for $n = 0$ but not for $n = 1$.

We say $e^\varepsilon a^I[v^K]b^J$ is nm**-plus allowable** if $e^\varepsilon a^I[v^K]b^{J+\Delta_0}$ is nm-allowable. We will denote the set of such (K, J) by $\mathcal{A}^+_{nm}$. Note that $\mathcal{A}^+_{nm} \subset \mathcal{A}_{nm}$.

Define the shift operator s on J by

$$b^{s(J)} = b^{\circ j_0}_{(1)} \circ b^{\circ j_1}_{(2)} \circ \cdots . \tag{3.2}$$

Theorem 3.3. *Let $H_*(-)$ be the standard mod p homology with p the prime associated with $P(n, m)$.*

For $n > 0$ and $* \le g(n, m)$. $H_*(\underline{P(n, m)}_*)$ is the same as $H_*(\underline{P(n)}_*)$ stated in [RW96, Theorem 1.3, p. 1045] *except we replace the n and n-plus allowable with nm and nm-plus allowable. For $p = 2$ there is one more minor modification described in the appendix to this paper.*

For $n = 0$ and $* < g(0, m)$,

$$H_*(\underline{P(0, m)}_*) \simeq \bigotimes_{(K,J)\in\mathcal{A}_{0m}} E(e[v^K]b^J) \bigotimes_{(K,J)\in\mathcal{A}_{0m}} P([v^K]b^J).$$

For $n = 0$ and $k = g(0, m)$, as a coalgebra, $H_*(\underline{P(0, m)}_k)$ is the divided power coalgebra

$$\bigotimes_{(K,J)\in\mathcal{A}_{0m}} \Gamma([v^K]b^{J+\Delta_0}).$$

The elements $\gamma_{p^i}([v^K]b^{J+\Delta_0})$ represent $[v^K]b^{s^i(J+\Delta_0)}$.

Remark 3.4. Of course we insist that one only uses the elements which actually lie in the appropriate spaces. The element $e^\varepsilon a^I[v^K]b^J$ is in $H_s(\underline{P(n, m)}_k)$ where $s = \varepsilon + \sum 2p^{i_q} + \sum 2p^{j_q}$ and $k = \varepsilon + \sum i_q + 2\sum j_q - \sum 2(p^q - 1)k_q$.

Remark 3.5. For $n = 0$ and $* < g(0, m)$, this is the same as in [RW77], replacing allowable with $0m$-allowable.

Proof of Corollary 2.1 . The Atiyah-Hirzebruch spectral sequence respects the two products, $\circ$ and $*$, and all elements in the $P(n)_*(\underline{P(n, m)}_*)$ we are considering are constructed using these two products from the basic elements e, $a_{(i)}$, $[v_i]$, and $b_{(i)}$ which are all permanent cycles by Proposition 3.1. Thus the spectral sequence collapses. □

Primitives are calculated simultaneously as in [RW96, Theorem 1.4, p. 1046] and [RW77].

The $p = 2$ case deserves some discussion. The spectra $P(n)$ and $P(n,m)$ are not commutative ring spectra. However, the standard homology is still a commutative Hopf ring, see the explanation in [Wil84, pages 1030-31]. There are no concerns raised by this lack of commutativity in the collapse of the Atiyah-Hirzebruch spectral sequence, the use of duality to compute the cohomologies or their application to get the splittings. The lack of commutativity could make things very bad for some applications but it doesn't interfere in the slightest with what we are doing. It does make the proof of the splitting in [BW] much harder.

The proof of our theorem relies on being able to identify elements in the bar spectral sequence, compute differentials and solve multiplicative extension problems, all using Hopf ring techniques. The $n = 0$ case has no differentials but does have extension problems.

Let Q stand for the indecomposables.

Theorem 3.6. *In $QH_*(\underline{P(n,m)}_k)$, $k \leq g(n,m)$, any $e^\varepsilon a^I[v^K]b^J$ can be written in terms of nm-allowable elements.*

Proof. The construction and proof of an algorithm for the reduction of non-allowable elements is done on pages 273–275 of [RW77]. The proof applies with only notational modification to the case of nm-allowable when $I = 0$. We can then circle multiply by a^I to get our result. □

The homology and primitives are calculated simultaneously by induction on degree in the bar spectral sequence. Recall that for a loop space X with classifying space BX the bar spectral sequence converges to $H_*(BX)$, and its E^2-term is

$$\mathrm{Tor}_{*,*}^{H_*(X)}(\mathbf{Z}/(p), \mathbf{Z}/(p)).$$

When BX is also a loop space, we have a spectral sequence of Hopf algebras.

The E^2 term of the bar spectral sequence for $n > 0$ is calculated inductively from Theorem 3.3 just as in [RW96, Lemma 3.6, p. 1056]. For $n = 0$ the calculation is trivial as in [RW77]. Both cases must use the modified definition of allowable.

The complete behavior of the bar spectral sequence, using the modified definition of allowable, is given in [RW96, Theorem 3.7, p. 1057] for $n > 0$ and in [RW77] for $n = 0$. For $n > 0$ it is a gruesome description of all differentials and identification of elements in terms of the Hopf ring. There are no differentials for $n = 0$ and the identifications are much easier as well.

This does not complete the calculation of the homology but only the E^∞ term of the spectral sequence. Extension problems must be solved. However, first, since we claim that the proofs of all parts of the calculation are exactly the same as for the spaces in the Omega spectra for $P(n)$ and BP as in [RW96] and [RW77], some explanation is clearly needed in order to explain the differences between the two cases and to see why we cannot proceed up the Omega spectrum with $P(n,m)$. That difference comes about in the calculation of the differentials. For $n = 0$ there are no differentials so we do not see any difference at this step. For

$n > 0$ the differentials are not really calculated but inferred. Certain elements are shown to disappear and it is proven that they must be targets of differentials. They are counted and shown to be in one-to-one correspondence with possible sources of differentials. Thus, all possible sources must kill all necessary targets. The difference comes in the counting process. We use the following lemma and the difference is found in the proof, which we include.

Lemma 3.7. *Let $n > 0$. In $H_*\underline{P(n,m)}_{*<g(n,m)}$, there is a one-to-one correspondence between the set*

$$\{[v^{K+\Delta_n}]b^{J+(p^n-1)\Delta_0}\colon (K,J) \in \mathcal{A}_{nm}, j_0 = 0\}$$

and the set

$$\{[v^{K'}]b^{J'}\colon (K',J') \in \mathcal{A}_{nm} - \mathcal{A}^+_{nm}\}.$$

Remark 3.8. Recall that for $n = 0$ we have $0m$ and $1m$ give the same thing.

Proof. To see this, write

$$J = p^n\Delta_{d_n} + p^{n+1}\Delta_{d_{n+1}} + \cdots + p^q\Delta_{d_q} + J''$$

where q is maximal (this can be vacuous, i.e. $J = J''$, in which case we set $q = n-1$) and $d_n \le d_{n+1} \le \cdots \le d_q$ and J'' is non-negative. Now let

$$J' = J'' + (p^n - 1)\Delta_0 + p^{n+1}\Delta_{d_n-1} + p^{n+2}\Delta_{d_{n+1}-1} + \cdots + p^{q+1}\Delta_{d_q-1}$$

and $K' = K + \Delta_{q+1}$. □

If we are not observant, we can find ourselves letting q get too large in this proof and creating a $[v_{m+1}]$, which does not exist. Looking closely, we find that the smallest space this could happen in is $k = g(n,m)$. However, the counting for the $n > 0$ case that matters here, is done for differentials and the targets all have an e with them which throws this problem up to the $k = g(n,m)+1$ space. We are not working in this range so this doesn't affect us. It does tell us that the splitting cannot be delooped though since it does say we cannot have as many differentials on our smaller space as we would need to get the size of the homology down to where it splits off. That is, all of our necessary targets are not there in the next space so some possible sources will survive. They do not survive in the space for $P(n)$ though, so the homology cannot split off.

The final problem which arises is the solving of the extension problems to give us the proper homology. Again, the proofs are the same for the $P(n,m)$ and $P(n)$ cases. The counting argument of the previous proof is used again in this proof. Here we need to solve various extension problems. First, we show that certain elements cannot be generators and then we show that the only thing that can prevent them from being generators is if they are p-th powers. We then show that they are in one-to-one correspondence with the only elements which could possibly have non-trivial p-th powers. We use the same counting Lemma as in the previous proof. This time, in the $n > 0$ case, we must always have an $a_{(i)}$ involved in the p-th powers which again throws it up to the $g(n,m) + 1$ space before we see the creation of an unwanted $[v_{m+1}]$. However, in the $n = 0$ case, we see our p-th power extensions can, and do, actually occur in the $g(0, m)$ space. Our result, in this case, is only correct as coalgebras and in the small space the homology is not a polynomial algebra and so cannot split off of the homology of the larger space as Hopf algebras, thus preventing the splitting from being as H-spaces. This completes the proof.

4. Appendix: $p = 2$

This paper requires the results of [RW96]. However, in that paper the results were not stated explictly for $p = 2$. Because $p = 2$ is such an important part of the contribution of this paper, we must rectify that enough to do the $p = 2$ case here. The lack of precision with $p = 2$ in [RW96] originates in a similar vagueness in [Wil84]. First, we will straighten out the situation in [Wil84] and then we will do the same for [RW96].

The key to the solution is mentioned in the $p = 2$ comments in [Wil84, page 1030], namely, the element e must be included in the coproduct of the elements a_i. In particular, the Verschiebung is evaluated as $V(a_{(0)}) = e$. Using the mod 2 homology formula, $a^{*2}_{(n-1)} = a_{(0)} \circ [v_n] \circ b^{2^n-1}_{(0)}$, we can compute

$$(e \circ a_{(n-1)})^{*2} = a_{(0)} \circ (a_{(n-1)})^{*2} = a_{(0)} \circ a_{(0)} \circ [v_n] \circ b^{2^n-1}_{(0)}.$$

We note that

$$V(a_{(0)} \circ a_{(0)}) = V(a_{(0)}) \circ V(a_{(0)}) = e \circ e = b_1 = b_{(0)}.$$

Since $V(b_{(1)}) = b_{(0)}$ also and $b_{(1)}$ is the only element in degree 4 of this space, we must have $a_{(0)} \circ a_{(0)} = b_{(1)}$. Thus, we have $(e \circ a_{(n-1)})^{*2} = [v_n] \circ b^{2^n-1}_{(0)} \circ b_{(1)}$. For p odd, all elements with e in them were exterior. However, for $p = 2$, we need to look at all elements containing $e \circ a_{(n-1)}$ in both the cases we are considering.

For odd primes, we recall the result of [Wil84, Theorem 1] as

$$H_* \underline{K(n)}_* \simeq \bigotimes_{j_0 < p^n - 1} E(a^I b^J \circ e_1) \bigotimes_{\substack{I \neq I(1) \\ \text{if } i_0 = 1, \\ \text{then } j_0 < p^n - 1}} TP_{\rho(I)}(a^I b^J) \bigotimes_{\substack{I = I(1) \\ j_0 < p^n - 1}} P(a^I b^J)$$

where H is the mod p standard homology, $I(1)$ denotes the sequence of all ones and all $j_k < p^n$. The number $\rho(I) > 0$ is the smallest k with $i_{n-k} = 0$.

For $p = 2$ we must modify this to take into account the additional nontrivial squares which we have already identified. In this case our description is precise.

Theorem 4.1. *For $p = 2$, with the above constraints, $H_*\underline{K(n)}_* \simeq$*

$$\bigotimes_{\substack{j_0 < 2^n - 1 \\ i_{n-1} = 0}} E(a^I b^J \circ e_1) \bigotimes_{\substack{j_0 < 2^n - 1 \\ i_{n-1} = 1}} TP_{\rho(s(I-\Delta_{n-1}))+1}(a^I b^J e_1)$$

$$\bigotimes_{\substack{I \neq I(1) \\ \text{if } i_0 = 1, \\ \text{then } j_0 < 2^n - 1 \\ \text{if } i_0 = 0 \text{ and } j_0 = 2^n - 1 \\ \text{then } j_1 = 0}} TP_{\rho(I)}(a^I b^J) \bigotimes_{\substack{I = I(1) \\ j_0 < 2^n - 1}} P(a^I b^J)$$

The essentials of the proof remain unchanged. We have taken the exterior generators which should be truncated polynomial generators and we have taken away the generators which are their squares. The size remains the same in either description. This is significantly easier than our next case because here we can solve our extension problems precisely.

It is tedius to reproduce the results of [RW96] here and then modify them slightly. We will keep the flavor of the rest of the paper and only produce the modifications.

Theorem 4.2. *For $p = 2$, the standard mod 2 homology, $H_*(\underline{P(n)}_*)$, fits in a short exact sequence of Hopf algebras with the associated graded algebra being given by* [RW96, Theorem 1.3] *(the odd prime answer). The quotient Hopf algebra is just the exterior algebra on generators $ea^I[v^K]b^J$ as in* [RW96, Theorem 1.3] *with $i_{n-1} = 1$. These elements, in the actual algebra, all have nontrivial squares which are contained in the set of generators of the subalgebra given by $a^I[v^K]b^J$ with $i_0 = 0$ and $(K, J) \in \mathcal{A}_n - \mathcal{A}_n^+$.*

Proof. In the Morava K-theory case we could evaluate the necessary squares directly. Here we cannot. It is essential that we know the squares are all nontrivial and linearly independent but it is not obvious how to do that directly. However, the proof in [RW96] need only be modified slightly. In particular, we can still work in the same bar spectral sequence with the same elements. We must consider the elements $a^I[v^K]b^J$ with $i_0 = 0$ and $(K, J) \in \mathcal{A}_n - \mathcal{A}_n^+$. After double suspension to $a^I[v^K]b^{J+\Delta_0}$ we know this is zero mod $*$ since $(K, J) \notin \mathcal{A}_n^+$. However, it cannot be a square because it has no $a_{(0)}$ in it and its degree is a multiple of 4. (If it was the square of an elements with e in it, thus making the $a_{(0)}$ unnecessary, then it would have to have degree 2 mod 4). All differential targets must be odd degree so our double suspended element must be zero which implies something happened to it in the previous spectral squence as $ea^I[v^K]b^J$. Since it is odd degree it cannot be a square so it either is the target of a differential or it was already zero. Both happen. The counting argument of [RW96, page 1061-2] pairs these up with the potential source of differentials. For $p = 2$, the count, as given, uses $\gamma_2(\sigma ea^I[v^K]b^J)$ when $i_{n-1} = 1$ and $(K, J) \in \mathcal{A}_n^+$. However, at $p = 2$, this would have to be a d_1 differential which does not exist. We know that the elements which should be targets must be zero so the ones associated with these γ_2 must have already been zero. These are in 1-1 correspondence with our γ_2, or our $ea^I[v^K]b^J$ when $i_{n-1} = 1$ and

$(K, J) \in \mathcal{A}_n^+$. These are precisely the elements we wish to have non-trivial squares! Thus, the only solution to our problem is that they have non-trivial sauares, linearly independent, among the $a^I[v^K]b^J$ with $i_0 = 0$ and $(K, J) \in \mathcal{A}_n - \mathcal{A}_n^+$. Because they are squares, they never show up as $\sigma a^I[v^K]b^J$ in the next spectral sequence so they do not have to be killed there. Likewise, because $ea^I[v^K]b^J$ is not exterior, the γ_2 we worried about in the spectral sequence does not exist so those unwanted elements go away as well. This concludes the the discussion of the differences for $p = 2$. □

References

[BJW95] J. M. Boardman, D. C. Johnson, and W. S. Wilson. Unstable operations in generalized cohomology. In I. M. James, editor, *The Handbook of Algebraic Topology*, chapter 15, pages 687–828. Elsevier, 1995.

[BW] J. M. Boardman and W. S. Wilson. $k(n)$-torsion-free H-spaces and $P(n)$-cohomology. In preparation.

[Qui71] D. G. Quillen. Elementary proofs of some results of cobordism theory using Steenrod operations. *Adv. in Math.*, 7:29–56, 1971.

[RW77] D. C. Ravenel and W. S. Wilson. The Hopf ring for complex cobordism. *Journal of Pure and Applied Algebra*, 9:241–280, 1977.

[RW96] D. C. Ravenel and W. S. Wilson. The Hopf ring for $P(n)$. *Canadian Journal of Mathematics*, 48(5):1044–1063, 1996.

[RWY98] D. C. Ravenel, W. S. Wilson, and N. Yagita. Brown-Peterson cohomology from Morava K-theory. *K-Theory*, 15(2):149–199, 1998.

[SY76] N. Shimada and N. Yagita. Multiplications in complex bordism theory with singularities. *Publications of Research Institute of Mathematical Sciences, Kyoto University*, 12:259–293, 1976.

[Wil73] W. S. Wilson. The Ω-spectrum for Brown-Peterson cohomology, Part I. *Commentarii Mathematici Helvetici*, 48:45–55, 1973.

[Wil75] W. S. Wilson. The Ω-spectrum for Brown-Peterson cohomology, Part II. *American Journal of Mathematics*, 97:101–123, 1975.

[Wil84] W. S. Wilson. The Hopf ring for Morava K-theory. *Publications of Research Institute of Mathematical Sciences, Kyoto University*, 20:1025–1036, 1984.

Johns Hopkins University, Baltimore, Maryland 21218
E-mail address: `jmb@math.jhu.edu, wsw@math.jhu.edu`