$K(n+1)$ equivalence implies $K(n)$ equivalence

W. Stephen Wilson

Abstract. We give an entirely different proof of a recent result of Bousfield's which states that if there is a map of spaces inducing an isomorphism on the $(n+1)^{\text{st}}$ Morava K-theory then it also induces an isomorphism on the n^{th} Morava K-theory. The result relies heavily on the fundamentals introduced to prove the results of [**RWY**] which in turn relies on the Boardman-Wilson, [**BW**], generalization of Quillen's theorem that $MU^*(X)$ is generated by non-negative degree elements when X is a finite complex.

1. Introduction

Bousfield has shown that a map which induces an isomorphism on the Morava K-theory, $K(n+1)_*(-)$, also induces an isomorphism on $K(n)_*(-)$. This is easily seen, by taking the cofibre, to be equivalent to showing that if $K(n+1)_*(X)$ is acyclic then so is $K(n)_*(X)$. We state a cohomology version which is what we will prove. It is equivalent to the homology version by duality, i.e. X is $K(n)$ homology acyclic if and only if it is $K(n)$ cohomology acyclic. Our version is weaker than Bousfield's because we assume X is of finite type and he has no restrictions on X.

Theorem 1.1 (Bousfield, [**Bou**]). *Let X by a space of finite type with $K(n+1)^*(X)$ trivial, then $K(n)^*(X)$ is trivial for $n > 0$.*

Our proof has little, if anything, in common with Bousfield's and so we hope it will be of independent interest. Ravenel, in [**Rav84**, Theorem 2.11], proves this result for X finite. Our proof makes heavy use of work on phantom maps and the Atiyah-Hirzebruch spectral sequence in [**RWY**] which was preparatory to the main theorems of that paper. This work in turn was made possible by the generalized Quillen theorem in [**BW**] which says that $P(n)^*(X)$, for X finite, is generated by non-negative degree elements.

We use reduced theories throughout. Special thanks go out to Pete Bousfield, Dan Christensen, Douglas Ravenel, Hal Sadofsky, and Neil Strickland, all of whom have help me to clarify my thoughts on these matters.

2. Proof

We recall the generalized cohomology theory $E(n,n+1)$, $n > 0$, with coefficient ring $E(n,n+1)^* \simeq \mathbf{F}_p[v_n, v_{n+1}, v_{n+1}^{-1}]$.

LEMMA 2.1. *Let X be a space of finite type. If $K(n+1)^*(X)$ is trivial then $E(n,n+1)^*(X)$ is trivial.*

PROOF. There is a stable cofibration $E(n,n+1) \xrightarrow{v_n} E(n,n+1) \to K(n+1)$. This gives rise to a long exact sequence in cohomology theories and since $K(n+1)^*(X)$ is trivial, multiplication by v_n is an isomorphism on $E(n,n+1)^*(X)$. However, by [**RWY**, Corollary 4.11] there are no infinitely v_n divisible elements in $E(n,n+1)^*(X)$ so this must be trivial. □

PROOF OF THEOREM 1.1. By [**RWY**, Corollary 4.8] we have no phantom maps for either $K(n)^*(X)$ or $E(n,n+1)^*(X)$ and so the Atiyah-Hirzebruch spectral sequence for these theories converges. By the same result, we can obtain the Atiyah-Hirzebruch spectral sequence for these theories by tensoring $K(n)^*$ or $E(n,n+1)^*$ with the Atiyah-Hirzebruch spectral sequence for $P(n)^*(X)$ where $P(n)$ is the theory with coefficient ring BP^*/I_n where $BP^* \simeq \mathbf{Z}_{(p)}[v_1, v_2, \dots]$ and $I_n = (p, v_1, \dots, v_{n-1})$. Combining this with [**RWY**, Lemma 4.4] we have, for an arbitrary fixed s, some m for which $E_m^{s,*} \simeq E_\infty^{s,*}$ for all of the theories $P(n)$, $K(n)$ and $E(n,n+1)$. Our acyclic assumption and Lemma 2.1 tell us that $E_m^{s,*}(P(n)^*(X)) \otimes_{P(n)^*} E(n,n+1)^*$ is zero. $E_m^{s,*}(P(n)^*(X))$ is a finitely presented $P(n)^*(P(n))$ module and so has a Landweber filtration (from [**Yag76**] and [**Yos76**]), see [**RWY**, Theorem 3.10]. Again from Yagita and Yosimura, tensoring with $K(n)^*$ or $E(n,n+1)^*$ is exact, see [**RWY**, Theorem 3.9]. The quotients of the Landweber filtration are $P(n)^*/I_{q,n}$ where $I_{q,n} = (v_n, \dots, v_{q-1})$. By exactness we know that $E(n,n+1)^*$ tensored with each quotient must be zero. That means that each q must be greater than $n+1$. Tensoring those quotients with $K(n)^*$ also gives zero. Thus we have shown that $E_m^{s,*}(P(n)^*(X)) \otimes_{P(n)^*} K(n)^*$ is zero. This tells us the whole Atiyah-Hirzebruch spectral sequence converges to zero and our result is complete. □

References

[Bou] A. K. Bousfield. On Morava K-equivalences of spaces. In preparation.

[BW] J. M. Boardman and W. S. Wilson. $k(n)$-torsion-free H-spaces and $P(n)$-cohomology. In preparation.

[Rav84] D. C. Ravenel. Localization with respect to certain periodic homology theories. *American Journal of Mathematics*, 106:351–414, 1984.

[RWY] D. C. Ravenel, W. S. Wilson, and N. Yagita. Brown-Peterson cohomology from Morava K-theory. *K-Theory*. To appear.

[Yag76] N. Yagita. The exact functor theorem for BP_*/I_n-theory. *Proceedings of the Japan Academy*, 52:1–3, 1976.

[Yos76] Z. Yosimura. Projective dimension of Brown-Peterson homology with modulo $(p, v_1, \dots, v_{n-1})$ coefficients. *Osaka Journal of Mathematics*, 13:289–309, 1976.

JOHNS HOPKINS UNIVERSITY, BALTIMORE, MARYLAND 21218
E-mail address: wsw@math.jhu.edu