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Brown–Peterson Cohomology from Morava K-Theory, II

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Abstract. We improve on some results with Ravenel and Yagita in a paper by the same name. In particular, we generalize when injectivity, surjectivity, and exactness of Morava K-theory implies the same for Brown–Peterson cohomology. A type of flatness is no longer assumed, but instead it is a consequence of weaker assumptions. The main application is an easier proof that QS^{2k+1} has this flatness property. In addition, we show that if elements in the Brown–Peterson cohomology generate all of the Morava K-theories, then they also generate the Brown–Peterson cohomology and it is *Landweber flat*.

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1. Introduction

This paper follows [5] and improves several results contained there. [5] was quite narrowly focussed on spaces with even degree Morava K-theory. One of the main theorems was that this implies Landweber flatness for the Brown–Peterson cohomology. Towards the end of the writing of [5] it was realized that some of the hypotheses could be weakened to assume Landweber flat rather than even Morava K-theory, which implied it. Some of these results made it into [5], some were noticed independently by Kashiwabara, [3], and some appear here. The difficult parts of most of the proofs are still contained in [5]. This paper consists mostly of a rethinking of those results with the advantage of hindsight. Upon rethinking, it appears we essentially proved much stronger results than we had realized. The contribution here is to significantly weaken hypotheses and strengthen the conclusions. Not only do we replace even Morava K-theory with Landweber flatness, but sometimes we can do without the Landweber flatness as well.

Recall that the coefficient ring for Brown–Peterson cohomology is $BP^* \simeq \mathbf{Z}_{(p)}[v_1, v_2, \dots]$ where the degree of v_n is $-2(p^n - 1)$. Let I_n be the ideal $(p, v_1, \dots, v_{n-1})$. Let $BP\langle q\rangle^* = \mathbf{Z}_{(p)}[v_1, \dots, v_q]$. There are theories $P(n)$ and $E(k, n)$ with coefficient rings BP^*/I_n and $v_n^{-1}BP\langle n\rangle^*/I_k$, respectively. Let $P(0)$ be BP if $\lim^1 BP^*(X^m) = 0$ for each space X under discussion, and the p-adic completion of BP, $BP_p^\wedge$, if any of the spaces do not have this property. Likewise, if we have chosen $P(0)$ to be $BP_p^\wedge$ then we choose $E(0, n) = E(n)$ to be the p-adic completion as well. We say a $P(k)^*$-module, M, is Landweber flat if it is a flat $P(k)^*$-

module for the category of $P(k)^*(P(k))$-modules which are finitely presented over $P(k)^*$. This is equivalent to having the map v_q on M/I_qM be injective for all $q \geqslant k$ ([4] for $k = 0$, [7], and [6] for $k > 0$). When M is $P(k)^*(X)$ for X a space, this is equivalent to the following short exact sequences for all $n \geqslant k$:

$$0 \to P(n)^*(X) \xrightarrow{v_n} P(n)^*(X) \to P(n+1)^*(X) \to 0. \tag{1.1}$$

We have, from [5, Corollary 4.8], that $E^*(X) \simeq \lim^0 E^*(X^i) \simeq \lim^0 E^*(X)/F^i$ for $E = P(k)$ and $E(k, n)$, i.e. there are no phantom maps and the associated $\lim^1$ terms are all zero. F^i is the skeletal filtration. We work only with spaces with $\mathbf{Z}_{(p)}$-homology of finite type.

Our results are:

THEOREM 1.2 (Surjection). *Let $k \geqslant 0$. If $P(k)^*(X_2)$ is Landweber flat, and $f\colon X_1 \to X_2$ has $f^*\colon K(n)^*(X_2) \to K(n)^*(X_1)$ surjective for all $n \geqslant k$ $(n > 0)$, then $f^*\colon P(k)^*(X_2) \to P(k)^*(X_1)$ is also surjective and $P(k)^*(X_1)$ is Landweber flat.*

THEOREM 1.3 (Injection). *Let $k \geqslant 0$. If $P(k)^*(X_1)$ is Landweber flat, and $f\colon X_1 \to X_2$ has $f^*\colon K(n)^*(X_2) \to K(n)^*(X_1)$ injective for all $n \geqslant k$ $(n > 0)$, then $f^*\colon P(k)^*(X_2) \to P(k)^*(X_1)$ is also injective and $P(k)^*(X_2)$ is Landweber flat.*

THEOREM 1.4 (Exactness). *Let $k \geqslant 0$. Let spaces X_i, $i = 2, 3$, have $P(k)^*(X_i)$ Landweber flat. If $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3$ has $f_2 \circ f_1 \simeq 0$ and gives rise to exact sequences (as $K(n)^*$ modules)*

$$0 \longleftarrow K(n)^*(X_1) \xleftarrow{f_1^*} K(n)^*(X_2) \xleftarrow{f_2^*} K(n)^*(X_3)$$

for all $n \geqslant k$ $(n > 0)$, then we get an exact sequence:

$$0 \longleftarrow P(k)^*(X_1) \xleftarrow{f_1^*} P(k)^*(X_2) \xleftarrow{f_2^*} P(k)^*(X_3)$$

and $P(k)^*(X_1)$ is Landweber flat.

THEOREM 1.5 (Hopf exactness). *Let $k \geqslant 0$. Let spaces X_i, $i = 2, 3$, have $P(k)^*(X_i)$ Landweber flat. Assume that*

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3$$

has $f_2 \circ f_1 \simeq 0$ and all spaces are H-spaces and all maps are H-space maps. If we have exact sequences of bicommutative Hopf algebras for all $n \geqslant k$ $(n > 0)$:

$$K(n)_* \longrightarrow K(n)_*(X_1) \xrightarrow{f_{1*}} K(n)_*(X_2) \xrightarrow{f_{2*}} K(n)_*(X_3),$$

then $P(k)^*(X_1) \simeq P(k)^*(X_2)/(f_2^*)$ and $P(k)^*(X_1)$ is Landweber flat.

In [5], these theorems were proven under the assumption that all spaces involved had even Morava K-theory. The main result of [5] says that even Morava K-theory implies Landweber flat. Kashiwabara noticed, [3], that the proofs did not use the even Morava K-theory, which results in $P(k)^*(-)$ being even, but only the Landweber flatness property. We weaken the conditions one step further and show that it isn't necessary to assume Landweber flatness for one of the spaces. Landweber flatness for that space becomes our main result instead of our assumption! The main application is to show the Landweber flatness of $BP^*(QS^{2k+1})$ which follows immediately from the results above and Kashiwabara's paper [2]. This is something we could not do in [5] but which Kashiwabara managed to do in [3]. Kashiwabara's proof is much more difficult, but his paper has more applications at this time. In addition, in future joint work with Takuji Kashiwabara, we expect to use these results to show the Landweber flatness of $P(k)^*(\underline{BP\langle n\rangle}_*)$.

We thank the referee for leading us to the following result by pointing out a circular argument in a proof.

THEOREM 1.6. *Let $k \geqslant 0$ and let $T_k \subset P(k)^*(X)$ be almost all in F^s, the sth skeletal filtration, for all s. The following three conditions are equivalent and all imply that $P(k)^*(X)$ is Landweber flat. Furthermore, if $P(k)^*(X)$ is Landweber flat, then such a T_k exists.*

(i) *For each $n \geqslant k$, $P(n)^*(X)$ is generated topologically as a $P(n)^*$-module by the image of T_k.*
(ii) *For each $n \geqslant k$ ($n > 0$), $K(n)^*(X)$ is generated topologically as a $K(n)^*$-module by the image of T_k.*
(iii) *For each $n \geqslant k$ ($n > 0$), $v_n^{-1}P(n)^*(X)$ is generated topologically as a $v_n^{-1}P(n)^*$-module by the image of T_k.*

Remark 1.7. Theorem 1.21 of [5] says that if $P(k)^*(X)$ is Landweber flat and T_k generates all of the Morava K-theories, then T_k also generates $P(k)^*(X)$. What we see now is that we need not assume $P(k)^*(X)$ is Landweber flat; it is one of our conclusions. The first condition gives us that $P(n)^*(X)$ surjects to $P(n+1)^*(X)$ which gives us Landweber flatness. The fact that in this case the elements generate the Morava K-theories is implicit in [5]. The existence of T_k is done in [5, Theorem 1.20] but we state it here for completeness.

We have another item of some interest which, as the referee pointed out, we used without proof in our first version.

THEOREM 1.8. *If $P(k)^*(X)$ and $P(k)^*(Y)$ are both Landweber flat then $P(k)^*(X \times Y)$, which is $P(k)^*(X)\hat{\otimes}P(k)^*(Y)$, is also Landweber flat.*

PROPOSITION 1.9. *For $n \geqslant k > 0$ or if there are no phantom maps, $k = 0$, $n > 0$,*

$$E(k,n)^*(X) \simeq P(k)^*(X)\hat{\otimes}E(k,n)^*$$

and

$$v_n^{-1}P(k)^*(X) \simeq P(k)^*(X)\hat{\otimes}v_n^{-1}P(k)^*.$$

This is of some independent interest because it shows, for example, how to calculate the Morava K-theory, $K(n)^*(X)$, from $P(n)^*(X)$ for infinite complexes ($E(n,n) = K(n)$). This definitely requires the completed tensor product. This is not the algebraic localization, i.e. the standard tensor product, which gives the wrong answer. Here is where the Morava Structure Theorem of [1] fails for the cohomology of infinite complexes. There are some complications with the proof when attempted for the p-adically completed $k = 0$ case when there are phantom maps. However, we don't really need that case.

2. Proofs

Proof of Proposition 1.9. We have, from [5, Corollary 4.8], that $E^*(X) \simeq \lim^0 E^*(X^i) \simeq \lim^0 E^*(X)/F^i$ for $E = P(k)$, $v_n^{-1}P(k)$, and $E(k,n)$.

$$\begin{aligned} E(k,n)^*(X) &= \lim{}^0 E(k,n)^*(X^j) \\ &= \lim{}^0 (P(k)^*(X^j) \otimes E(k,n)^*) && \text{because } E(k,n)^* \text{ is Landweber flat} \\ &= \lim{}^0 (P(k)^*(X)/F^i \otimes E(k,n)^*) && \text{from [5, Corollary 4.6]} \\ &= P(k)^*(X)\hat{\otimes}E(k,n)^* && \text{by definition.} \end{aligned}$$

The same proof works for $v_n^{-1}P(k)$. □

Many of the equivalent versions of Landweber flatness are nicely written down in [3]. We will use the following observation:

PROPOSITION 2.1. *$P(k)^*(X)$ is Landweber flat if and only if*

$$0 \to E(q,n)^*(X) \xrightarrow{v_q} E(q,n)^*(X) \to E(q+1,n)^*(X) \to 0$$

is exact for all $n > q \geqslant k$.

Proof. Landweber flatness is equivalent to having the exact sequences of Equation (1.1). Thus, to get Landweber flatness it is enough to show that any element of $P(q)^*(X)$, $q \geqslant k$, is not v_q torsion. By Proposition 4.12 of [5] we know that for any $x \in P(q)^*(X)$ there is an N such that x maps nontrivially to $E(q,n)^*(X)$ for $n \geqslant N$. Since we are assuming that $E(q,n)^*(X)$ does not have v_q torsion, then our element is not v_q torsion and we are done with the 'if' part. For the 'only if' part we can use Proposition 1.9 to see that the image of $P(q+1)^*(X)$ generates $E(q+1,n)^*(X)$. Since $P(q)^*(X) \to P(q+1)^*(X)$ is surjective, its image must generate $E(q+1,n)^*(X)$ as well. This map factors through $E(q,n)^*(X)$ which forces surjectivity $E(q,n)^*(X) \to E(q+1,n)^*(X)$. □

Proof of Theorem 1.6. Theorem 1.20 of [5] gives the existence of a set T_k satisfying (i) if $P(k)^*(X)$ is Landweber flat. If we have a set T_k which satisfies condition (i) then we have surjections $P(n)^*(X) \to P(n+1)^*(X)$ for all $n \geqslant k$ and so we have $P(k)^*(X)$ is Landweber flat by (1.1).

We will show that (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). Letting $k = n$ in Proposition 1.9 we see that (i) $\Rightarrow$ (iii). Likewise, still using Proposition 1.9, it follows that (iii) $\Rightarrow$ (ii).

We will now show that (ii) implies Landweber flatness, which in turn implies (i). We use Proposition 2.1 and show that $E(q,n)^*(X) \to E(q+1,n)^*(X)$ is surjective for $k \leqslant q < n$. Landweber flatness then follows. Our proof is by downward induction on q. We first show that $E(q,n)^*(X) \to E(q+1,n)^*(X)$ is surjective and then we will show that T_k generates $E(q,n)^*(X)$. We can ground our induction because $E(n,n)^*(X)$ is just $K(n)^*(X)$. So, by our inductive assumption, (the image of) T_k generates $E(q+1,n)^*(X)$. Since T_k factors through $E(q,n)^*(X)$ we get the surjection. All we need now is to show that T_k generates $E(q,n)^*(X)$. We have the exact sequence:

$$0 \to E(q,n)^*(X) \xrightarrow{v_q} E(q,n)^*(X) \xrightarrow{\rho} E(q+1,n)^*(X) \to 0.$$

Recall that our set $T_k = \{t_1, t_2, \dots\} \subset E(q,n)^*(X)$ has the property that all but a finite number are in F^s. Given an arbitrary $y_0 \in E(q,n)^*(X)$, we want to show that $y_0 = \sum b_i t_i$, $b_i \in E(q,n)^*$. Since there are no phantom maps this sequence converges. Since the $\rho(t_i)$ generate $E(q+1,n)^*(X)$ we can write $\rho(y_0) = \sum a_{0,i}\rho(t_i)$. Abusing notation and picking lifts for the $a_{0,i}$, we can see that $\rho(y_0 - \sum a_{0,i}t_i) = 0$ and by the exact sequence there is a y_1 with $v_q y_1 = y_0 - \sum a_{0,i}t_i$ so $y_0 = \sum a_{0,i}t_i + v_q y_1$. Repeating this process we get $y_n = \sum a_{n,i}t_i + v_q y_{n+1}$. Substituting, we have, $y_0 = \sum a_{0,i}t_i + v_q a_{1,i}t_i + v_q^2 a_{2,i}t_i + v_q^3 a_{3,i}t_i + \cdots$. Modulo F^s this is a finite sum. This must be because the higher powers of v_q force the t_i to be in higher and higher degrees, so the low ones are not used when there is a high power of v_q. Since there are no phantom maps, the group is the inverse limit and so the series converges to y_0. (They are two well defined elements which agree modulo each F^s.) We do not need to show that the set T_0 generates $E(0,q)^*(X)$ since we already have all of the surjections we need.

Now we have that $P(k)^*(X)$ is flat. Combined with our assumption (ii), Theorem 1.21 of [5] tells us that T_k generates $P(k)^*(X)$. Theorem 1.20 of [5] then tells us that T_k generates $P(n)^*(X)$, and we have (i). $\square$

Remark 2.2. Theorem 1.21 of [5] shows that if T_k satisfies (ii) and $P(k)^*(X)$ is already Landweber flat, then T_k generates $P(k)^*(X)$. The proof is somewhat long and complex. However, the only place the Landweber flatness assumption is used in the proof is at the beginning of the proof of Lemma 6.3 where it is used to prove

$$v_n^{-1}P(n)^*(X) \simeq v_n^{-1}P(n)^* \widehat{\otimes}_{v_n^{-1}P(k)^*} v_n^{-1}P(k)^*(X).$$

Indeed, flatness makes this easy, but (ii) will get it for us just the same.

Proof of Theorem 1.8. By Theorem 1.6 we have sets $T_k^X \subset P(k)^*(X)$ and $T_k^Y \subset P(k)^*(Y)$ which generate all the Morava K-theories for $n \geqslant k$. Because $K(n)^*(-)$ has a Künneth isomorphism, the set $T_k^X \otimes T_k^Y$ generates the Morava K-theories of $X \times Y$. By Theorem 1.6, $P(k)^*(X \times Y)$ is Landweber flat and we are done. Note that Theorem 1.11 of [5] gives us the Künneth isomorphism for $P(k)$ in our case. □

Proof of Theorem 1.2, surjectivity. Using Theorem 1.20 of [5] we can find generators for $P(k)^*(X_2)$ which map to generators of $K(n)^*(X_2)$ for all $n \geqslant k$. These generators map, by the surjection, to generators of $K(n)^*(X_1)$. By commutativity of the maps, our generators map to elements of $P(k)^*(X_1)$ which generate the $K(n)^*(X_1)$. By Theorem 1.6 we see that these elements generate $P(k)^*(X_1)$, which implies surjectivity. Furthermore, Theorem 1.6 implies that $P(k)^*(X_1)$ is Landweber flat. □

Proof of Theorem 1.3, injectivity. We start by using downward induction to show that $E(q,n)^*(X_2) \to E(q,n)^*(X_1)$ is injective if $K(n)^*(-)$ is, $n \geqslant q \geqslant k$. Landweber flatness implies the short exact sequences (Proposition 2.1):

$$0 \to E(q,n)^*(X_1) \xrightarrow{v_q} E(q,n)^*(X_1) \to E(q+1,n)^*(X_1) \to 0.$$

To show flatness for X_2 it is enough to show that there is a similar exact sequence which injects to this one (Proposition 2.1). Given an element $y \in E(q,n)^*(X_2)$, write it as $y = v_q^m x$ with x reducing nontrivially to $E(q+1,n)^*(X_2)$ (Corollary 4.11 of [5] tells us that there are no infinitely divisible elements). By our induction, x maps nontrivially to $E(q+1,n)^*(X_1)$ and thus to $E(q,n)^*(X_1)$. This last group is v_q-torsion free so x cannot be v_q-torsion and y must map nontrivially to $E(q,n)^*(X_1)$. This gives us our short exact sequence and thus flatness. To get injectivity for $P(k)$ we use Proposition 4.12 of [5] which states that for any $x \in P(k)^*(X_2)$ there is an N such that x maps nontrivially to $E(k,n)^*(X_2)$ for $n \geqslant N$. Since $E(k,n)^*(X_2)$ injects to $E(k,n)^*(X_1)$ we get injectivity for $P(k)^*(X_2)$ into $P(k)^*(X_1)$ as well. □

The proofs of Theorems 1.4 and 1.5 are the same as the proofs of Theorems 1.18 and 1.19 of [5] except we must use the injectivity and surjectivity results of Theorems 1.2 and 1.3 instead of Theorem 1.17 of [5]. As the referee pointed out, there is an additional requirement in the proof of Theorem 1.5. In the proof it is necessary to use Theorem 1.8 rather than the even Morava K-theory argument of Theorem 1.19 of [5].

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