CHAPTER 2

Unstable Operations in Generalized Cohomology

J. Michael Boardman

Johns Hopkins University
Baltimore, Maryland, U.S.A.

David Copeland Johnson

University of Kentucky
Lexington, Kentucky, U.S.A.

W. Stephen Wilson

Johns Hopkins University
Baltimore, Maryland, U.S.A.

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1. Introduction

A multiplicative generalized cohomology theory $E^*(-)$ on spaces is represented by the spaces $\underline{E}_n$ of its Ω-spectrum, as described in detail in [8, Thm. 3.17]. We denote its coefficient ring by E^*. Our five examples are ordinary cohomology $H^*(-;\mathbb{F}_p)$, unitary cobordism $MU^*(-)$, Brown-Peterson cohomology $BP^*(-)$, complex K-theory $KU^*(-)$, and Morava K-theory $K(n)^*(-)$. (They were properly introduced in [8, §2].) Recent work [25] shows that a sixth example, the cohomology theory $P(n)^*(-)$, also satisfies our hypotheses.

We are interested in three kinds of cohomology operation: *stable* operations, which form the endomorphism ring $E^*(E,o)$ of E (in our notation) and were studied in [8]; *unstable* operations, defined on $E^n(X)$ for spaces X and fixed n, which form $E^*(\underline{E}_n)$; and *additive* unstable operations r on $E^n(-)$ (that satisfy $r(x+y) = r(x)+r(y)$), which form the subset $PE^*(\underline{E}_n)$. Since a stable operation restricts to an additive unstable operation on any degree, these are related by

$$E^*(E,o) \longrightarrow PE^*(\underline{E}_n) \subset E^*(\underline{E}_n) \ .$$

Each of these is an E^*-module in the usual way, by $(r+s)(x) = r(x) + s(x)$ and $(vr)(x) = v\,r(x)$ (for any $v \in E^*$). We can compose, $(sr)(x) = (s \circ r)(x) = s(r(x))$, whenever the sources and targets match. We can also multiply unstable operations together by $(r \smile s)(x) = r(x)s(x)$.

In the classical case $E = H(\mathbb{F}_p)$, for which $E^*(E,o)$ is the Steenrod algebra, it is true that: (a) every additive operation comes from a stable operation; (b) the additive operations generate multiplicatively all the unstable operations. Our difficulties stem from the fact that for MU and BP, both (a) and (b) are *false*. (See [27] for more discussion of the differences.) We propose to describe completely the algebraic structure that has to be present on an E^*-module or E^*-algebra to make it an unstable object, with particular attention to the case $E = BP$. Our definitions lead to structure theorems.

Stable BP-operations have been available for quite some time and are well established. Less has been done with unstable BP-operations, owing to their complexity, but we do have the work [4, 5] of Bendersky, Curtis, Davis, and Miller. The algebraic structure on an *additively* unstable module is described in [27] and (without proofs) in [6].

Our major task, therefore, is to set up precise algebraic descriptions of the unstable structures we need on modules and algebras, along the lines of the stable structures in [8]. Part of the difficulty is that one is forced to work in the unfamiliar context of nonadditive operations; but the real problem turns out to be Thm. 9.4, that unstable modules (as distinct from unstable algebras) simply *do not exist* compatibly with our other objects! When we limit attention to the less exotic *additive* operations, this difficulty does not arise and we have both modules and algebras.

In fact, there is a huge amount of data to be codified in an unstable algebra. The key idea is that given an E^*-algebra M, we define $(UM)^k$ for each k as the set of all algebra homomorphisms $E^*(\underline{E}_k) \to M$; each such homomorphism may be thought of as a candidate for the values of all operations on a typical element

of M^k. Apparently merely a graded set, UM becomes an E^*-algebra for suitable E, thanks to extra structure on the spaces $\underline{E}_n$. Then an unstable structure on M is a homomorphism $\rho_M\colon M \to UM$ of E^*-algebras, which selects for each $x \in M^k$ the function $\rho_M(x)\colon E^*(\underline{E}_k) \to M$; then we define $r(x) = \rho_M(x)r$. This is not enough; in order to compose operations correctly, it is necessary to know the E-cohomology homomorphism $r^*\colon E^*(\underline{E}_m) \to E^*(\underline{E}_k)$ induced by each operation $r\colon E^k(-) \to E^m(-)$. This extra structure makes the functor U a *comonad*, and (M, ρ_M) a *coalgebra* over this comonad. We have a similar construction for additive operations, and can compare with the stable constructions of [8].

This is our elegant but extremely terse answer, and we do not believe that it can be efficiently expressed without using comonads. But it does have the effect that the work consists largely of definitions. In section 10, we translate this answer into practical language, in the context of Hopf rings, that we can use for computation. This includes Cartan formulae for $r(x+y)$ as well as $r(xy)$, and related formulae for $r_*(b*c)$ and $r_*(b\circ c)$ that we use to compute the induced E-homology homomorphism $r_*\colon E_*(\underline{E}_k) \to E_*(\underline{E}_m)$ dual to r^*.

Landweber filtrations. We recall that $BP^* = BP^*(T)$, the BP-cohomology of the one-point space T, is the polynomial ring $\mathbb{Z}_{(p)}[v_1, v_2, v_3 \ldots]$, with $\deg(v_n) = -2(p^n-1)$ (under our degree conventions). It contains the well-known ideals

$$I_n = (p, v_1, v_2, \ldots, v_{n-1}) \subset BP^* \tag{1.1}$$

for $0 \leqslant n \leqslant \infty$ (with the convention that $I_\infty = (p, v_1, v_2, \ldots)$, $I_1 = (p)$, and $I_0 = 0$).

The significance [8, Lemma 15.8] of I_n is that it is *invariant* under the action of the stable operations on $BP^*(T)$. Indeed, Landweber [15] and Morava [20] showed that the I_n for $0 \leqslant n < \infty$ are the only finitely generated invariant prime ideals in BP^*. Landweber used this fact to show (see [16, Thm. 3.3′] or [8, Thm. 15.11]) that a stable (co)module M that is finitely presented as a BP^*-module, including $BP^*(X)$ for any finite complex X, admits a finite filtration by invariant submodules

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_m = M \tag{1.2}$$

in which each quotient M_i/M_{i-1} is generated (as a BP^*-module) by a single element x_i whose annihilator ideal $\mathrm{Ann}(x_i) = I_{n_i}$ for some n_i. Thus $M_i/M_{i-1} \cong BP^*/I_{n_i}$.

The first unstable result on BP-cohomology, due to Quillen [22] (see Thm. 20.2), was that for a finite complex X, $BP^*(X)$ is generated, as a BP^*-module, by elements of non-negative degree. What started this project was the observation that if an unstable object M is generated by a single element x, there is an unstable operation (see Prop. 1.14 or the Remark following Cor. 20.9) that takes $v_n x$ to x, provided $\deg(x)$ is small enough; it follows that $v_n x \neq 0$ and that M cannot be isomorphic to BP^*/I_{n+1}.

The proof of Landweber's theorem depends on the concept of *primitive* element in a comodule M. Given any $x \in M$, there is the obvious homomorphism of BP^*-modules $f\colon BP^* \to M$, defined by $fv = vx$. It is a morphism of stable modules if and only if x is primitive, and if so, we have the isomorphism $BP^*/\mathrm{Ann}(x) \cong (BP^*)x \subset M$ of stable modules. An important example (see [8, Thm. 15.10]) is that the only nonzero primitives in BP^*/I_n, for $n > 0$, are the (images of the)

elements λv_n^i, where $\lambda \in \mathbb{F}_p$, $\lambda \neq 0$, and $i \geqslant 0$. For additive unstable operations, the appropriate definition of primitive becomes more restrictive.

Theorem 1.3. (This is included in Thm. 20.10.) *Let M be the BP^*-module generated by a single element x with* $\mathrm{Ann}(x) = I_n$, *where $n > 0$. Then M admits an additively unstable module structure* (as defined in section 5) *if and only if* $\deg(x) \geqslant f(n) - 2$, *and it is unique.*

The only nonzero primitive elements in M are those of the form $\lambda v_n^i x$, where $\lambda \in \mathbb{F}_p$, and $\deg(v_n^i x) \geqslant f(n)$ *if $i > 0$.*

Here, and everywhere, we need the numerical function

$$f(n) = \frac{|\deg(v_n)|}{p-1} = \frac{2(p^n-1)}{p-1} = 2(p^{n-1} + p^{n-2} + \ldots + p + 1) \tag{1.4}$$

for $n > 0$; it is reasonable to define also $f(0) = 0$.

We use this result in Thm. 20.11 to construct a Landweber filtration (1.2) of an appropriate module M, including $BP^*(X)$ for any finite complex X, in which each quotient M_i/M_{i-1} has the form in Thm. 1.3 (or is BP^*-free). Once our machinery is in working order, we are able to give a one-line proof of Thm. 20.3, the weak form of Quillen's theorem.

In our main structure theorem, we do one better by allowing all unstable operations instead of only the additive ones. One complication is that the unstable analogue of Thm. 1.3 has to be stated for algebras only, owing to the nonexistence of unstable modules.

Theorem 1.5. (This is stated precisely as Thm. 21.12.) *Let M be an unstable BP^*-algebra such as $BP^*(X)$ for a finite complex X. Then M admits a filtration* (1.2) *by invariant ideals M_i, in which each quotient M_i/M_{i-1} is generated, as a BP^*-module, by a single element x_i such that* $\mathrm{Ann}(x_i) = I_{n_i}$ *for some $n_i \geqslant 0$, where* $\deg(x_i) \geqslant \max(f(n_i)-1, 0)$.

Splittings of *BP*-cohomology. Another application of our machinery yields idempotent operations that split unstable BP-cohomology into indecomposable pieces. Such splittings were constructed in [26] by means of Postnikov systems. What is new is that explicit definitions of everything allow us to carry out computations. Our results are logically independent of [26] and rely on it only to recognize the summands as known objects; nevertheless, it is a valuable guide as to what the summands look like and where to find them. In a sequel [9], two of the authors go on to apply the structure theorems of [25] to establish analogous (but slightly different) splitting theorems for the cohomology theory $P(n)^*(-)$, whose coefficient ring is BP^*/I_n.

For each $n \geqslant 0$, we define the ideal

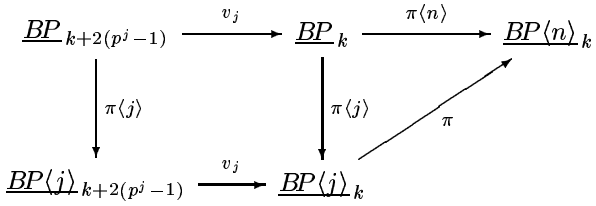
$$J_n = (v_{n+1}, v_{n+2}, v_{n+3}, \ldots) \subset BP^*. \tag{1.6}$$

In [26], Baas-Sullivan theory [2] was used to construct a cohomology theory $BP\langle n\rangle^*(-)$ having coefficients $BP^*/J_n \cong \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_n]$. In particular,

$BP\langle 0\rangle^*(-) = H^*(-;\mathbb{Z}_{(p)})$. The desired splitting is

$$BP^k(X) \cong BP\langle n\rangle^k(X) \oplus \prod_{j>n} BP\langle j\rangle^{k+2(p^j-1)}(X)\ . \tag{1.7}$$

The representing spectrum $BP\langle n\rangle$ is (at least) a BP-module spectrum, and comes equipped with a canonical map of BP-module spectra that we shall call $\pi\langle n\rangle\colon BP \to BP\langle n\rangle$. There is also a canonical map $\pi\colon BP\langle j\rangle \to BP\langle n\rangle$ whenever $j > n$. (Geometrically, $BP\langle n\rangle$ allows more singularities than $BP\langle j\rangle$.) Everything we need to know about $BP\langle n\rangle$ is contained in the commutative diagram



(1.8)

of H-spaces and H-maps, where $j > n$.

Although the cohomology theory $BP\langle n\rangle^*(-)$ may be unfamiliar, in the range of degrees of interest it is easily described in terms of BP-cohomology. It is clear by construction that $\pi\langle n\rangle_*\colon BP^*(X) \to BP\langle n\rangle^*(X)$ kills $J_nBP^*(X)$.

Theorem 1.9. *Assume that $k \leqslant f(n+1)$, where $n \geqslant 0$, and that X is finite-dimensional. Then $\pi\langle n\rangle$ induces a natural isomorphism of BP^*-modules*

$$BP^k(X) \Big/ \sum_{j>n} v_j BP^{k+2(p^j-1)}(X) \cong BP\langle n\rangle^k(X)\ . \tag{1.10}$$

We derive this below as an immediate consequence of Thm. 1.12. It is best possible, as [26] shows that $\pi\langle n\rangle_*$ is not surjective in general for $k > f(n+1)$.

Lemma 1.11. (This is included in Lemma 22.1.) *Given $k < f(n+1)$, where $n \geqslant 0$, there is an H-space splitting $\overline{\theta}_n\colon \underline{BP\langle n\rangle}_k \to \underline{BP}_k$ of $\pi\langle n\rangle\colon \underline{BP}_k \to \underline{BP\langle n\rangle}_k$ which naturally embeds $BP\langle n\rangle^k(X) \subset BP^k(X)$ as a summand* (as abelian groups).

If also $k \geqslant f(n)$, the H-space $\underline{BP\langle n\rangle}_k$ does not decompose further.

Remark. The splittings $\overline{\theta}_n$ are not canonical or unique. The ideal J_n, unlike I_n, is in no way canonical, but depends on the choice of the polynomial generators of BP^*. Although the BP-module structure of $BP\langle n\rangle$ obviously depends on J_n, it follows from the Lemma that the resulting H-space structure on $\underline{BP\langle n\rangle}_k$ is well defined. Even for fixed J_n, we find there are many choices for $\overline{\theta}_n$, and no preferred choice is apparent.

We establish Lemma 1.11 in section 22 by constructing a suitable idempotent operation θ_n on $BP^*(-)$. The second assertion implies that the first is best possible. We insert these splittings into diag. (1.8) to decompose BP-cohomology.

Theorem 1.12. *Assume $n \geqslant 0$. Then:*

(a) *For $k < f(n+1)$, the injections $\overline{\theta}_n$ and $v_j \circ \overline{\theta}_j$ from Lemma 1.11 induce the natural abelian group decomposition* (1.7)*, which is maximal if $k \geqslant f(n)$;*

(b) *For $k = f(n+1)$, we have instead the natural short exact sequence of abelian groups*

$$0 \longrightarrow \prod_{j>n} BP\langle j\rangle^{k+2(p^j-1)}(X) \longrightarrow BP^k(X) \xrightarrow{\pi\langle n\rangle_*} BP\langle n\rangle^k(X) \longrightarrow 0, \quad (1.13)$$

where none of the groups decomposes further naturally, and $\pi\langle n\rangle_*$ admits a nonadditive *natural splitting $\overline{\theta}_n\colon BP\langle n\rangle^k(X) \to BP^k(X)$, so that we have eq.* (1.7) *as a bijection of sets.*

Remark. The simplified description of $BP\langle -\rangle$-cohomology in Thm. 1.9 applies everywhere (when X is finite-dimensional). These splittings definitely do not preserve the BP^*-module structure. We plan to return to this point in future work.

Proof of Thm. 1.9. For finite-dimensional X, the sum in eq. (1.10) is in fact finite. It is clear from eq. (1.7) or (1.13) that the sum contains $\operatorname{Ker}\pi\langle n\rangle_*$. On the other hand, $\pi\langle n\rangle_*$ is a homomorphism of BP^*-modules which kills J_n. □

Projection to the first factor of the product in eq. (1.7) yields an interesting operation

$$r\colon BP^k(X) \longrightarrow BP\langle n+1\rangle^{k'}(X) \subset BP^{k'}(X),$$

where $k' = k + 2(p^{n+1}-1) = k + (p-1)f(n+1)$, which roughly has the effect of dividing by v_{n+1}. Precisely, $r(v_{n+1}y) = y$ whenever $y \in BP\langle n+1\rangle^{k'}(X) \subset BP^{k'}(X)$. Given any $x \in BP^{k'}(X)$, we can put $y = \theta_{n+1}x$; then by Thm. 1.12(a), applied to $BP^{k'}(X)$, we have $y \equiv x \bmod J_{n+1}$. For convenience, we reindex.

Proposition 1.14. *If $k \leqslant pf(n)$, there is an operation $r\colon BP^{k-2(p^n-1)}(X) \to BP^k(X)$, which is additive if $k < pf(n)$, with the property that given any element $x \in BP^k(X)$, where X is finite-dimensional, there exists y such that $y \equiv x \bmod J_nBP^*(X)$ and $r(v_ny) = y$.* □

Equivalently, we can represent eq. (1.7) by the decomposition of spaces

$$\underline{BP}_k \simeq \underline{BP\langle n\rangle}_k \times \prod_{j>n} \underline{BP\langle j\rangle}_{k+2(p^j-1)} \ . \quad (1.15)$$

Theorem 1.16. *Assume $n \geqslant 0$. Then:*

(a) *For $k < f(n+1)$, we have the H-space decomposition* (1.15)*, which is maximal if $k \geqslant f(n)$;*

(b) *For $k = f(n+1)$, we have the fibration*

$$\prod_{j>n} \underline{BP\langle j\rangle}_{k+2(p^j-1)} \longrightarrow \underline{BP}_k \xrightarrow{\pi\langle n\rangle} \underline{BP\langle n\rangle}_k \quad (1.17)$$

of H-spaces and H-maps, which admits a section (not an H-map), so that eq. (1.15) *holds as an equivalence of spaces (but not as H-spaces), and none of the spaces decomposes further as a product of spaces.* (In other words, $BP^k(-)$ is represented by the right side of eq. (1.15), equipped with a different H-space multiplication.)

We use Lemma 1.11 to prove parts (a) of Thms. 1.12 and 1.16 in section 22. For parts (b), the necessary idempotent θ_n has to be nonadditive, and we construct it in section 23. We need the full strength of our machinery just to prove that θ_n is idempotent.

History. Our real motivation for this study is what is called the Johnson Question, which is stated in [24, p. 745]. Rephrased as a conjecture, it is:

Conjecture. *If $x \neq 0$ in $BP_n(X)$, where X is a space, then $v_n^i x \neq 0$ for all $i > 0$.*

No counter-examples are known, although examples exist [13, 14, 24] where $v_j x = 0$ for all $j < n$. It holds if x reduces nontrivially to homology, therefore for $n < 2p$. We hoped to circumvent our lack of knowledge of unstable homology operations by working instead with the rather better understood unstable BP-cohomology operations and using the (not at all unstable) duality spectral sequence

$$\mathrm{Ext}^{**}_{BP^*}(BP^*(X), BP^*) \Longrightarrow BP_*(X)$$

of Adams [1] (see also [12]). The reason for optimism is that if we substitute $\Sigma^k(BP^*/I_n)$ for $BP^*(X)$, a standard calculation shows that the only surviving Ext group is $\mathrm{Ext}^n = \Sigma^m(BP^*/I_n)$, with $m = f(n) - k - n$; so that $k \geqslant f(n) - 1$ implies $-m \geqslant n - 1$, almost what we want. If we confine ourselves to additive operations, we obtain $-m \geqslant n - 2$, off by one more. We can hope to work our way up from $\Sigma^k(BP^*/I_n)$ to a general $BP^*(X)$ by extension and the filtration (1.2).

This is all grounds for our suspicion that for a *geometric* unstable algebra, i. e. $M = BP^*(X)$ for some space X, the bounds in Thm. 1.5 should be one better (thus giving us $-m \geqslant n$ in the above discussion). Again, there are no known counter-examples, although spaces are known which have $\deg(x_i) = f(n_i)$, thus showing that the bounds cannot be improved by more than one.

Recently, with the help of Mike Hopkins, a new approach to the Johnson Question has been developed. It requires a much better understanding of the unstable splittings of BP. Now that we have so much explicit information on these splittings, this method of attack seems promising.

Outline. There are two main threads running through this work: the theory of additive unstable operations, which closely resembles the stable theory of [8], and the theory of all unstable operations, which is radically different. The comonad tent is big enough to accommodate both, as well as the stable theory. We have kept the additive material in separate sections so that it can be read independently.

In section 2, we discuss several classes of cohomology operation. In sections 3 and 4, we study the E-(co)homology of group objects, in preparation for sections 5 and 7, where we study modules and algebras from the additive point of view. In section 6, we consider additive operations as linear functionals. In sections 12

and 14, we study suspensions and complex orientation. In section 16, we present the additive structure for each of our five examples E.

It turns out that much of the stable machinery does not extend to all unstable operations, because it relies too heavily on the bilinearity of tensor products. However, the approach in terms of comonads does work, and in section 8 we develop the requisite comonad U. We also show in section 9 that the corresponding comonad for unstable modules does not exist and compare the various stable and unstable structures. In section 10, we convert the categorical elegance into machinery we can use; specifically, cohomology operations become linear functionals on Hopf rings. In Thm. 10.47, we display in full detail the definition of an unstable algebra from this point of view.

In sections 11, 13, and 15, we revisit the cohomology of a point, sphere, and complex projective space $\mathbb{C}P^\infty$ from this new Hopf ring point of view. These spaces alone yield almost enough generators and relations to specify the Hopf rings for our five examples E, as we discuss in detail in section 17. The case $E = KU$ is used to determine the structure of $KU_*(KU, o)$, as quoted in [8, §14]. From a sufficiently elevated perspective, the results of section 17, the additive results of section 16, and the stable results of [8] all fit into a grand master plan.

In section 20, we restrict attention to the case $E = BP$ and use the additive operations to recover Quillen's theorem and prove Thm. 20.11. This relies on the relations developed in section 18. In section 21, we use nonadditive operations to improve Thm. 20.11 by one dimension to Thm. 21.12, which is Thm. 1.5.

In section 22, we construct additive idempotent operations θ_n which yield the desired factorizations (1.7) in all except the top degree. In section 23, we finish off Thms. 1.12 and 1.16 by constructing nonadditive idempotent operations. To do this, it is necessary in section 19 to develop the notion of a Hopf ring ideal.

An index of symbols is included at the end.

This work is also notable for what it does *not* contain. There are no spectral sequences, except implicitly in the references. There are no explicit Steenrod operations, except in a few examples; in our wholesale approach, most individual operations never even acquire names. There are no formal indeterminates anywhere; the elements that are sometimes treated as such are really Chern classes x; but when $x^i = 0$, we can no longer take the coefficients of x^i.

Notation. We make heavy use of the notation and machinery developed in [8]. Topologically, we generally work in the homotopy category *Ho* of *unbased* spaces. For compatibility with the unstable notation, the E-cohomology and E-homology of a *spectrum* X are written $E^*(X, o)$ and $E_*(X, o)$. Algebraically, our most important categories are the categories *FMod* and *FAlg* of filtered E^*-modules and algebras. These and the other categories we need were introduced in [8, §6]. We make frequent use of Yoneda's Lemma. All tensor products are taken over E^* unless otherwise stated.

For reasons discussed in [8], we always give cohomology $E^*(X)$ the *profinite* topology [8, Defn. 4.9], and complete it as in [8, Defn. 4.11] to $E^*(X)\hat{}$ as necessary. In contrast, the homology $E_*(X)$ is always discrete. Because we emphasize coho-

mology, we invariably assign the degree i to elements of $E^i(X)$; this forces elements of $E_i(X)$ to have degree $-i$.

One theorem provides all the duality and Künneth isomorphisms we need.

Theorem 1.18. *Assume that $E_*(X)$ is a free E^*-module. Then we have:*

(a) $d\colon E^*(X) \cong DE_*(X)$ *in* FMod*, the strong duality homeomorphism;*

(b) $E_*(X \times Y) \cong E_*(X) \otimes E_*(Y)$*, the Künneth isomorphism in homology;*

(c) $E^*(X \times Y) \cong E^*(X) \mathbin{\widehat{\otimes}} E^*(Y)$ *in* FMod*, the Künneth homeomorphism in cohomology, provided $E_*(Y)$ is also a free E^*-module.*

Proof. We collect Thms. 4.2, 4.14, and 4.19 from [8]. Indeed, (c) follows from (a) and (b). □

Acknowledgements. The genesis of this paper is that the last two authors had worked out much of the unstable *BP* structure theorems, without having a precise definition of unstable algebra, when the first author supplied a suitable framework, of which [7] is an early version. In fact, this is an oversimplification: the various contributions are more intermingled than this might suggest. In the proper context, several of the proofs simplify significantly. We thank Martin Bendersky for pointing out Lemma 19.32, which is vastly simpler than our previous treatment.

The last two authors wish to thank the topologists at the University of Manchester and the Science and Engineering Research Council (SERC) for support during the summer of 1985 when this project got its start. The last author wishes to thank Miriam and Harold Levy for their hospitality and the peace they provided for the writing of the first draft.

2. Cohomology operations

In this section, we consider several kinds of unstable cohomology operation. Yoneda's Lemma allows us to identify the following:

(i) The *cohomology operation* $r\colon E^k(-) \to E^m(-)$;
(ii) The *cohomology class* $r = r(\iota_k) \in E^m(\underline{E}_k)$; (2.1)
(iii) The representing *map* $r\colon \underline{E}_k \to \underline{E}_m$ in Ho.

We write any of these more succinctly as $r\colon k \to m$. We use all three interpretations. Some care is needed with degrees and signs, as (i) has degree $m-k$ and (ii) has degree m, while (iii) has no degree at all.

Based operations. The following mild but useful condition can be interpreted many ways. The space T is the one-point space.

Definition 2.2. We call the operation r *based* if $r(0) = 0$ in $E^*(T) = E^*$.

Lemma 2.3. *The following conditions on an operation $r\colon k \to m$ are equivalent:*

(a) $r(0) = 0$ *in $E^*(T)$, i. e. r is a based operation;*

(b) *For any based space* (X, o), r *restricts to the* reduced *operation*

$$r\colon E^k(X, o) \longrightarrow E^m(X, o); \tag{2.4}$$

(c) *As a cohomology class,* $r \in E^m(\underline{E}_k, o) \subset E^m(\underline{E}_k)$*;*

(d) *The map* $r\colon \underline{E}_k \to \underline{E}_m$ *is (homotopically) based.*

Proof. The short exact sequence [8, (3.2)] shows that (a) and (c) are equivalent, also that (a) implies (b); but (c) is the special case of (b) for $\iota_k \in E^k(\underline{E}_k, o)$. Part (d) is just a restatement of (a). □

Given any (good) pair of spaces (X, A), we can use (b) to make based operations $r\colon k \to m$ act on *relative* cohomology as in [8, (3.4)] by

$$E^k(X, A) = E^k(X/A, o) \xrightarrow{r} E^m(X/A, o) = E^m(X, A)\ . \tag{2.5}$$

Additive operations. An *additive* operation $r\colon k \to m$ is one that satisfies $r(x+y) = r(x) + r(y)$ for any $x, y \in E^k(X)$. The universal example is

$$X = \underline{E}_k \times \underline{E}_k, \qquad \text{with } x = \iota_k \times 1,\ y = 1 \times \iota_k,\ x + y = \mu_k, \tag{2.6}$$

which gives $r(\mu_k) = r\times 1 + 1\times r$ in $E^*(\underline{E}_k \times \underline{E}_k)$. (The addition map $\mu_k\colon \underline{E}_k \times \underline{E}_k \to \underline{E}_k$ was defined in [8, Thm. 3.6].) This allows us to recognize additive operations three ways.

Proposition 2.7. *The following conditions on an operation* $r\colon k \to m$ *are equivalent, and define the* E^**-submodule* $PE^*(\underline{E}_k) \subset E^*(\underline{E}_k, o) \subset E^*(\underline{E}_k)$*:*

(a) *The operation* $r\colon E^k(-) \to E^m(-)$ *is additive;*

(b) *The class* $r \in E^m(\underline{E}_k)$ *satisfies* $\mu_k^* r = p_1^* r + p_2^* r$ *in* $E^m(\underline{E}_k \times \underline{E}_k)$*, i. e.*

$$PE^*(\underline{E}_k) = \mathrm{Ker}[\mu_k^* - p_1^* - p_2^*\colon E^*(\underline{E}_k) \longrightarrow E^*(\underline{E}_k \times \underline{E}_k)]; \tag{2.8}$$

(c) *The map* $r\colon \underline{E}_k \to \underline{E}_m$ *is a morphism of group objects in* **Ho**. □

Corollary 2.9. *Assume that* $E_*(\underline{E}_k)$ *is a free* E^**-module. Then* $PE^*(\underline{E}_k)$ *is complete Hausdorff and so an object of* **FMod**.

Proof. In eq. (2.8), $E^*(\underline{E}_k)$ and $E^*(\underline{E}_k \times \underline{E}_k)$ are complete Hausdorff by Thm. 1.18. □

When $E_*(\underline{E}_k)$ is free, the Künneth homeomorphism for $E^*(\underline{E}_k \times \underline{E}_k)$ makes $E^*(\underline{E}_k)$ a completed Hopf algebra; then (b) agrees with the primitives in the sense of [8, (6.13)], completed. However, we need no hypotheses on E to define $PE^*(\underline{E}_k)$.

On some spaces, all operations are additive.

Lemma 2.10. *On the suspension* ΣX *of any based space* (X, o)*, we have* $r(x+y) = r(x) + r(y)$ *in* $E^m(\Sigma X, o)$ *for any based operation* $r\colon k \to m$ *and any elements* $x, y \in E^k(\Sigma X, o)$*.*

Proof. By [8, Lemma 7.6(c)], $r\colon E^k(\Sigma X, o) \to E^m(\Sigma X, o)$ preserves the group structure defined from the cogroup object ΣX in **Ho′**. By [8, Prop. 7.3], this structure coincides with the given E-cohomology addition. □

Products of operations. Given operations $r\colon k \to m$ and $s\colon k \to n$, the *product* operation $r \smile s\colon k \to m+n$, defined by $(r \smile s)x = (rx)(sx)$, corresponds to the cup product in $E^*(\underline{E}_k)$, which may be constructed using the diagonal map $\Delta\colon \underline{E}_k \to \underline{E}_k \times \underline{E}_k$. We often wish to neglect such operations; if r and s are additive, $r \smile s$ is clearly not additive, but conveys no new information.

The map Δ, together with $q\colon \underline{E}_k \to T$, makes $\underline{E}_k$ a monoid object in the symmetric monoidal category $(Ho^{\mathrm{op}}, \times, T)$. We therefore dualize eq. (2.8) and introduce the quotient E^*-module

$$QE^*(\underline{E}_k) = \mathrm{Coker}[\Delta^* - i_1^* - i_2^*\colon E^*(\underline{E}_k \times \underline{E}_k) \longrightarrow E^*(\underline{E}_k)] \tag{2.11}$$

of *"indecomposables"* of $E^*(\underline{E}_k)$, where i_1 and i_2 are the inclusions (using the basepoint). (We shall not need a topology on this module.) When $E_*(\underline{E}_k)$ is a free E^*-module, we have by Thm. 1.18(c) a Künneth homeomorphism for $E^*(\underline{E}_k \times \underline{E}_k)$, and $QE^*(\underline{E}_k)$ is the quotient of $E^*(\underline{E}_k, o)$ by all finite (or infinite) sums of products of two based operations.

Looping of operations. On restriction to spaces, a stable operation r on $E^*(-,o)$ of degree h induces a sequence of additive operations $r_k\colon k \to k+h$. It is clear from [8, fig. 2 in §9] that r_{k+1} determines r_k. We generalize this construction to unstable operations (but omit the sign, in order to make it a homomorphism of E^*-modules).

Proposition 2.12. *Given a based unstable operation $r\colon k \to m$, we can define the* looped operation $\Omega r\colon k-1 \to m-1$ *in any of three equivalent ways:*

(a) *The operation that makes the diagram commute (with no sign),*

$$\begin{array}{ccccc} E^{k-1}(X) & \xrightarrow[\Sigma]{\cong} & E^k(S^1\times X, o\times X) & \xleftarrow{\cong} & E^k(\Sigma(X^+), o) \\ \vdots\ \Omega r & & \downarrow r & & \downarrow r \\ E^{m-1}(X) & \xrightarrow[\Sigma]{\cong} & E^m(S^1\times X, o\times X) & \xleftarrow{\cong} & E^m(\Sigma(X^+), o) \end{array} \tag{2.13}$$

which we can express algebraically as

$$\Sigma(\Omega r)x = r\Sigma x; \tag{2.14}$$

(b) *The image of r under the E^*-module homomorphism*

$$\Omega\colon E^m(\underline{E}_k, o) \xrightarrow{(-1)^{k-1} f_{k-1}^*} E^m(\Sigma\underline{E}_{k-1}, o) \cong E^{m-1}(\underline{E}_{k-1}, o)$$

induced by the structure map $f_{k-1}\colon \Sigma\underline{E}_{k-1} \to \underline{E}_k$ of [8, Defn. 3.19]*;*

(c) *The map*

$$\Omega r\colon \underline{E}_{k-1} \simeq \Omega\underline{E}_k \xrightarrow{(-1)^{m-k}\Omega r} \Omega\underline{E}_m \simeq \underline{E}_{m-1},$$

where we use the right adjunct equivalences to f_{k-1} and f_{m-1}.

Proof. For a based space X, diag. (2.13) simplifies by naturality to

$$\begin{array}{ccc} E^{k-1}(X,o) & \xrightarrow[\Sigma]{\cong} & E^k(\Sigma X,o) \\ \vdots\,\Omega r \downarrow & & \downarrow r \\ E^{m-1}(X,o) & \xrightarrow[\Sigma]{\cong} & E^m(\Sigma X,o) \end{array} \tag{2.15}$$

If we evaluate on the universal case $\iota_{k-1} \in E^{k-1}(\underline{E}_{k-1}, o)$ by eq. (2.14), we find

$$\Sigma(\Omega r)\iota_{k-1} = r\Sigma\iota_{k-1} = (-1)^{k-1} r f_{k-1}^* \iota_k = (-1)^{k-1} f_{k-1}^* r,$$

which gives (b). Further, by [8, Lemma 3.21], the class $\Sigma(\Omega r)\iota_{k-1} \in E^*(\Sigma\underline{E}_{k-1}, o)$ corresponds, up to the sign $(-1)^{m-1}$, to the lower route in the square

$$\begin{array}{ccc} \Sigma\underline{E}_{k-1} & \xrightarrow{f_{k-1}} & \underline{E}_k \\ {\scriptstyle \Sigma(\Omega r)}\downarrow & {\scriptstyle (-1)^{m-k}} & \downarrow r \\ \Sigma\underline{E}_{m-1} & \xrightarrow{f_{m-1}} & \underline{E}_m \end{array} \qquad \text{in } \mathit{Ho} \tag{2.16}$$

which therefore commutes up to sign. We take adjuncts of this to get (c). □

We recall from [8, Defn. 9.3] the stabilization map $\sigma_k\colon \underline{E}_k \to E$ of spectra.

Corollary 2.17. $\Omega \circ \sigma_k^* = \sigma_{k-1}^*\colon E^*(E,o) \to E^*(\underline{E}_{k-1}, o)$.

Proof. Suppose the stable operation $r \in E^h(E,o)$ restricts to give the additive operations $r_k\colon k \to k+h$ and $r_{k-1}\colon k-1 \to k+h-1$. By [8, (9.8)], $\sigma_k^* r = (-1)^{kh} r_k$ and $\sigma_{k-1}^* r = (-1)^{(k-1)h} r_{k-1}$. We compare diag. (2.16) with [8, (9.2)] to see that $\Omega r_k = (-1)^h r_{k-1}$. □

Corollary 2.18. *The loop construction in Prop. 2.12(b) factors as*

$$\Omega\colon E^*(\underline{E}_k, o) \longrightarrow QE^*(\underline{E}_k) \longrightarrow PE^*(\underline{E}_{k-1}) \subset E^*(\underline{E}_{k-1}, o)\ . \tag{2.19}$$

Proof. It is clear from Prop. 2.12(c), or from eq. (2.14) and Lemma 2.10, that Ωr is always additive. The construction factors through $QE^*(\underline{E}_k)$ by Prop. 2.12(b) and naturality of Q, since $QE^*(\Sigma\underline{E}_{k-1}) \cong E^*(\Sigma\underline{E}_{k-1}, o)$. (Loosely, there are no products in $E^*(\Sigma X, o)$.) □

These results allow us to rewrite the Milnor short exact sequence [8, (9.7)] in the more useful form (which does not change any terms)

$$0 \longrightarrow \lim_k{}^1 PE^*(\underline{E}_k) \longrightarrow E^*(E,o) \longrightarrow \lim_k PE^*(\underline{E}_k) \longrightarrow 0\ . \tag{2.20}$$

It remains true that the projection from $E^*(E,o)$ is an open map, and therefore a homeomorphism whenever it is a bijection. The kth component is the E^*-module homomorphism

$$\sigma_k^*\colon E^*(E,o) \longrightarrow PE^*(\underline{E}_k) \subset E^*(\underline{E}_k) \tag{2.21}$$

induced by the stabilization map σ_k. It sends a stable operation r to the induced additive operation r_k on $E^k(-)$ (but with a sign; see [8, (9.9)]).

The factorization (2.19) raises two obvious questions:

(a) Can every additive operation be delooped?
(b) Does $\Omega r = 0$ imply that r decomposes? (2.22)

Both hold precisely when we have an isomorphism $\Omega\colon QE^*(\underline{E}_k) \cong PE^*(\underline{E}_{k-1})$. We discuss this further in section 4.

3. Group objects and E-cohomology

Before we can discuss additive E-cohomology operations adequately, it is necessary to generalize section 2. We extend Prop. 2.7 by defining the primitives $PE^*(X)$ for any group object X in the homotopy category Ho. Dually, we extend the definition of the indecomposables $QE^*(X)$ to any based space X.

Coalgebra primitives. We start from the definition (2.8) of $PE^*(\underline{E}_k)$.

Definition 3.1. Given any group object (or H-space) X in Ho, with multiplication $\mu\colon X \times X \to X$, we define the E^*-submodule $PE^*(X)$ of *coalgebra primitives* in $E^*(X)$ as

$$PE^*(X) = \{x \in E^*(X) : \mu^* x = p_1^* x + p_2^* x \text{ in } E^*(X \times X)\}.$$

Remark. As in Prop. 2.7(c), the class $x \in E^k(X)$ is primitive if and only if the associated map $x\colon X \to \underline{E}_k$ is a morphism of group objects in Ho.

We note that $PE^*(X)$ is defined even if $E^*(X)$ is not a (completed) coalgebra. Thus $PE^*(-)\colon \mathsf{Gp}(\mathsf{Ho})^{\mathrm{op}} \to \mathsf{Mod}$ is a functor defined on the dual of the category of group objects in Ho. We topologize $PE^*(X)$ as a subspace of $E^*(X)$.

If Y is another group object in Ho, we construct the product group object $X \times Y$ in the obvious way. The one-point space T is trivially a group object, and is terminal in $\mathsf{Gp}(\mathsf{Ho})$. Lemma 6.14 of [8] carries over to this situation.

Lemma 3.2. *For the product $X \times Y$ of two group objects X and Y in Ho, we have $PE^*(X \times Y) \cong PE^*(X) \oplus PE^*(Y)$ in FMod. Also, $PE^*(T) = 0$.*

In other words, the functor $PE^*(-)$ takes finite products in $\mathsf{Gp}(\mathsf{Ho})$ to coproducts (direct sums) in FMod.

Remark. No Künneth formula is needed for this result.

Proof. We dualize the proof of [8, Lemma 6.11]. Let us write $Z = X \times Y$ for the product group object and $\omega_Y\colon T \to Y$ for the unit (or zero) map of Y. We note first that the maps $j_1 = 1_X \times \omega_Y\colon X \cong X \times T \to X \times Y = Z$, $j_2\colon Y \to Z$ (defined similarly), $p_1\colon Z = X \times Y \to X$, and $p_2\colon Z \to Y$ are all morphisms of group objects and therefore send primitives to primitives. Define the map

$$f\colon Z = X \times Y \cong (X \times T) \times (T \times Y) \longrightarrow (X \times Y) \times (X \times Y) = Z \times Z$$

using $(1_X \times \omega_Y) \times (\omega_X \times 1_Y)$. Then $\mu_Z \circ f = 1_Z$ and $P_s \circ f = j_s \circ p_s$ (for $s = 1, 2$), where $P_s\colon Z \times Z \to Z$ denotes the projection for Z. Any element $z \in PE^*(Z)$ satisfies $\mu_Z^* z = P_1^* z + P_2^* z$, by definition. When we apply f^*, we obtain $z = p_1^* x + p_2^* y$, where $x = j_1^* z \in E^*(X)$ and $y = j_2^* z \in E^*(Y)$ must be primitive. Conversely, any primitives x and y determine a primitive z by this formula. We have a homeomorphism because j_s^* and p_s^* are continuous.

We compute $PE^*(T) = \{v \in E^* : v = v + v\} = 0$. □

Since the unit map $\omega\colon T \to X$ of X is a morphism of group objects, $PE^*(T) = 0$ implies that $PE^*(X) \subset E^*(X, o)$.

The space $\underline{E}_k$ is more than just a group object. By [8, Cor. 7.8], we have the E^*-module object $n \mapsto \underline{E}_n$ in Ho, on which $v \in E^h$ acts by the maps $\xi v\colon \underline{E}_k \to \underline{E}_{k+h}$ that represent scalar multiplication by v. Clearly, ξv is additive.

Lemma 3.3. *Assume that $E^*(\underline{E}_k)$ is Hausdorff for all k. Then:*

(a) *We have the E^*-module object $n \mapsto PE^*(\underline{E}_n)$ in the ungraded category $\mathsf{FMod}^{\mathrm{op}}$, with the action of $v \in E^h$ given by $P(\xi v)^*\colon PE^*(\underline{E}_{k+h}) \to PE^*(\underline{E}_k)$;*

(b) *The object in (a) is related to the stable E^*-module object $E^*(E, o)$ of* [8, Prop. 11.3] *by the following diagram, which commutes up to sign for any $v \in E^h$,*

$$\begin{array}{ccc} E^*(E,o) & \xrightarrow{(\xi v)^*} & E^*(E,o) \\ \Big\downarrow{\scriptstyle \sigma^*_{k+h}} & (-1)^{hk} & \Big\downarrow{\scriptstyle \sigma^*_k} \\ PE^*(\underline{E}_{k+h}) & \xrightarrow{P(\xi v)^*} & PE^*(\underline{E}_k) \end{array} \tag{3.4}$$

Proof. In (a), the object $n \mapsto \underline{E}_n$ is in fact an E^*-module object in $\mathsf{Gp}(\mathsf{Ho})$. We apply [8, Lemma 7.6(a)] to the functor $PE^*(-)$; it preserves finite products by Lemma 3.2.

For (b), we apply E-cohomology to diag. [8, (9.8)], taking $r = \xi v$. □

Indecomposables. Dually, we extend eq. (2.11) to any based space X by defining the quotient E^*-module

$$QE^*(X) = \mathrm{Coker}[\Delta^* - i_1^* - i_2^*\colon E^*(X \times X) \longrightarrow E^*(X)] \tag{3.5}$$

of *"indecomposables"* of $E^*(X)$. (We shall not need a topology on it.)

4. Group objects and E-homology

We dualize section 2 by defining the indecomposables $QE_*(\underline{E}_k)$ and primitives $PE_*(\underline{E}_k)$ in E-homology. This will prove useful because $E_*(\underline{E}_k)$ is usually smaller and more manageable than $E^*(\underline{E}_k)$. As in section 3, we need to handle more general X. However, some properties that were immediate in section 2 become less intuitive and have to be proved.

The structure map $f_k\colon \Sigma \underline{E}_k \to \underline{E}_{k+1}$ (see [8, Defn. 3.19]) of the spectrum E induces the important *suspension* homomorphism

$$E_*(\underline{E}_k) \longrightarrow E_*(\underline{E}_k, o) \cong E_*(\Sigma \underline{E}_k, o) \xrightarrow{f_{k*}} E_*(\underline{E}_{k+1}, o), \tag{4.1}$$

dual (apart from sign) to the looping Ω in Prop. 2.12(b). Again, suspended elements behave better. We dualize Lemma 2.10.

Lemma 4.2. *For any elements $x, y \in E^k(\Sigma X, o)$, the induced E-homology homomorphisms satisfy*

$$(x+y)_* = x_* + y_*\colon E_*(\Sigma X, o) \longrightarrow E_*(\underline{E}_k, o)\ .$$

Proof. By [8, Lemma 7.6(c)], E-homology induces a homomorphism

$$\mathit{Ho}'(\Sigma X, \underline{E}_k) \longrightarrow \mathit{Mod}(E_*(\Sigma X, o), E_*(\underline{E}_k, o))$$

of groups, where both group structures are induced by the cogroup structure on ΣX in Ho'. By [8, Prop. 7.3], they agree with the obvious group structures. □

Indecomposables. We dualize Defn. 3.1.

Definition 4.3. Given any group object (or H-space) X in Ho, we define the E^*-*module $QE_*(X)$ of "indecomposables"* of $E_*(X)$ as

$$QE_*(X) = \mathrm{Coker}[\mu_* - p_{1*} - p_{2*}\colon E_*(X \times X) \longrightarrow E_*(X)].$$

It comes equipped with a canonical projection $E_*(X) \to QE_*(X)$.

When $E_*(X)$ is free, we have the Künneth isomorphism Thm. 1.18(b) for $E_*(X \times X)$ and this agrees with the usual definition for the algebra $E_*(X)$. We need one easy example.

Lemma 4.4. *Let G be a discrete abelian group. Then $QE_*(G) \cong E^* \otimes_{\mathbb{Z}} G$ as an E^*-module.*

Proof. We recognize $E_*(G)$ as the group algebra of G over E^*, with an E^*-basis element $[g]$ for each $g \in G$. The correspondence we seek is induced by $v[g] \leftrightarrow v \otimes g$, and is well defined in both directions. □

Lemma 3.2 dualizes without difficulty; again, no Künneth formula is needed. Then we will be able to dualize Lemma 3.3.

Lemma 4.5. *For the product $X \times Y$ of two group objects X and Y in Ho, we have $QE_*(X \times Y) \cong QE_*(X) \oplus QE_*(Y)$. Also, $QE_*(T) = 0$. In other words, the functor $QE_*(-)\colon \mathit{Gp}(\mathit{Ho}) \to \mathit{Mod}$ preserves finite products.* □

We have an immediate application to the Hopf bundle.

Lemma 4.6. *Assume E has a complex orientation. Then the inclusion $\mathbb{C}P^\infty \to \mathbb{Z} \times BU$* (see [8, (5.8)]) *defined by the Hopf line bundle ξ over $\mathbb{C}P^\infty$ induces an isomorphism of E^*-modules*

$$E_*(\mathbb{C}P^\infty) \xrightarrow{\cong} QE_*(\mathbb{Z} \times BU) \cong E^* \oplus QE_*(BU)\ .$$

Proof. The second isomorphism comes from Lemmas 4.5 and 4.4. We compare Lemmas 5.4 and 5.6 of [8]; the generators β_i corrrespond, except that $\beta_0 \mapsto (1,0)$. □

Lemma 4.7. *For any ring spectrum E:*

(a) $n \mapsto QE_*(\underline{E}_n)$ *is an E^*-module object in the ungraded category* Mod *of E^*-modules;*

(b) *The suspension* (4.1) *factors through $QE_*(\underline{E}_k)$;*

(c) *The stabilization $\sigma_{k*}\colon E_*(\underline{E}_k, o) \to E_*(E, o)$ factors through $QE_*(\underline{E}_k)$.*

Proof. The proof of (a) is like Lemma 3.3(a), except that we use the functor $QE_*(-)$ and Lemma 4.5.

For (c), we use $\sigma_k^*\iota = \iota_k$ to restate the universal example (2.6) as

$$\sigma_k \circ \mu_k = \sigma_k \circ p_1 + \sigma_k \circ p_2 \colon \underline{E}_k \times \underline{E}_k \longrightarrow E \qquad \text{in } \mathit{Stab}^*.$$

We apply E-homology to see that σ_{k*} factors as desired. Similarly for (b), except that we use Lemma 4.2 with $X = \Sigma(\underline{E}_k \times \underline{E}_k)$, $x = \Sigma p_1$, and $y = \Sigma p_2$. □

Dually to the short exact sequence (2.20), we may use (b) and (c) to rewrite [8, (9.22)] in the more convenient form

$$E_*(E, o) = \operatorname*{colim}_k E_*(\underline{E}_k, o) = \operatorname*{colim}_k QE_*(\underline{E}_k) \ . \tag{4.8}$$

There is a multiplication, analogous to the stable multiplication on $E_*(E, o)$.

Lemma 4.9. *There is a bilinear multiplication*

$$Q\phi\colon QE_*(\underline{E}_k) \otimes QE_*(\underline{E}_m) \longrightarrow QE_*(\underline{E}_{k+m}),$$

which may be defined as a quotient of

$$E_*(\underline{E}_k) \otimes E_*(\underline{E}_m) \xrightarrow{\times} E_*(\underline{E}_k \times \underline{E}_m) \xrightarrow{\phi_*} E_*(\underline{E}_{k+m}) \ .$$

Proof. The only difficulty is to prove that $Q\phi$ is well defined. We express the distributive law for the E^*-algebra object $n \mapsto \underline{E}_n$ as the commutative square

$$\begin{array}{ccc} \underline{E}_k \times \underline{E}_k \times \underline{E}_m & \xrightarrow{\phi_L} & \underline{E}_{k+m} \times \underline{E}_{k+m} \\ \downarrow{\scriptstyle f\times 1} & & \downarrow{\scriptstyle g} \\ \underline{E}_k \times \underline{E}_m & \xrightarrow{\phi} & \underline{E}_{k+m} \end{array} \tag{4.10}$$

in which $f = \mu_k$, $g = \mu_{k+m}$, and ϕ_L has the components $\phi \circ (p_1 \times 1)$ and $\phi \circ (p_2 \times 1)$. (Cohomologically, ϕ_L represents the operation $(x, y, z) \mapsto (xz, yz)$.) We deduce the commutative diagram in homology

$$\begin{array}{ccccc} E_*(\underline{E}_k \times \underline{E}_k) \otimes E_*(\underline{E}_m) & \xrightarrow{\times} & E_*(\underline{E}_k \times \underline{E}_k \times \underline{E}_m) & \xrightarrow{\phi_{L*}} & E_*(\underline{E}_{k+m} \times \underline{E}_{k+m}) \\ \downarrow{\scriptstyle f_* \otimes 1} & & \downarrow{\scriptstyle (f\times 1)_*} & & \downarrow{\scriptstyle g_*} \\ E_*(\underline{E}_k) \otimes E_*(\underline{E}_m) & \xrightarrow{\times} & E_*(\underline{E}_k \times \underline{E}_m) & \xrightarrow{\phi_*} & E_*(\underline{E}_{k+m}) \end{array}$$

(4.11)

By Defn. 4.3, we have the exact sequence

$$E_*(\underline{E}_k \times \underline{E}_k) \xrightarrow{\mu_{k*} - p_{1*} - p_{2*}} E_*(\underline{E}_k) \longrightarrow QE_*(\underline{E}_k) \longrightarrow 0 \ .$$

After tensoring with $E_*(\underline{E}_m)$, this remains exact. We note that diag. (4.10) and hence diag. (4.11) also commute if we take $f = p_1$ and $g = p_1$, or $f = p_2$ and $g = p_2$. Then diag. (4.11), with these three choices for f and g, shows that its bottom row induces a quotient pairing $QE_*(\underline{E}_k) \otimes E_*(\underline{E}_m) \to QE_*(\underline{E}_{k+m})$.

A second similar step, on the right, uses this pairing to produce $Q\phi$. □

Coalgebra primitives. We also dualize eq. (3.5) in the obvious way. If X is a based space, we construct the E^*-module homomorphism

$$\Delta_* - i_{1*} - i_{2*} \colon E_*(X) \longrightarrow E_*(X \times X) \ . \tag{4.12}$$

Definition 4.13. Given any based space X, we define the *E^*-submodule of coalgebra primitives* $PE_*(X) = \mathrm{Ker}[\Delta_* - i_{1*} - i_{2*}] \subset E_*(X)$.

Again, the definition is meaningful even without a Künneth formula for $E_*(X \times X)$. The companion result to Lemma 4.4 is elementary.

Proposition 4.14. *For any discrete based space X, we have $PE_*(X) = 0$.* □

The suspension (4.1) factors, with the help of Lemma 4.7(b), as

$$E_*(\underline{E}_k, o) \longrightarrow QE_*(\underline{E}_k) \longrightarrow PE_*(\underline{E}_{k+1}) \subset E_*(\underline{E}_{k+1}, o) \ . \tag{4.15}$$

Again we ask whether $QE_*(\underline{E}_k) \to PE_*(\underline{E}_{k+1})$ is an isomorphism.

Duality. Under reasonable assumptions, the sequence (2.19) is dual to (4.15). One can see from Lemma 4.17 and section 17 that this holds for each of our five examples E. Moreover, in each case there are isomorphisms $QE_*(\underline{E}_k) \cong PE_*(\underline{E}_{k+1})$ in (4.15), thus answering the questions (2.22) affirmatively.

Lemma 4.16. *Assume that $E_*(X)$ is a free E^*-module.*

(a) *If X is a group object in* Ho *(or an H-space), then d induces a homeomorphism $d\colon PE^*(X) \cong DQE_*(X)$ in* FMod*;*

(b) *If X is a based space and the image of the homomorphism* (4.12) *splits off both $E_*(X)$ and $E_*(X \times X)$, then d induces a bijection $d\colon QE^*(X) \cong DPE_*(X)$.*

Proof. In (a), d induces the commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & PE^*(X) & \xrightarrow{\subset} & E^*(X) & \xrightarrow{\mu^* - p_1^* - p_2^*} & E^*(X \times X) \\
 & & \downarrow d & & \downarrow d & & \downarrow d \\
0 & \longrightarrow & DQE_*(X) & \longrightarrow & DE_*(X) & \xrightarrow{D(\mu_* - p_{1*} - p_{2*})} & DE_*(X \times X)
\end{array}$$

whose rows are exact by Defns. 3.1 and 4.3, because D automatically takes cokernels to kernels. Strong duality for X and $X \times X$ from Thm. 1.18 provides two homeomorphisms d. The third d is therefore also a homeomorphism, because $DQE_*(X)$ has the subspace topology from $DE_*(X)$ by [8, Lemma 6.15(c)].

The proof of (b) is analogous, except that we assume the splittings to ensure that the bottom row of the relevant diagram is (split) exact, use [8, Lemma 6.15(a)] instead, and have no topology to check. □

We clearly need information on when $E_*(\underline{E}_k)$ is free.

Lemma 4.17. *For $E = H(\mathbb{F}_p)$, BP, MU, K(n), or KU:*

(a) *$E_*(\underline{E}_k)$ and $QE_*(\underline{E}_k)$ are free E^*-modules for all k;*

(b) *$E^*(\underline{E}_k)$ and $PE^*(\underline{E}_k)$ are complete Hausdorff for all k.*

Proof. For $E = H(\mathbb{F}_p)$ or $K(n)$, all E^*-modules are free and (a) is trivial.

We consider the remaining three cases together. For odd k, $E_*(\underline{E}_k)$ is an exterior algebra over E^* by [23] (for BP or MU) or [8, Cor. 5.12] (for KU, when $\underline{E}_k = U$), and (a) is clear.

For even k, we write $\underline{E}_k = E^k \times \underline{E}'_k$ as in [8, (3.7)], where $\underline{E}'_k$ denotes the zero component and E^k is treated as a discrete group. Then $E_*(\underline{E}'_k)$ is a polynomial algebra over E^*, by [23] (for BP or MU) or [8, Lemma 5.6(c)] (for KU, when $\underline{E}'_k = BU$), so that $E_*(\underline{E}'_k)$ (and hence $E_*(\underline{E}_k)$) and $QE_*(\underline{E}'_k)$ are free modules.

To finish (a), we note that by Lemmas 4.5 and 4.4,

$$QE_*(\underline{E}_k) = (E^* \otimes_{\mathbb{Z}} E^k) \oplus QE_*(\underline{E}'_k) \ .$$

The first summand is free, because $E^k = \mathbb{Z}$ (for KU), or is $\mathbb{Z}$-free (for MU), or is $\mathbb{Z}_{(p)}$-free (for BP).

Part (b) is immediate from (a) by Thm. 1.18(a) and Cor. 2.9. □

5. What is an additively unstable module?

In this section, we give various interpretations of what it means to have a module over the additive unstable operations on E-cohomology. All four stable answers in [8] generalize.

We recall from [8, Cor. 7.8] that each $\underline{E}_k$ is an abelian group object in Ho and therefore also in $\mathsf{Gp}(\mathsf{Ho})$, and that $n \mapsto \underline{E}_n$ is an E^*-module object in Ho, with $v \in E^h$ acting by the map $\xi v\colon \underline{E}_k \to \underline{E}_{k+h}$. From Prop. 2.7 we have the submodule $PE^*(\underline{E}_k)$ of additive operations defined on $E^k(-)$.

We assume throughout that $E_*(\underline{E}_k)$ is a free E^*-module. Then by Cor. 2.9, $PE^*(\underline{E}_k)$ is complete Hausdorff and an object of FMod.

First Answer. The additive operations $r\colon k \to m$ act on $E^*(X)$ by composition

$$\circ\colon PE^m(\underline{E}_k) \times E^k(X) \longrightarrow E^m(X) \tag{5.1}$$

in Ho. We recover the stable action [8, (10.1)] by using $\sigma_k^*\colon E^*(E, o) \to PE^*(\underline{E}_k)$.

This composition is already biadditive. Given $x \in E^k(X)$ and $v \in E^h$, the commutative square

$$\begin{array}{ccc} PE^m(\underline{E}_{k+h}) \times E^k(X) & \xrightarrow{1\times v} & PE^m(\underline{E}_{k+h}) \times E^{k+h}(X) \\ \downarrow{\scriptstyle P(\xi v)^* \times 1} & & \downarrow{\scriptstyle \circ} \\ PE^m(\underline{E}_k) \times E^k(X) & \xrightarrow{\quad\circ\quad} & E^m(X) \end{array} \tag{5.2}$$

expresses the identity $(r \cdot v)x = rvx = r(vx)$ for operations $r\colon k+h \to m$. It suggests that we should make the action (5.1) more closely resemble the stable action by introducing a formal shift and rewriting it with a tensor product as

$$\lambda_X\colon \Sigma^{-k} PE^m(\underline{E}_k) \otimes_k E^k(X) \longrightarrow E^m(X)\ . \tag{5.3}$$

(Here, unlike [8], the action scheme is clearly visible: the notation $\otimes_k$ indicates that the tensor product is to be formed using the two E^*-actions indexed by k.)

This approach was initiated in [27, §11]. However, it presents even more problems than in the stable case, and we do not pursue it further here.

Second Answer. Our hypotheses ensure that $PE^*(\underline{E}_k)$ is dual to $QE_*(\underline{E}_k)$. We can convert the action of $PE^*(\underline{E}_k)$ into a coaction

$$E^k(X) \longrightarrow E^*(X) \mathbin{\widehat{\otimes}} QE_*(\underline{E}_k)\ .$$

These are clearly not the components of an E^*-module homomorphism, because the degree varies.

In section 6, as suggested by (5.3), we shall shift degrees by introducing $Q(E)^k_* = \Sigma^k QE_*(\underline{E}_k)$, which will allow us to write the coaction as an E^*-module homomorphism with components

$$\rho_X\colon E^k(X) \longrightarrow E^*(X) \mathbin{\widehat{\otimes}} Q(E)^k_* \tag{5.4}$$

and the same action scheme as stably. We shall construct a comultiplication $Q(\psi)$ and counit $Q(\epsilon)$ that make $Q(E)^*_*$ a coalgebra and allow us to interpret $E^*(X)$ as a $Q(E)^*_*$-comodule.

Third Answer. We write our Second Answer more functorially. Given any E^*-module M, we construct the graded group $A'M$ having the component

$$(A'M)^k = M^i \mathbin{\widehat{\otimes}_i} Q(E)^k_i = (M \mathbin{\widehat{\otimes}} Q(E)^k_*)^k$$

in degree k. In section 6 we shall make $A'M$ an E^*-module. Then $M \otimes Q(\psi)$ and $M \otimes Q(\epsilon)$ define natural transformations $\psi'\colon A' \to A'A'$ and $\epsilon'\colon A' \to I$, which will make A' a comonad in *FMod* and $E^*(X)\hat{}$ an A'-coalgebra.

Fourth Answer. Still imitating the stable case, we eliminate all tensor products by converting the First Answer to adjoint form. This will make everything very much cleaner, evidence that this is the natural answer (although the Second Answer is undeniably convenient for computation).

Any element $x \in E^k(X)$ may be regarded as a map $x\colon X \to \underline{E}_k$, which induces the morphism $x^*\colon E^*(\underline{E}_k) \to E^*(X)\hat{}$ in FMod. Generally, given any object M in FMod, we define for each integer k the abelian group

$$A^k M = \mathsf{FMod}(PE^*(\underline{E}_k), M) \tag{5.5}$$

of all continuous E^*-module homomorphisms $PE^*(\underline{E}_k) \to M$. (There is no need to shift degrees.) Then we convert the action (5.1) to the coaction

$$\rho_X\colon E^k(X) \longrightarrow A^k(E^*(X)\hat{}) = \mathsf{FMod}(PE^*(\underline{E}_k), E^*(X)\hat{}) \tag{5.6}$$

by defining $\rho_X x = x^*|PE^*(\underline{E}_k)$.

We assemble the $A^k M$, as k varies, to form the graded group AM with components $(AM)^k = A^k M$, and the coactions ρ_X into the single homomorphism $\rho_X\colon E^*(X) \to A(E^*(X)\hat{})$ of graded groups of degree zero.

The destabilization $\sigma_k^*\colon E^*(E,o) \to PE^*(\underline{E}_k)$ (see [8, Defn. 9.3]) induces

$$A^k M = \mathsf{FMod}(PE^*(\underline{E}_k), M) \longrightarrow \mathsf{FMod}^k(E^*(E,o), M) = (SM)^k, \tag{5.7}$$

if we also assume that $E^*(E,o)$ is Hausdorff. As k varies, we take these as the components of the *stabilization* natural transformation $\sigma M\colon AM \to SM$, of degree zero. It allows us to compare with the stable case.

Theorem 5.8. *Assume that $E_*(\underline{E}_k)$ is a free E^*-module for all k* (as is true for $E = H(\mathbb{F}_p)$, BP, MU, KU, or $K(n)$ by Lemma 4.17(a)). *Then:*

(a) *We can make the functor A, defined in eq.* (5.5), *a comonad in the category FMod of complete Hausdorff filtered E^*-modules;*

(b) *If $E^*(E,o)$ is also Hausdorff, the stabilization $\sigma\colon A \to S$* (defined in eq. (5.7)) *is a morphism of comonads in FMod.*

The relevant definitions are now clear.

Definition 5.9. An *additively unstable (E-cohomology) module* is an A-coalgebra in FMod, i. e. a complete Hausdorff filtered E^*-module M equipped with a morphism $\rho_M\colon M \to AM$ in FMod that satisfies the coaction axioms [8, (8.7)]. We then define the action of $r \in PE^m(\underline{E}_k)$ on $x \in M^k$ by $rx = \rho_M(x)r \in M$ (with no sign).

A closed submodule $L \subset M$ is called *(additively unstably) invariant* if ρ_M restricts to give $\rho_L\colon L \to AL$. Then the quotient M/L inherits an additively unstable module structure.

This is a stronger structure than a stable module (when $E^*(E,o)$ is Hausdorff, so that stable modules exist). Given a coaction ρ_M as above, Thm. 5.8(b) shows that the coaction

$$M \xrightarrow{\rho_M} AM \xrightarrow{\sigma M} SM \tag{5.10}$$

makes M a stable module.

One may think of $A^k M$ as the set of all candidates for the action of $PE^*(\underline{E}_k)$ on a typical element of M^k, and ρ_M as the selection of a candidate for each $x \in M^k$. The coaction axioms translate into the usual action axioms $(sr)x = s(rx)$ and $\iota_k x = x$.

As stably, it is sometimes useful to fix $r\colon k \to m$ and express the first axiom as the commutative square

$$\begin{array}{ccc} M^k & \xrightarrow{r} & M^m \\ \downarrow{\scriptstyle \rho_M} & & \downarrow{\scriptstyle \rho_M} \\ A^k M & \xrightarrow{\omega_r M} & A^m M \end{array} \tag{5.11}$$

where $\omega_r M$ denotes composition with $Pr^*\colon PE^*(\underline{E}_m) \to PE^*(\underline{E}_k)$.

Theorem 5.12. *Assume that $E_*(\underline{E}_k)$ is a free E^*-module for all k* (as is true for $E = H(\mathbb{F}_p)$, BP, MU, KU, or $K(n)$ by Lemma 4.17(a)). *Then:*

(a) ρ_X (defined in eq. (5.6)) *factors through $E^*(X)\hat{}$ as $\rho_X\colon E^*(X)\hat{} \to A(E^*(X)\hat{})$ to make $E^*(X)\hat{}$ an additively unstable module for any space X;*

(b) *If $E^*(E, o)$ is Hausdorff, we recover the stable coaction in* [8, Thm. 10.16(a)] *from ρ_X by diag.* (5.10)*;*

(c) *ρ is universal: given an object N of FMod and an integer k, any additive natural transformation of abelian groups $\theta X\colon E^k(X) \to \mathsf{FMod}(N, E^*(X)\hat{})$ (or $\widehat{\theta} X\colon E^k(X)\hat{} \to \mathsf{FMod}(N, E^*(X)\hat{})$) that is defined on all spaces X is induced from ρ_X by a unique morphism $f\colon N \to PE^*(\underline{E}_k)$ in FMod, as*

$$\begin{aligned} \theta X\colon E^k(X) \xrightarrow{\rho_X} A^k(E^*(X)\hat{}) = \mathsf{FMod}(PE^*(\underline{E}_k), E^*(X)\hat{}) \\ \xrightarrow{\mathrm{Hom}(f,1)} \mathsf{FMod}(N, E^*(X)\hat{}) \ . \end{aligned}$$

Proof of Thms. 5.8 and 5.12. We prove parts (a) and (b) of both Theorems together, in the same seven steps as the stable proof of Thms. 10.12 and 10.16 of [8]. As most steps are more or less repetitions of that proof, except for the insertion of indices everywhere, we indicate only the substantive changes for (a) and the additions needed to handle σ for the (b) parts. Instead of $\iota \in E^*(E, o)$, we have $\iota_k \in PE^*(\underline{E}_k)$. Instead of $\mathrm{id}_{\mathcal{A}}$, we have the identity map $\mathrm{id}_k\colon PE^*(\underline{E}_k) = PE^*(\underline{E}_k)$, considered as an element of $A^k PE^*(\underline{E}_k)$. We write ρ_k for ρ_X when $X = \underline{E}_k$.

Step 1. We construct an E^*-module structure on the graded group AM we defined in eq. (5.5). We start with the E^*-module object $n \mapsto PE^*(\underline{E}_n)$ in $\mathsf{FMod}^{\mathrm{op}}$ from Lemma 3.3(a), with $v \in E^*$ acting by $P(\xi v)^*$. We apply the additive functor $\mathrm{Mor}(-, M)\colon \mathsf{FMod}^{\mathrm{op}} \to \mathsf{Ab}$ to obtain by [8, Lemma 7.6(a)] the E^*-module object $n \mapsto A^n M$ in Ab, i. e. make AM an E^*-module.

Despite appearances, the square (3.4) does commute in the dual category $\mathsf{FMod}^{*\,\mathrm{op}}$, to show that $\sigma M\colon AM \to SM$ is an E^*-module homomorphism.

Step 2. We have defined ρ_X as a natural transformation of sets. For fixed X, the cohomology functor $E^*(-)\hat{}\colon \mathsf{Ho} \to \mathsf{FMod}^{\mathrm{op}}$ induces the natural transformation

$$\mathsf{Ho}(X, -) \longrightarrow \mathsf{FMod}(PE^*(-)\hat{}, E^*(X)\hat{})\colon \mathsf{Gp}(\mathsf{Ho}) \longrightarrow \mathsf{Set}.$$

We apply [8, Lemma 7.6(c)] to the E^*-module object $n \mapsto \underline{E}_n$ to see that ρ_X is a morphism of E^*-module objects, i.e. takes values in Mod.

For Thm. 5.12(b), we note that given $x \in E^k(X)$, we have $(\sigma(E^*(X)\hat{}))x^*_U = x^*_U \circ \sigma^*_k = x^*_S$, by [8, (9.4)].

If X is a group object in Ho and $x \in PE^k(X)$, the associated map $x\colon X \to \underline{E}_k$ is a morphism of group objects (as remarked after Defn. 3.1) and so induces $x^*\colon PE^*(\underline{E}_k) \to PE^*(X)$. If $E^*(X)$ (and hence $PE^*(X)$) is Hausdorff, ρ_X restricts to define

$$P\rho_X\colon PE^*(X) \longrightarrow APE^*(X)\ . \tag{5.13}$$

Step 3. We filter AM exactly as we did SM in [8, §10], by the submodules $F^a(AM) = A(F^aM)$, using naturality. The proof that AM is complete Hausdorff is formally the same as for SM. Our choice of filtrations and the naturality of ρ clearly make ρ_X and σM continuous, so that ρ_X factors through $E^*(X)\hat{}$ and σ takes values in FMod.

Step 4. Whenever X is a group object in Ho and $E^*(X)$ is Hausdorff, we convert the object $PE^*(X)$ of FMod to the corepresented functor

$$F_{PX} = \mathsf{FMod}(PE^*(X), -)\colon \mathsf{FMod} \longrightarrow \mathsf{Ab}$$

and the coaction $P\rho_X$ in (5.13) to a natural transformation $\rho_{PX}\colon F_{PX} \to F_{PX}A\colon \mathsf{FMod} \to \mathsf{Ab}$. Given M, $\rho_{PX}M\colon F_{PX}M \to F_{PX}AM$ is the homomorphism

$$\rho_{PX}M\colon \mathsf{FMod}(PE^*(X), M) \longrightarrow \mathsf{FMod}(PE^*(X), AM) \tag{5.14}$$

that is defined on $f\colon PE^*(X) \to M$ as the composite

$$(\rho_{PX}M)f\colon PE^*(X) \xrightarrow{P\rho_X} APE^*(X) \xrightarrow{Af} AM\ .$$

Step 5. To construct $\psi = \psi_A\colon A \to AA$, we take $X = \underline{E}_k$ in (5.14) and define

$$(\psi M)^k\colon \mathsf{FMod}(PE^*(\underline{E}_k), M) \longrightarrow \mathsf{FMod}(PE^*(\underline{E}_k), AM)$$

on the element $f\colon PE^*(\underline{E}_k) \to M$ of A^kM as the composite

$$(\psi M)^k f\colon PE^*(\underline{E}_k) \xrightarrow{P\rho_k} APE^*(\underline{E}_k) \xrightarrow{Af} AM\ .$$

When we substitute the E^*-module object $n \mapsto \underline{E}_n$ for X in (5.14), [8, Lemma 7.6(c)] shows that $(\psi M)^k\colon A^kM \to A^kAM$ lies in Mod. As k varies, we obtain the natural transformation $\psi\colon A \to AA$. Naturality in M also shows that ψM is filtered and so lies in FMod.

Step 6. The other required natural transformation, $\epsilon\colon A \to I$, is defined on M simply as the evaluation

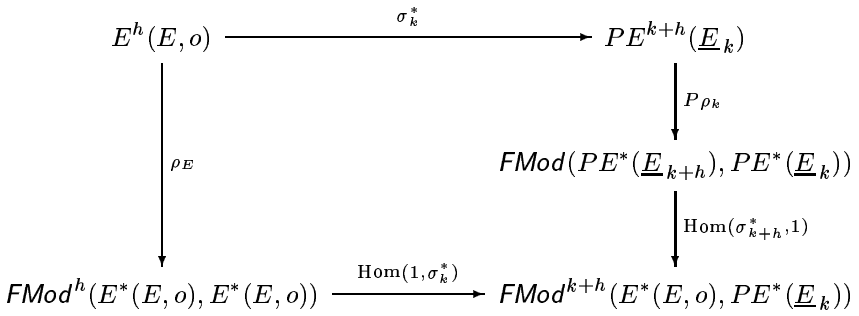
$$(\epsilon M)^k = (\epsilon_A M)^k\colon A^kM = \mathsf{FMod}(PE^*(\underline{E}_k), M) \longrightarrow M \tag{5.15}$$

on $\iota_k \in PE^k(\underline{E}_k)$. It is continuous by naturality. It is compatible with the stable version, $\epsilon_A = \epsilon_S \circ \sigma\colon A \to I$, since given $f \in A^kM$, we have

$$(\epsilon_S M)(\sigma M)f = ((\sigma M)f)\iota = f\sigma^*_k\iota = f\iota_k = (\epsilon_A M)f\ .$$

Step 7. To see that ρ_X is a coaction on $E^*(X)$, we use [8, Lemma 8.20] (adapted to graded objects). We use $R = PE^*(\underline{E}_n)$ (really, the graded object $n \mapsto PE^*(\underline{E}_n)$), $1_R = \iota_n$, and $\rho_R = P\rho_n$. By [8, Lemma 8.22], A is a comonad in FMod.

To see that $\sigma\colon A \to S$ is a morphism of comonads, we apply [8, Lemma 8.24]. The first condition on $u = \sigma_k^*\colon E^*(E,o) \to PE^*(\underline{E}_k)$ is the commutative diagram



A stable operation $r_S \in E^h(E,o)$ restricts to an additive operation $r_U\colon k \to k+h$. On r_S, the lower route gives by diag. [8, (9.8)]

$$\sigma_k^* \circ r_S^* = (-1)^{hk}(r_S \circ \sigma_k)^* = (-1)^{hk}(\sigma_{k+h} \circ r_U)^* = (-1)^{hk} r_U^* \circ \sigma_{k+h}^* \ .$$

This agrees with the upper route, because $\sigma_k^* r_S = (-1)^{hk} r_U$ by [8, (9.9)]. The second condition needed is $\sigma_k^* \iota = \iota_k$, which holds by the definition of σ_k.

For Thm. 5.12(c), as in [8, Thm. 10.16(b)], it is enough to consider θX. Because $\underline{E}_k$ represents $E^k(-)$, natural transformations θ are classified by the elements $f = \theta\iota_k\colon N \to E^*(\underline{E}_k)$, i. e. morphisms in FMod. The additivity $(\theta X)(x+y) = (\theta X)(x) + (\theta X)(y)$ of θX on the universal example (2.6) yields

$$\mu_k^* \circ f = p_1^* \circ f + p_2^* \circ f\colon N \longrightarrow E^*(\underline{E}_k \times \underline{E}_k).$$

By Prop. 2.7(b), f factors through $PE^*(\underline{E}_k)$. □

6. Unstable comodules

Although the Fourth Answer of section 5 is the cleanest and most general, the Second Answer, in terms of unstable comodules, is usually the most practical and is available in the cases of interest. The parallel with the stable theory of [8] is extremely close, in spite of the very different provenance of the two theories. Some of the machinery was used in [6]; here we supply the missing definitions.

We assume throughout this section that $E_*(\underline{E}_k)$ and $QE_*(\underline{E}_k)$ are free E^*-modules for all k, so that we have available all the results of section 5.

The bigraded group $Q(E)^*_*$. As noted in section 5, tensor products do not work correctly because the groups $QE_*(\underline{E}_k)$ have the wrong degree; we therefore shift degrees. We also adopt more efficient notation, that hides the details of construction and emphasizes the algebraic aspects and the formal similarity to stable comodules. (We remind that homology $E_i(X)$ has degree $-i$ under our conventions.)

Definition 6.1. We define the bigraded group $Q(E)^*_*$ as having the components $Q(E)^k_i = QE_i(\underline{E}_k)$ (the component of $QE_*(\underline{E}_k)$ in degree $-i$), *except* that we assign

the degree $k-i$ (instead of $-i$) to elements of $Q(E)^k_i$. (This is the degree that governs signs in formulae. We thus have the formal isomorphism $\Sigma^k\colon QE_*(\underline{E}_k) \cong Q(E)^k_*$ of degree k.)

We define the left action of $v \in E^h$ on $\Sigma^k c \in Q(E)^k_*$, for $c \in QE_*(\underline{E}_k)$, by $v(\Sigma^k c) = (-1)^{hk}\Sigma^k vc$, as in [8, (6.7)], to make $\Sigma^k\colon QE_*(\underline{E}_k) \cong Q(E)^k_*$ an isomorphism of E^*-modules of degree k.

We equip $Q(E)^k_*$ with the projection

$$q_k\colon E_*(\underline{E}_k) \longrightarrow QE_*(\underline{E}_k) \xrightarrow{\Sigma^k} Q(E)^k_* \ . \tag{6.2}$$

We define the *stabilization*

$$Q(\sigma)\colon Q(E)^k_* \xrightarrow{\Sigma^{-k}} QE_*(\underline{E}_k) \xrightarrow{Q\sigma_{k*}} E_*(E,o), \tag{6.3}$$

where Lemma 4.7(c) provides the factorization $Q\sigma_{k*}$ of σ_{k*} .

We thus have the factorization into E^*-module homomorphisms

$$\sigma_{k*} = Q(\sigma) \circ q_k\colon E_*(\underline{E}_k) \longrightarrow Q(E)^k_* \longrightarrow E_*(E,o), \tag{6.4}$$

where we arranged for $Q(\sigma)$ to have degree zero and q_k to have degree k.

Definition 6.5. Given an additive operation $r\colon k \to m$, i.e. an element $r_A \in PE^m(\underline{E}_k)$, we define the associated E^*-linear functional

$$\langle r_Q, -\rangle\colon Q(E)^k_* \xrightarrow{\Sigma^{-k}} QE_*(\underline{E}_k) \xrightarrow{\langle r_A, -\rangle} E^* \tag{6.6}$$

of degree $m-k$ (with no sign).

Now we can make the degree shift suggested by eq. (5.4). We have the strong duality $PE^*(\underline{E}_k) \cong DQE_*(\underline{E}_k)$ from Lemma 4.16(a). Given an object M of FMod, we use [8, Lemma 6.16(b)] and the freeness of $QE_*(\underline{E}_k)$ to define the natural isomorphism of degree k

$$\mathsf{FMod}^*(PE^*(\underline{E}_k), M) \cong M \mathbin{\widehat{\otimes}} QE_*(\underline{E}_k) \xrightarrow{M\otimes\Sigma^k} M \mathbin{\widehat{\otimes}} Q(E)^k_* \ . \tag{6.7}$$

Lemma 6.8. *Given an additive operation $r\colon k \to m$ and an object M of FMod, the composite (formed using* (6.7)*)*

$$\mathsf{FMod}^*(PE^*(\underline{E}_k), M) \cong M \mathbin{\widehat{\otimes}} Q(E)^k_* \xrightarrow{M\otimes\langle r_Q,-\rangle} M \otimes E^* \cong M$$

coincides with the evaluation homomorphism $e_r\colon \mathsf{FMod}^*(PE^*(\underline{E}_k), M) \to M$ defined by $e_r f = (-1)^{m\deg(f)} f r_A$.

Proof. We choose $x \in M$, $c \in QE_*(\underline{E}_k)$, and evaluate. □

With Defn. 6.5 in hand, we extend Prop. 2.7 and identify:

(6.9)

(i) the *additive operation* $r\colon E^k(-) \to E^m(-)$;

(ii) the *cohomology class* $r = r_A = r\iota_k \in PE^m(\underline{E}_k)$;

(iii) the *morphism of group objects* $r\colon \underline{E}_k \to \underline{E}_m$ in Ho;

(iv) the *E^*-linear functional* $\langle r, -\rangle = \langle r_Q, -\rangle\colon Q(E)^k_* \to E^*$, of degree $m-k$, defined by eq. (6.6).

(We drop the decorations $_A$ and $_Q$ on r except when we need to compare different versions.) As $Q(E)^*_*$ is smaller than $PE^*(\underline{E}_*)$, (iv) is the preferred choice. We do have to be careful with degrees, as (ii) has a different degree from (i) and (iv), while (iii) has no degree at all.

Scholium on signs. We construct the duality diagram in *FMod**

$$\begin{array}{ccccc} & r_S & & r_A & & r_U & \\ E^*(E,o) & \xrightarrow{\sigma_k^*} & PE^*(\underline{E}_k) & \xrightarrow{\subset} & E^*(\underline{E}_k) & \\ \downarrow \cong & (-1)^k & \downarrow \cong & (-1)^k & \downarrow \cong & \\ DE_*(E,o) & \xrightarrow{DQ(\sigma)} & D(Q(E)^k_*) & \xrightarrow{Dq_k} & DE_*(\underline{E}_k) & \\ \langle r_S, - \rangle & & \langle r_Q, - \rangle & & \langle r_U, - \rangle & \end{array} \tag{6.10}$$

whose center isomorphism is taken as

$$PE^*(\underline{E}_k) \xrightarrow{d} DQE_*(\underline{E}_k) \xrightarrow{D(\Sigma^{-k})} D(Q(E)^k_*) \ .$$

Because D is contravariant, each square commutes up to the sign $(-1)^k$.

On restriction to spaces, a stable operation r of degree h yields an additive unstable operation $r\colon k \to k+h$, and we obtain elements r_S, r_A, and r_U lying in the indicated groups. From these, we get the linear functionals $\langle r_S, - \rangle$, $\langle r_U, - \rangle$, and by eq. (6.6) also $\langle r_Q, - \rangle$. We note that r_S and r_Q have degree h, while r_A and r_U have degree $k+h$. The algebra forces us to work with the element r_A and the functional $\langle r_Q, - \rangle$; we are not really interested in the functional $\langle r_A, - \rangle$, which appears only in the definition of $\langle r_Q, - \rangle$, and the element r_Q will occur nowhere.

The complication is that these six elements do *not* all correspond in obvious ways under the morphisms of diag. (6.10). The first surprise was [8, (9.9)], that $\sigma_k^* r_S = (-1)^{kh} r_A$. Of course, r_A and r_U do correspond, because they are the same element regarded as being in different groups. The second surprise is that r_A does not correspond to $\langle r_Q, - \rangle$, because the definition [8, (6.4)] of $D(\Sigma^k)$ requires the sign $(-1)^{k(h+k)}$, which is absent from Defn. 6.5. In fact, matters are simpler if we work with elements and refrain from turning everything into E^*-module homomorphisms.

Proposition 6.11. *In diag.* (6.10)*:*

(a) *Given a stable operation r, the homomorphism $DQ(\sigma)$ takes $\langle r_S, - \rangle$ to $\langle r_Q, - \rangle$, or in elements,*

$$\langle r_Q, c \rangle = \langle r_S, Q(\sigma)c \rangle \qquad \text{for } c \in Q(E)^k_*, \tag{6.12}$$

and also

$$\langle r_U, c \rangle = \langle r_S, \sigma_{k*} c \rangle \qquad \text{for } c \in E_*(\underline{E}_k); \tag{6.13}$$

(b) *Given an additive operation $r\colon k \to m$, the homomorphism Dq_k takes $\langle r_Q, - \rangle$ to $(-1)^{k(m-k)} \langle r_U, - \rangle$, or equivalently, in elements,*

$$\langle r_U, c \rangle = \langle r_Q, q_k c \rangle \qquad \text{for } c \in E_*(\underline{E}_k). \tag{6.14}$$

Proof. We just proved (a), except for eq. (6.13), which combines eqs. (6.12) and (6.14). In (b), $\langle r_A, -\rangle$ is simply the restriction of $\langle r_U, -\rangle$, so that

$$\langle r_U, c\rangle = \langle r_A, \Sigma^{-k} q_k c\rangle = \langle r_Q, q_k c\rangle.$$

But the definition of Dq_k adds the unwanted sign $(-1)^{k(m-k)}$. □

$Q(E)^*_*$ as an algebra. There is much structure on $Q(E)^*_*$. First, it is by construction a left E^*-module.

Proposition 6.15. *For any ring spectrum E, $Q(E)^*_*$ has the properties:*

(a) *$Q(E)^*_*$ is a bigraded E^*-algebra, with multiplication $Q(\phi)$ defined by the commutative diagram* (6.16)

$$\begin{array}{ccccc}
E_*(\underline{E}_k) \otimes E_*(\underline{E}_m) & \xrightarrow{\times} & E_*(\underline{E}_k \times \underline{E}_m) & \xrightarrow{\phi_{U*}} & E_*(\underline{E}_{k+m}) \\
\downarrow{\scriptstyle q_k \otimes q_m} & & & & \downarrow{\scriptstyle q_{k+m}} \\
Q(E)^k_* \otimes Q(E)^m_* & & \xrightarrow{\quad Q(\phi) \quad} & & Q(E)^{k+m}_* \\
\downarrow{\scriptstyle Q(\sigma)\otimes Q(\sigma)} & & & & \downarrow{\scriptstyle Q(\sigma)} \\
E_*(E,o) \otimes E_*(E,o) & \xrightarrow{\times} & E_*(E \wedge E, o) & \xrightarrow{\phi_{S*}} & E_*(E,o)
\end{array} \tag{6.16}$$

and unit $Q(\eta)$ defined by the commutative diagram

$$\begin{array}{ccccc}
E_*(T) & \xrightarrow{=} & E^* & \xrightarrow{=} & E_*(T^+, o) \\
\downarrow{\scriptstyle \eta_{U*}} & & \downarrow{\scriptstyle Q(\eta)} & & \downarrow{\scriptstyle \eta_{S*}} \\
E_*(\underline{E}_0) & \xrightarrow{q_0} & Q(E)^0_* & \xrightarrow{Q(\sigma)} & E_*(E,o)
\end{array} \tag{6.17}$$

(b) *The stabilization $Q(\sigma)\colon Q(E)^*_* \to E_*(E,o)$ is a homomorphism of E^*-algebras.*

Proof. $Q(\phi)$ is inherited, with a shift, from the multiplication on $QE_*(\underline{E}_*)$ constructed by Lemma 4.9. It thus fills in diag. (6.16), which is derived from [8, (9.15)] by applying E-homology and the factorization (6.4). We simply define $Q(\eta) = q_0 \circ \eta_{U*}$, to fill in diag. (6.17). This comes from diag. [8, (9.4)] by taking $x = 1_T \in E^*(T)$. The algebraic properties of $Q(\phi)$ and $Q(\eta)$ are inherited from the E^*-algebra object $n \mapsto \underline{E}_n$ in Ho. Part (b) is clear from the diagrams. □

$Q(E)^*_*$ as a bimodule. We also need the right E^*-action. By Lemma 4.5, the functor $QE_*(-)\colon \mathit{Gp}(\mathit{Ho}) \to \mathit{Mod}$ preserves finite products. We apply [8, Lemma 7.6(a)]

to the E^*-module object $n \mapsto \underline{E}_n$ in $\mathsf{Gp}(\mathsf{Ho})$, to obtain, for each $v \in E^h$, homomorphisms $Q(\xi v)$ that fill in the commutative diagram

$$\begin{array}{ccccc} E_*(\underline{E}_k) & \xrightarrow{q_k} & Q(E)^k_* & \xrightarrow{Q(\sigma)} & E_*(E,o) \\ \downarrow{\scriptstyle (\xi_U v)_*} & & \downarrow{\scriptstyle Q(\xi v)} & & \downarrow{\scriptstyle (\xi_S v)_*} \\ E_*(\underline{E}_{k+h}) & \xrightarrow{q_{k+h}} & Q(E)^{k+h}_* & \xrightarrow{Q(\sigma)} & E_*(E,o) \end{array} \tag{6.18}$$

and make $Q(E)^*_*$ a module object in Mod^*, i. e. an E^*-bimodule. This diagram came from diag. [8, (9.8)] by taking $r = \xi v$.

We have the additive analogue of the stable right unit.

Definition 6.19. We define the *right unit* function $\eta_R\colon E^* \to Q(E)^*_*$ on $v \in E^h = E^h(T)$ by $\eta_R v = q_h v_* 1 \in Q(E)^h_0$, using the homology homomorphism $v_*\colon E^* \cong E_*(T) \to E_*(\underline{E}_h)$ induced by the map $v\colon T \to \underline{E}_h$.

It is clear from [8, (9.4)] and the factorization (6.4) that composition with $Q(\sigma)$ yields the stable right unit $\eta_R\colon E^* \to E_*(E,o)$ of [8, Defn. 11.2].

Proposition 6.20. *For any ring spectrum E, the algebra $Q(E)^*_*$ has the properties:*

(a) *It is a bigraded E^*-bimodule, with components $Q(E)^k_i = QE_i(\underline{E}_k)$ which are assigned the degree $k-i$;*

(b) *It has the well-defined unit element $1 = Q(\eta)1 = \eta_R 1 \in Q(E)^0_0$;*

(c) *The left action of $v \in E^h$ is left multiplication by $v1 \in Q(E)^0_{-h}$;*

(d) *The right action of $v \in E^h$ is right multiplication by $\eta_R v \in Q(E)^h_0$;*

(e) *The stabilization $Q(\sigma)\colon Q(E)^*_* \to E_*(E,o)$ is a homomorphism of E^*-bimodules.*

Remark. Props. 6.15 and 6.20 are similar to [8, Prop. 11.3], except that $Q(E)^*_*$ is bigraded and the conjugation χ is conspicuous by its absence. The examples of section 16 show that χ *does not exist*, at least, not in any obvious sense. (This is why we eschewed χ in [8].)

Proof. Most of the proof is formally identical to the stable case [8, Prop. 11.3]. For (d), we apply E-homology to the factorization [8, (3.27)] of ξv. Part (e) is clear from diag. (6.18). □

We write the left and right E^*-actions as $\lambda_L\colon E^h \otimes Q(E)^k_i \to Q(E)^k_{i-h}$ and $\lambda_R\colon Q(E)^k_i \otimes E^h \to Q(E)^{k+h}_i$. Explicitly, the signs for λ_R are

$$\lambda_R(c \otimes v) = c \cdot v = c(\eta_R v) = (-1)^{h \deg(c)}(\eta_R v)c = (-1)^{h \deg(c)} Q(\xi v)c, \tag{6.21}$$

where $v \in E^h$ and $c \cdot v$ denotes the right action. For future use, we rewrite (d) as the commutative square

$$\begin{array}{ccc} Q(E)_*^k \otimes Q(E)_*^m & \xrightarrow{Q(\phi)} & Q(E)_*^{k+m} \\ \downarrow{\scriptstyle Q(\xi v)\otimes 1} & & \downarrow{\scriptstyle Q(\xi v)} \\ Q(E)_*^{k+h} \otimes Q(E)_*^m & \xrightarrow{Q(\phi)} & Q(E)_*^{k+m+h} \end{array} \tag{6.22}$$

The functor A'. Given an E^*-module M, we define (as promised in section 5) the graded group $A'M$ as having the components

$$(A'M)^k = M^i \mathbin{\widehat{\otimes}_i} Q(E)_i^k = (M \mathbin{\widehat{\otimes}} Q(E)_*^k)^k \tag{6.23}$$

(where the tensor product $\widehat{\otimes}_i$ is formed using the two E^*-actions indexed by i. We have no use for the rest of $M \mathbin{\widehat{\otimes}} Q(E)_*^k$!) We use the isomorphism (6.7) to define the isomorphism $AM \cong A'M$ as having the components

$$(AM)^k = A^k M = \mathsf{FMod}(PE^*(\underline{E}_k), M) \cong M^i \mathbin{\widehat{\otimes}_i} Q(E)_i^k = (A'M)^k. \tag{6.24}$$

We use *this* isomorphism to transfer all the structure of section 5 from A to A' and make A' a comonad, just as we did stably in [8]. (We generally drop the decorations $'$ except when comparing different versions.)

In particular, we use (6.24) to convert modules to comodules. If M is an additively unstable module with coaction $\rho_M\colon M \to AM$ (as in Defn. 5.9), we deduce the equivalent coaction $\rho'_M\colon M \to A'M$ with components

$$\rho'_M\colon M^k \longrightarrow (A'M)^k = M^i \mathbin{\widehat{\otimes}_i} Q(E)_i^k \subset M \mathbin{\widehat{\otimes}} Q(E)_*^k \ . \tag{6.25}$$

In particular, for a space X, we convert the action ρ_X in (5.6) to

$$\rho'_X\colon E^k(X) \longrightarrow E^i(X) \mathbin{\widehat{\otimes}_i} Q(E)_i^k \subset E^*(X) \mathbin{\widehat{\otimes}} Q(E)_*^k \ . \tag{6.26}$$

$Q(E)_*^*$ as a coalgebra. The stable discussion carries over, except that $Q(E)_*^*$ is bigraded. The comonad structure (ψ, ϵ) on A translates into a comonad structure (ψ', ϵ') on A'. By naturality and the case $M = \Sigma^i E^*$, $\psi' M\colon (A'M)^k \to (A'A'M)^k$ must take the form $M \mathbin{\widehat{\otimes}} \psi$ for a certain comultiplication

$$\psi = Q(\psi)\colon Q(E)_i^k \longrightarrow Q(E)_i^j \otimes_j Q(E)_j^k \tag{6.27}$$

(where we sum over j as in eq. (6.23)), and $\epsilon' M\colon (A'M)^k \to M^k$ must take the form $M \mathbin{\widehat{\otimes}} \epsilon$ for a certain counit

$$\epsilon = Q(\epsilon)\colon Q(E)_i^k \longrightarrow E^{k-i}. \tag{6.28}$$

By construction, these are both E^*-bimodule homomorphisms of degree zero.

Proposition 6.29. *Assume that $E_*(\underline{E}_k)$ and $QE_*(\underline{E}_k)$ are free E^*-modules for all k. Then:*

(a) *The homomorphisms $\psi = Q(\psi)$ and $\epsilon = Q(\epsilon)$ in diags.* (6.27) *and* (6.28) *make $Q(E)_*^*$ a coalgebra over E^*;*

(b) *If $E_*(E,o)$ is also free, the stabilization $Q(\sigma)\colon Q(E)^*_* \to E_*(E,o)$ is a morphism of coalgebras* (cf. [8, Lemma 11.8]).

Proof. By taking $M = \Sigma^i E^*$, the comonad axioms [8, (8.6)] for A' yield the coassociativity

$$\begin{array}{ccc} Q(E)^k_h & \xrightarrow{Q(\psi)} & Q(E)^j_h \otimes_j Q(E)^k_j \\ \downarrow{\scriptstyle Q(\psi)} & & \downarrow{\scriptstyle 1\otimes Q(\psi)} \\ Q(E)^i_h \otimes_i Q(E)^k_i & \xrightarrow{Q(\psi)\otimes 1} & Q(E)^i_h \otimes_i Q(E)^j_i \otimes_j Q(E)^k_j \end{array} \tag{6.30}$$

of $Q(\psi)$ and the two counit axioms

$$\begin{array}{ccccc} Q(E)^k_i & \xrightarrow{Q(\psi)} & Q(E)^j_i \otimes_j Q(E)^k_j & \quad & Q(E)^k_i & \xrightarrow{Q(\psi)} & Q(E)^j_i \otimes_j Q(E)^k_j \\ \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle Q(\epsilon)\otimes 1} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle 1\otimes Q(\epsilon)} \\ Q(E)^k_i & \xleftarrow{\lambda_L} & E^{j-i} \otimes_j Q(E)^k_j & & Q(E)^k_i & \xleftarrow{\lambda_R} & Q(E)^j_i \otimes_j E^{k-j} \end{array} \tag{6.31}$$

Part (b) is the translation of Thm. 5.8(b). □

Comodules. Now that we have the coalgebra $Q(E)^*_*$, we can convert Defn. 5.9 and Thm. 5.12.

Definition 6.32. An *unstable (E-cohomology) comodule* is an A'-coalgebra in *FMod*.

In detail, given a complete Hausdorff filtered E^*-module M (i. e. object of *FMod*), an unstable comodule structure on M consists of a coaction $\rho_M\colon M \to A'M$, with components $M^k \to M^i \mathbin{\widehat{\otimes}_i} Q(E)^k_i$ as in diag. (6.25), that is a continuous homomorphism of E^*-modules (i. e. morphism in *FMod*) and satisfies the axioms

$$\begin{array}{ccc} M & \xrightarrow{\rho_M} & M \mathbin{\widehat{\otimes}} Q(E)^*_* \\ \downarrow{\scriptstyle \rho_M} & & \downarrow{\scriptstyle M\otimes Q(\psi)} \\ M \mathbin{\widehat{\otimes}} Q(E)^*_* & \xrightarrow{\rho_M\otimes 1} & M \mathbin{\widehat{\otimes}} Q(E)^*_* \mathbin{\widehat{\otimes}} Q(E)^*_* \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\rho_M} & M \mathbin{\widehat{\otimes}} Q(E)^*_* \\ & \searrow^{\cong} & \downarrow{\scriptstyle M\otimes Q(\epsilon)} \\ & & M \otimes E^* \end{array} \tag{6.33}$$

(i) (ii)

This is a stronger structure than a stable comodule (assuming that $E_*(E,o)$ is free, so that stable comodules can be defined). Given a coaction ρ_M as above, Prop. 6.29(b) shows that the coaction

$$M \xrightarrow{\rho_M} M \mathbin{\widehat{\otimes}} Q(E)^*_* \xrightarrow{M\otimes Q(\sigma)} M \mathbin{\widehat{\otimes}} E_*(E,o) \tag{6.34}$$

makes M a stable comodule.

Remark. We regard comodules as essentially additive constructs, as we find no analogue in the fully unstable context. We therefore omit the adjective "additive" from comodules.

Theorem 6.35. *Assume that $E_*(\underline{E}_k)$ and $QE_*(\underline{E}_k)$ are free E^*-modules for all k* (which is true for $E = H(\mathbb{F}_p)$, BP, MU, KU, or $K(n)$ by Lemma 4.17(a).) *Then given a complete Hausdorff filtered E^*-module M (i. e. object of* FMod*), an additively unstable module structure on M in the sense of Defn. 5.9 is equivalent to an unstable comodule structure on M in the sense of Defn. 6.32.*

Proof. We have the isomorphism $AM \cong A'M$ in eq. (6.24). The axioms (6.33) are just the general coaction axioms [8, (8.7)] interpreted for A'. □

Theorem 6.36. *Assume that $E_*(\underline{E}_k)$ and $QE_*(\underline{E}_k)$ are free E^*-modules for all k* (which is true for $E = H(\mathbb{F}_p)$, BP, MU, KU, or $K(n)$ by Lemma 4.17(a).) *Then:*

(a) *For any space X, there is a natural coaction*

$$\rho_X \colon E^*(X) \longrightarrow E^*(X) \mathbin{\widehat{\otimes}} Q(E)^*_*$$

that makes $E^*(X)\hat{}$ an unstable comodule, which corresponds by Thm. 6.35 to the additive module structure given by Thm. 5.12;

(b) *If also $E_*(E, o)$ is free, we recover the stable coaction* [8, (11.15)] *on $E^*(X)$ from ρ_X as in diag.* (6.34)*;*

(c) *ρ is universal: given a discrete E^*-module N and an integer k, any additive natural transformation $\theta X \colon E^k(X) \to E^*(X) \mathbin{\widehat{\otimes}} N$ (or $\widehat{\theta} X \colon E^k(X)\hat{} \to E^*(X)\hat{} \mathbin{\widehat{\otimes}} N$) that is defined for all spaces X is induced from ρ_X by a unique homomorphism $f \colon Q(E)^k_* \to N$ of E^*-modules as*

$$\theta X \colon E^k(X) \xrightarrow{\rho_X} E^*(X) \mathbin{\widehat{\otimes}} Q(E)^k_* \xrightarrow{1 \otimes f} E^*(X) \mathbin{\widehat{\otimes}} N \ .$$

Proof. We deduce (a) from Thm. 5.12(a) and Thm. 6.35, just as we did stably in [8, Thm. 11.14]. In eq. (6.26), we defined the coaction ρ'_X as corresponding to ρ_X. In (b), the stabilization $Q(\sigma)$ clearly dualizes to $\sigma^*_k \colon E^*(E, o) \to PE^*(\underline{E}_k)$, which we used in eq. (5.7) to define the stabilization $\sigma \colon A \to S$ of comonads.

In (c), the natural transformation θ is classified by the element $u = \theta \iota_k \in E^*(\underline{E}_k) \mathbin{\widehat{\otimes}} N$. Additivity of θ for the universal example (2.6) states that

$$(\mu_k^* \otimes N)u = (p_1^* \otimes N)u + (p_2^* \otimes N)u \qquad \text{in } E^*(\underline{E}_k \times \underline{E}_k) \mathbin{\widehat{\otimes}} N.$$

By [8, Lemma 6.16(a)], u corresponds to a homomorphism $f \colon E_*(\underline{E}_k) \to N$ of E^*-modules. The above property dualizes to

$$f \circ \mu_{k*} = f \circ p_{1*} + f \circ p_{2*} \colon E_*(\underline{E}_k \times \underline{E}_k) \longrightarrow N,$$

which shows that f factors through $Q(E)^k_*$ as required. □

Remark. Just as stably, (c) allows us to use diags. (6.33) to *define* $Q(\psi)$ and $Q(\epsilon)$ in terms of ρ. Three applications of the uniqueness in (c) show that $Q(\psi)$ is coassociative and has $Q(\epsilon)$ as a two-sided counit.

Linear functionals. Theorem 6.35 establishes the equivalence between unstable modules and comodules. For applications, we need the details. All our formulae stabilize to the corresponding formulae of [8, §11] by applying $Q(\sigma)$, which conveniently has degree zero.

Given an unstable comodule M, we recover the action of the additive operation $r\colon k \to m$ on M from Lemma 6.8 as

$$r\colon M^k \xrightarrow{\rho_M} M \mathbin{\widehat{\otimes}} Q(E)^k_* \xrightarrow{M\otimes\langle r,-\rangle} M \otimes E^* \cong M. \tag{6.37}$$

Because $\langle r,-\rangle$ has degree $m-k$, r takes values in M^m. To make this action explicit, let us choose $x \in M^k$ and write

$$\rho_M x = \sum_\alpha (-1)^{\deg(x_\alpha)\deg(c_\alpha)} x_\alpha \otimes c_\alpha \qquad \text{in } M \mathbin{\widehat{\otimes}} Q(E)^k_*, \tag{6.38}$$

where the sum may be infinite, and of course $\deg(x_\alpha) = k - \deg(c_\alpha)$. (As in [8], we insert signs here to keep the next formula simple.) Then

$$rx = \sum_\alpha \langle r, c_\alpha\rangle\, x_\alpha \qquad \text{in } M, \text{ for all } r\colon k \to m, \tag{6.39}$$

where the c_α and x_α depend only on x, not on r. Because M is assumed complete, this sum converges if it is infinite. (Recall that $Q(E)^k_*$ always has the discrete topology.)

Remark. It is important for our applications *not* to require the c_α to form a basis of $Q(E)^k_*$, or even be linearly independent; but if they do form a basis, the x_α are uniquely determined by eq. (6.39) as $x_\alpha = c^*_\alpha x$, where c^*_α denotes the operation dual to c_α.

The fact that ρ_M is an E^*-module homomorphism is expressed by

$$r(vx) = \sum_\alpha \langle r, (\eta_R v)c_\alpha\rangle x_\alpha = \sum_\alpha (-1)^{h\deg(c_\alpha)} \langle r, c_\alpha\, \eta_R v\rangle\, x_\alpha \quad \text{in } M, \tag{6.40}$$

for any $v \in E^h$ and all operations $r\colon k+h \to m$.

Because $Q(\epsilon)\colon Q(E)^k_* \to E^*$ corresponds to ϵ in eq. (5.15), which is evaluation on ι_k, we have immediately

$$\langle \iota_k, -\rangle = Q(\epsilon)\colon Q(E)^k_* \longrightarrow E^*, \tag{6.41}$$

as is obvious by comparing axiom (6.33)(ii) with eq. (6.37). In other words, in the list (6.9), the identity operation ι_k corresponds to the functional $Q(\epsilon)$.

The cohomology of a point. Our first test space is the one-point space T.

Proposition 6.42. *In the unstable comodule $E^*(T) = E^*$:*

(a) *The action of the additive operation $r\colon k \to m$ on $v \in E^k$ is given by*

$$rv = \langle r, \eta_R v\rangle \qquad \text{in } E^*(T) = E^*; \tag{6.43}$$

(b) *The coaction $\rho_T\colon E^* \to E^* \otimes Q(E)^*_* \cong Q(E)^*_*$ coincides with the right unit $\eta_R\colon E^* \to Q(E)^*_0$* (see Defn. 6.19).

Proof. We imitate [8, Prop. 11.22]. The map $v\colon T \to \underline{E}_k$ yields

$$rv = \langle rv, 1\rangle = \langle v^* r_U, 1\rangle = \langle r_U, v_* 1\rangle = \langle r_Q, q_k v_* 1\rangle = \langle r_Q, \eta_R v\rangle,$$

by eq. (6.14) and Defn. 6.19 of η_R. We compare eqs. (6.38) and (6.39) and rewrite this as $\rho_T v = 1 \otimes \eta_R v$, to give (b). □

Homology homomorphisms. A class $x \in E^k(X)$ may be regarded as a map $x\colon X \to \underline{E}_k$. We need information about the induced homology homomorphism $x_*\colon E_*(X) \to E_*(\underline{E}_k)$.

Proposition 6.44. *Assume that $E_*(\underline{E}_k)$ and $QE_*(\underline{E}_k)$ are free E^*-modules for all k. Given $x \in E^k(X)$, suppose that rx is given by eq.* (6.39)*. Then the homomorphism $q_k \circ x_*\colon E_*(X) \to Q(E)^k_*$ induced by the map $x\colon X \to \underline{E}_k$ is given on $z \in E_h(X)$ by*

$$q_k x_* z = \sum_\alpha (-1)^{\deg(c_\alpha)(\deg(x_\alpha)+h)} \langle x_\alpha, z\rangle\, c_\alpha = \sum_\alpha c_\alpha \langle x_\alpha, z\rangle \quad \text{in } Q(E)^k_* \ . \tag{6.45}$$

Proof. For any additive $r\colon k \to m$, we have $\langle r_Q, q_k x_* z\rangle = \langle r_U, x_* z\rangle$ by eq. (6.14). The rest of the proof is formally identical to the stable analogue [8, Prop. 11.26]. □

Conversely, we can recover $\rho_X x$ from x_* when X is well behaved, just as we did stably. If $E_*(X)$ is free, we have strong duality $E^*(X) \cong DE_*(X)$ by Thm. 1.18(a), and [8, Lemma 6.16(a)] supplies the isomorphism

$$E^*(X) \mathbin{\widehat{\otimes}} Q(E)^k_* \cong \mathit{Mod}^*(E_*(X), Q(E)^k_*) \ . \tag{6.46}$$

Proposition 6.47. *Assume that $E_*(X)$, $E_*(\underline{E}_k)$, and $QE_*(\underline{E}_k)$ are free E^*-modules for all k. Take $x \in E^k(X)$. Then under the isomorphism* (6.46)*, the element $\rho_X x$ corresponds to the homomorphism $q_k \circ x_*\colon E_*(X) \to E_*(\underline{E}_k) \to Q(E)^k_*$.*

Proof. We apply the isomorphism to eq. (6.38) and compare with eq. (6.45). □

In particular, it is important to know the homomorphism of E^*-modules

$$Q(r)\colon Q(E)^k_* \cong QE_*(\underline{E}_k) \xrightarrow{Qr_*} QE_*(\underline{E}_m) \cong Q(E)^m_* \tag{6.48}$$

induced by an additive operation $r\colon k \to m$ (which by Prop. 2.7(c) is a morphism of group objects in Ho). It has degree $m-k$. The $Q(r)$ provide a convenient faithful representation of the additive operations. The translation of diag. (5.11) is the commutative square

$$\begin{array}{ccc} M^k & \xrightarrow{\quad r \quad} & M^m \\ \downarrow{\scriptstyle \rho_M} & & \downarrow{\scriptstyle \rho_M} \\ M^i \otimes_i Q(E)^k_i & \xrightarrow{M \otimes Q(r)} & M^i \otimes_i Q(E)^m_i \end{array} \tag{6.49}$$

which stabilizes to diag. [8, (11.29)].

Just as stably, we easily recover the functional $\langle r, -\rangle$ from $Q(r)$ as

$$\langle r, -\rangle : Q(E)_*^k \xrightarrow{Q(r)} Q(E)_*^m \xrightarrow{Q(\epsilon)} E^*. \tag{6.50}$$

Conversely, we have the additive analogue of [8, Lemma 11.31].

Lemma 6.51. *Assume that $E_*(\underline{E}_k)$ and $QE_*(\underline{E}_k)$ are free E^*-modules for all k. If $r: k \to m$ is an additive operation, then the homology homomorphism $Q(r): Q(E)_*^k \to Q(E)_*^m$ in eq.* (6.48) *has the properties:*

(a) *The diagram*

$$\begin{array}{ccc} Q(E)_*^k & \xrightarrow{\quad Q(r) \quad} & Q(E)_*^m \\ \downarrow{\scriptstyle Q(\psi)} & & \downarrow{\scriptstyle Q(\psi)} \\ Q(E)_*^* \otimes Q(E)_*^k & \xrightarrow{1 \otimes Q(r)} & Q(E)_*^* \otimes Q(E)_*^m \end{array} \tag{6.52}$$

commutes; in other words, $Q(r)$ is a morphism of left $Q(E)_*^*$-comodules;

(b) *$Q(r): Q(E)_*^k \to Q(E)_*^m$ is the unique homomorphism of left E^*-modules that satisfies eq.* (6.50) *and is a morphism of left $Q(E)_*^*$-comodules in the sense of (a);*

(c) *$Q(r)$ is given in terms of the functional $\langle r, -\rangle$ as*

$$\begin{aligned} Q(r): Q(E)_i^k \xrightarrow{Q(\psi)} Q(E)_i^j \otimes_j Q(E)_j^k \xrightarrow{1 \otimes \langle r, -\rangle} Q(E)_i^j \otimes_j E^{m-j} \\ \xrightarrow{\lambda_R} Q(E)_i^m. \end{aligned}$$

□

We deduce from (c) that the composite $sr: k \to n$ of the operations $r: k \to m$ and $s: m \to n$ corresponds to the functional

$$\begin{aligned} \langle sr, -\rangle : Q(E)_i^k \xrightarrow{Q(\psi)} Q(E)_i^j \otimes_j Q(E)_j^k \xrightarrow{1 \otimes \langle r, -\rangle} Q(E)_i^j \otimes_j E^{m-j} \\ \xrightarrow{\lambda_R} Q(E)_i^m \xrightarrow{\langle s, -\rangle} E^{n-i}. \end{aligned} \tag{6.53}$$

Remark. From diags. (6.30) and (6.31)(ii) we observe that for fixed h, $Q(\psi)$ makes the graded group $n \mapsto Q_h^n$ an additively unstable comodule, if we use the right E^*-module action (6.21). Then by (c), the action of $r: k \to m$ is just $Q(r)$, and diag. (6.52) becomes a special case of diag. (6.49).

7. What is an additively unstable algebra?

In this section, we define an additively unstable algebra by enriching each of the four Answers in section 5 with multiplicative structure. The treatment is closely parallel to the stable case [8, §12] and we give only the significant additions. The logical sequence is made slightly complicated by the fact that the monoidal structure is

most easily described in the context of the Second (or Third) Answer, while the comonad structure prefers the Fourth Answer.

In Defn. 7.13 we introduce the collapse operation, which detects the connectedness of a space.

We assume throughout this section that $E_*(\underline{E}_k)$ *and* $QE_*(\underline{E}_k)$ *are free* E^**-modules for all* k, which is true for our five examples by Lemma 4.17(a). Then by Cor. 2.9, $PE^*(\underline{E}_k)$ is an object of *FMod*.

First Answer. We have, for any space X, the additively unstable action (5.1)

$$\circ\colon PE^m(\underline{E}_k) \times E^k(X) \longrightarrow E^m(X) \ .$$

Given $x \in E^k(X)$, $y \in E^m(X)$, and $r \in PE^*(\underline{E}_{k+m})$, we would like to have a Cartan formula

$$r(xy) = \sum_\alpha (r'_\alpha x)(r''_\alpha y) \qquad \text{in } E^*(X), \tag{7.1}$$

for suitably chosen operations r'_α and r''_α (depending on k and m as well as r). For the universal example

$$X = \underline{E}_k \times \underline{E}_m, \qquad \text{with } x = \iota_k \times 1,\ y = 1 \times \iota_m,\ xy = \phi = \iota_k \times \iota_m, \tag{7.2}$$

where $\phi\colon \underline{E}_k \times \underline{E}_m \to \underline{E}_{k+m}$ denotes the multiplication map of [8, Thm. 3.25], eq. (7.1) reduces to

$$\phi^* r = \sum_\alpha r'_\alpha \times r''_\alpha \qquad \text{in } E^*(\underline{E}_k \times \underline{E}_m) \ .$$

To ensure that $\phi^* r$ is expressible in this form, we need to allow infinite sums and use the Künneth homeomorphism $E^*(\underline{E}_k \times \underline{E}_m) \cong E^*(\underline{E}_k) \mathbin{\widehat{\otimes}} E^*(\underline{E}_m)$ from Thm. 1.18(c).

We need to know more, that $r'_\alpha, r''_\alpha \in PE^*(\underline{E}_*)$. We have enough duality isomorphisms to dualize the multiplication in Lemma 4.9 and define a comultiplication ψ_P by the commutative diagram

$$\begin{array}{ccccc}
PE^*(\underline{E}_{k+m}) & & \xrightarrow{\quad\psi_P\quad} & & PE^*(\underline{E}_k) \mathbin{\widehat{\otimes}} PE^*(\underline{E}_m) \\
\big\downarrow{\scriptstyle \subset} & & & & \big\downarrow{\scriptstyle \subset} \\
E^*(\underline{E}_{k+m}) & \xrightarrow{\phi^*} & E^*(\underline{E}_k \times \underline{E}_m) & \xleftarrow{\cong} & E^*(\underline{E}_k) \mathbin{\widehat{\otimes}} E^*(\underline{E}_m)
\end{array} \tag{7.3}$$

Then we write $\psi_P r = \sum_\alpha r'_\alpha \otimes r''_\alpha$, as required.

We must not forget the unit element $1_X \in E^*(X)$. We define the counit $\epsilon_P\colon PE^*(\underline{E}_0) \to E^*$ as the restriction of $\eta^*\colon E^*(\underline{E}_0) \to E^*(T) = E^*$, so that $r1_X = (\epsilon_P r)1_X$ in $E^*(X)$.

It is now clear what an additively unstable algebra should be. Given an E^*-algebra M, we need actions $PE^m(\underline{E}_k) \times M^k \to M^m$ that compose correctly, are biadditive and E^*-bilinear in the sense of diag. (5.2), satisfy the Cartan formula (7.1), and respect the unit in the sense that $r1_M = (\epsilon_P r)1_M$. In the classical case $E = H(\mathbb{F}_p)$, there is a good Cartan formula and this approach is useful. For more general E, such as MU and BP, this structure seems even more impractical than it was stably.

Second Answer. We have the coaction (6.26),

$$\rho_X \colon E^k(X) \longrightarrow E^i(X) \mathbin{\widehat{\otimes}_i} Q(E)^k_i.$$

In contrast to the Cartan formula of the First Answer, and just as stably in [8], all we have to do is observe that as k varies, ρ_X is a homomorphism of E^*-algebras, where we use the bigraded algebra structure on $Q^*_* = Q(E)^*_*$ from Prop. 6.15.

Explicitly, if for particular $x, y \in E^*(X)$ we have, as in eq. (6.39),

$$rx = \sum_\alpha \langle r, c_\alpha \rangle\, x_\alpha; \qquad ry = \sum_\beta \langle r, d_\beta \rangle\, y_\beta; \qquad \text{for all } r,$$

the Cartan formula (7.1) becomes (cf. the stable analogue [8, (12.5)])

$$r(xy) = \sum_\alpha \sum_\beta (-1)^{\deg(d_\beta)\deg(x_\alpha)} \langle r, c_\alpha d_\beta \rangle\, x_\alpha y_\beta \qquad \text{in } E^*(X)\hat{}, \text{ for all } r. \quad (7.4)$$

Lemma 7.5. *Assume that $E_*(\underline{E}_k)$ and $QE_*(\underline{E}_k)$ are free E^*-modules for all k. Then the homomorphisms $Q(\psi)$ and $Q(\epsilon)$ in* (6.27) *and* (6.28) *are multiplicative and respect the unit element.*

We defer the proofs until after Thm. 7.9, as the coalgebra structure on $Q(E)^*_*$ is not easily handled directly. The Lemma makes the following definition reasonable.

Definition 7.6. We call an unstable comodule M in the sense of Defn. 6.32 an *unstable (E-cohomology) comodule algebra* if M is a filtered algebra (i.e. object of FAlg) and its coaction $\rho_M \colon M \to M \mathbin{\widehat{\otimes}} Q(E)^*_*$ is a homomorphism of E^*-algebras.

In detail, M is a complete Hausdorff commutative filtered E^*-algebra, equipped with a structure map ρ_M that is a continuous homomorphism of E^*-algebras and makes diags. (6.33) commute.

Theorem 7.7. *Assume that $E_*(\underline{E}_k)$ and $QE_*(\underline{E}_k)$ are free E^*-modules for all k* (which is true for $E = H(\mathbb{F}_p)$, BP, MU, KU, or $K(n)$ by Lemma 4.17(a)). *Then:*

(a) *For any space X, ρ_X makes $E^*(X)\hat{}$ an unstable comodule algebra in the sense of Defn. 7.6;*

(b) *ρ is universal: given a (possibly bigraded) discrete E^*-algebra B, any natural transformation of rings $\theta X \colon E^*(X) \to E^*(X) \mathbin{\widehat{\otimes}} B$ (or $\widehat{\theta} X \colon E^*(X)\hat{} \to E^*(X)\hat{} \mathbin{\widehat{\otimes}} B$) that is defined for all spaces X is induced from ρ_X by a unique homomorphism $f \colon Q(E)^*_* \to B$ of left E^*-algebras as*

$$\theta X \colon E^*(X) \xrightarrow{\rho_X} E^*(X) \mathbin{\widehat{\otimes}} Q(E)^*_* \xrightarrow{1 \otimes f} E^*(X) \mathbin{\widehat{\otimes}} B\ .$$

Proof. This will follow from Thm. 7.9 in the same way that the stable result Thm. 12.8 followed from Thm. 12.10 in [8]. □

Third Answer. We use the multiplication $Q(\phi)\colon Q^k_* \otimes Q^m_* \to Q^{k+m}_*$ from Prop. 6.15 to make A' a symmetric monoidal functor $(A', \zeta_{A'}, z_{A'})$ in *FMod*, with

$$\zeta_{A'}(M,N)\colon (A'M)^k \mathbin{\widehat{\otimes}} (A'N)^m \longrightarrow (A'(M \mathbin{\widehat{\otimes}} N))^{k+m}$$

given by

$$\begin{aligned}\zeta_{A'}(M,N)\colon M \mathbin{\widehat{\otimes}} Q^k_* \mathbin{\widehat{\otimes}} N \mathbin{\widehat{\otimes}} Q^m_* &\cong M \mathbin{\widehat{\otimes}} N \mathbin{\widehat{\otimes}} (Q^k_* \otimes Q^m_*) \\ &\longrightarrow M \mathbin{\widehat{\otimes}} N \mathbin{\widehat{\otimes}} Q^{k+m}_*\end{aligned} \tag{7.8}$$

and $z_{A'} = \eta_R\colon E^h \to E^* \otimes Q^h_* \cong Q^h_*$. Thus when M is an E^*-algebra, so is $A'M$. We see that A', equipped with natural transformations $\psi'\colon A' \to A'A'$ and $\epsilon'\colon A' \to I$ constructed from $Q(\psi)$ and $Q(\epsilon)$, becomes a symmetric monoidal comonad in *FMod* and therefore a comonad in *FAlg*.

Fourth Answer. For suitable E, we can make A a comonad in *FAlg*.

Theorem 7.9. *Assume that $E_*(\underline{E}_k)$ and $QE_*(\underline{E}_k)$ are free E^*-modules for all k* (which is true for $E = H(\mathbb{F}_p)$, BP, MU, KU, or $K(n)$ by Lemma 4.17(a)). *Then:*

(a) *We can enrich A to make it a symmetric monoidal comonad in *FMod* and therefore a comonad in *FAlg*;*

(b) *If also $E_*(E,o)$ is free, the stabilization $\sigma\colon A \to S$ is a monoidal natural transformation in *FMod*.*

The relevant definition is now clear.

Definition 7.10. An *additively unstable (E-cohomology) algebra* is an A-coalgebra in *FAlg*, i. e. a complete Hausdorff commutative filtered E^*-algebra M equipped with a morphism $\rho_M\colon M \to AM$ in *FAlg* that satisfies the coaction axioms [8, (8.7)].

If the closed ideal $L \subset M$ is invariant, the quotient algebra M/L inherits a well-defined A-coalgebra structure.

Theorem 7.11. *Assume that $E_*(\underline{E}_k)$ and $QE_*(\underline{E}_k)$ are free E^*-modules for all k* (which is true for $E = H(\mathbb{F}_p)$, BP, MU, KU, or $K(n)$ by Lemma 4.17(a)). *Then given a complete Hausdorff commutative filtered E^*-algebra M* (i. e. object of *FAlg*), *an unstable comodule algebra structure on M in the sense of Defn. 7.6 is equivalent to an additively unstable algebra structure on M in the sense of Defn. 7.10.*

Theorem 7.12. *Assume that $E_*(\underline{E}_k)$ and $QE_*(\underline{E}_k)$ are free E^*-modules for all k* (which is true for $E = H(\mathbb{F}_p)$, BP, MU, KU, or $K(n)$ by Lemma 4.17(a)). *Then:*

(a) *For any space X, the coaction $\rho_X\colon E^*(X) \to A(E^*(X)\hat{})$ in diag.* (5.6) *is a homomorphism of E^*-algebras and makes $E^*(X)\hat{}$ an additively unstable algebra;*

(b) *ρ is universal: given a graded monoid object $n \mapsto C^n$ in $\mathit{FMod}^{\mathrm{op}}$, so that (by* [8, Lemma 7.9]*) $n \mapsto G^n(X) = \mathit{FMod}(C^n, E^*(X)\hat{})$ is a graded ring, any natural*

transformation of graded rings $\theta X\colon E^*(X) \to G^*(X)$ *(or* $\widehat{\theta}X\colon E^*(X)\hat{} \to G^*(X)$*), that is defined for all spaces* X*, is induced from* ρ_X *by a unique morphism in* $\mathsf{FMod}^{\mathrm{op}}$ *of graded monoid objects with components* $f^n\colon C^n \to PE^*(\underline{E}_n)$ *in* FMod*, as*

$$\theta X\colon E^n(X) \xrightarrow{\rho_X} \mathsf{FMod}(PE^*(\underline{E}_n), E^*(X)\hat{}) \xrightarrow{\mathrm{Hom}(f^n,1)} \mathsf{FMod}(C^n, E^*(X)\hat{}) \ .$$

Proof of Thms. 7.9 and 7.12. The main proof proceeds by the same five steps as stably for [8, Thms. 12.10, 12.13], except based on Thm. 5.8 instead of [8, Thm. 10.12]. We give only the major changes. We recall the universal class $\iota_k \in E^k(\underline{E}_k)$, element $\mathrm{id}_k \in A^k PE^*(\underline{E}_k)$, and ρ_k from the proof of Thm. 5.8.

Step 1. We construct the symmetric monoidal functor

$$(A, \zeta_A, z_A)\colon (\mathsf{FMod}, \widehat{\otimes}, E^*) \longrightarrow (\mathsf{Mod}, \otimes, E^*) \ .$$

Then A will take monoid objects in FMod (i.e. objects of FAlg) to monoid objects in Mod (i.e. E^*-algebras).

By Lemma 4.16(a), we can construct the diagram (7.3) that defines ψ_P and verify its properties, which are dual to those of $Q(\phi)$ in Props. 6.15 and 6.20. The counit $\epsilon_P\colon PE^*(\underline{E}_0) \to E^*$ is the restriction of $\eta^*\colon E^*(\underline{E}_0) \to E^*(T) = E^*$. These make $n \mapsto PE^*(\underline{E}_n)$ an E^*-algebra object in $\mathsf{FMod}^{*\,\mathrm{op}}$, to which we apply [8, Lemma 7.14]. The necessary compatibility axiom [8, (7.13)] is the dual of diag. (6.22). As stably, we use [8, (7.15)] to identify z_A with $\rho_T\colon E^*(T) \to AE^*(T)$.

If $E_*(E, o)$ is also free, we can dualize Prop. 6.15(b) to see that the destabilizations $\sigma_n^*\colon E^*(E, o) \to PE^*(\underline{E}_n)$ form a morphism of graded monoid objects in $(\mathsf{FMod}^{*\,\mathrm{op}}, \widehat{\otimes}, E^*)$. Then [8, Lemma 7.9(b)] shows that $\sigma\colon A \to S$ is monoidal.

Step 2. The proof that ρ is monoidal is similar to the stable case. Here, the universal example is $X = \underline{E}_k$ and $Y = \underline{E}_m$, with the element $\iota_k \otimes \iota_m$. The two elements of $A^{k+m}E^*(\underline{E}_k \times \underline{E}_m)$ to be compared are

$$PE^*(\underline{E}_{k+m}) \xrightarrow{\psi_P} PE^*(\underline{E}_k) \mathbin{\widehat{\otimes}} PE^*(\underline{E}_m) \subset E^*(\underline{E}_k) \mathbin{\widehat{\otimes}} E^*(\underline{E}_m) \xrightarrow{\times} E^*(\underline{E}_k \times \underline{E}_m)$$

and

$$PE^*(\underline{E}_{k+m}) \subset E^*(\underline{E}_{k+m}) \xrightarrow{\phi^*} E^*(\underline{E}_k \times \underline{E}_m).$$

These agree by diag. (7.3). The second condition needed is just $z_A = \rho_T$.

Step 3. The analogue of diag. [8, (12.17)] for this situation is fig. 1. To establish this, we proceed as in [8, Thm. 12.10]. Because ρ is monoidal and natural, we have the commutative diagram fig. 2 (cf. diag. [8, (12.16)]) which includes an isomorphism from Thm. 1.18(c). Figure 1 is obtained from this by restriction, using the coaction (5.13) and diag. (7.3).

Step 4. The monoidality of ψ follows formally from that of ρ, just as stably (cf. diags. [8, (12.18)]). The universal example is $M = PE^*(\underline{E}_m)$ and $N = PE^*(\underline{E}_n)$, with element $\mathrm{id}_m \otimes \mathrm{id}_n$. We use fig. 1 instead of diag. [8, (12.17)].

Figure 1. Additive operations and comultiplication

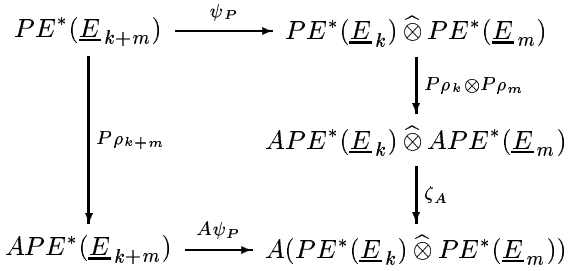
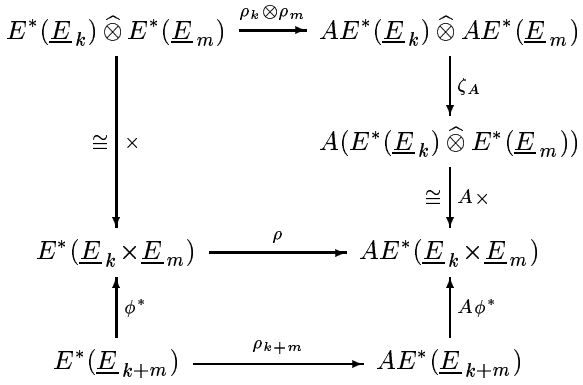


Figure 2. The monoidality of ρ



Step 5. The proof that ϵ is monoidal is formally the same as stably, except for the insertion of indices.

In Thm. 7.12(b), C has comultiplications $\psi_C\colon C^{k+m} \to C^k\,\widehat{\otimes}\,C^m$ and a counit $\epsilon_C\colon C^0 \to E^*$ which make $n \mapsto \mathit{FMod}(C^n, E^*(X)\hat{})$ a graded ring. For each n, Thm. 5.12(c) provides a morphism $f^n\colon C^n \to PE^*(\underline{E}_n)$ in FMod. For the universal example (7.2), the multiplicativity $(\theta X)(xy) = ((\theta X)x)((\theta X)y)$ reduces to the commutativity of the outside of the diagram in fig. 3. The lower rectangle is diag. (7.3). It follows that the upper square commutes, so that f preserves the comultiplication. Similarly, $(\theta T)1 = 1$ yields fig. 4, which shows that f preserves the counit. □

Proof of Thm. 7.11. We use the isomorphism (6.7) to translate the monoidal structure of A to A'. From ζ_A, which is given by [8, (7.11)], we obtain eq. (7.8). We have identified both z_A and $z_{A'}$ with the coaction ρ_T. □

Figure 3. Comparison of comultiplications

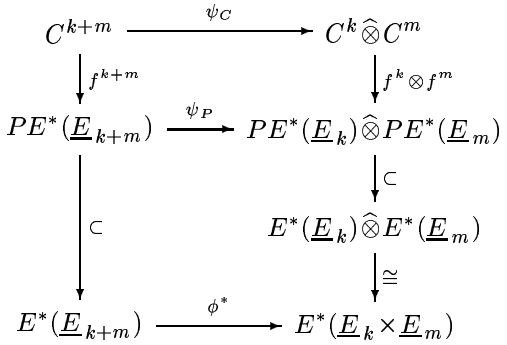
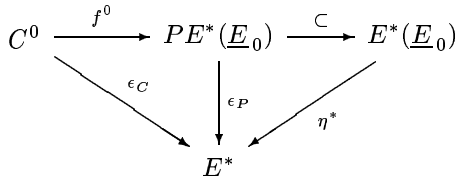


Figure 4. Comparison of counits



Proof of Lemma 7.5. Theorem 7.9(a) shows in particular that $\psi\colon A \to AA$ and $\epsilon\colon A \to I$ are monoidal natural transformations. By the isomorphism (6.24), so are $\psi'\colon A' \to A'A'$ and $\epsilon'\colon A' \to I$. Evaluation of the relevant diagrams involving ζ for $M = N = E^*$ show precisely that $Q(\psi)$ and $Q(\epsilon)$ are multiplicative. Since $z_{A'} = \eta_R\colon E^h \to E^* \otimes Q(E)^h_* \cong Q(E)^h_*$, the two diagrams involving z show that $\psi 1 = 1 \otimes 1$ and $\epsilon 1 = 1$, simply because $\eta_R 1$ is the unit element of $Q(E)^*_*$. □

Proof of Thm. 7.7. Part (a) follows from Thm. 7.12(a). In (b), Thm. 6.36(c) provides for each n the E^*-module homomorphism $f^n\colon Q(E)^n_* \to B$ that induces $\theta X\colon E^n(X) \to E^*(X) \widehat{\otimes} B$. As in the proof of [8, Thm. 12.8(b)], the resulting $f\colon Q(E)^*_* \to B$ is an E^*-algebra homomorphism. □

Connectedness. There is a particular operation that is useful for expressing the concept of connectedness in a cohomology algebra. It sees only the path components of a space.

Definition 7.13. For each n, we define the *collapse* operation $\kappa_n\colon n \to n$ as the map $\kappa_n\colon \underline{E}_n \to \underline{E}_n$ (well defined up to homotopy) that sends each path component of $\underline{E}_n$ to one point in that path component.

It is clearly additive, multiplicative ($\kappa(xy) = (\kappa x)(\kappa y)$), unital ($\kappa_0 1_X = 1_X$), and idempotent. It commutes with all operations in the sense that $\kappa_m \circ r = r \circ \kappa_k \colon k \to m$ for all $r \colon k \to m$; in particular, κ is E^*-linear. It is zero in any degree n for which $E^n = 0$. In spite of being defined in all degrees, it is not at all stable, as $\Omega\kappa_n = 0$. All these properties carry over to any additively unstable algebra M; in particular, we always have the E^*-module decomposition $M = \operatorname{Im}\kappa \oplus \operatorname{Ker}\kappa$, with $(E^*)1_M \subset \operatorname{Im}\kappa$.

For a connected space X with basepoint o, it is clear that the augmentation ideal $E^*(X, o) \subset E^*(X)$ is precisely $\operatorname{Ker}\kappa$. In general, $\operatorname{Ker}\kappa = F^1E^*(X)$ for any space X, the first stage of the skeleton filtration. This suggests the following definition.

Definition 7.14. We call the additively unstable algebra M *connected* if $\operatorname{Im}\kappa = (E^*)1_M$. We call M *spacelike* if it is a product (in FAlg) of connected algebras.

In particular, for a space X, $E^*(X)\hat{}$ is always spacelike, and is connected if and only if X is connected.

8. What is an unstable object?

In this section, we interpret what it means to have an algebra over all the unstable operations on E-cohomology. Tensor products rapidly become unworkable for non-additive operations, with the effect that only the First and Fourth Answers from section 5 survive intact.

We generally assume that $E_*(\underline{E}_k)$ is a free E^*-module for all k. Then Thm. 1.18 provides all the Künneth and duality isomorphisms and homeomorphisms we need. Of course, when we compare with the additive or stable theory, we impose the appropriate extra conditions.

As in (2.1), we identify:

(i) The *cohomology operation* $r \colon E^k(-) \to E^m(-)$;

(ii) The *class* $r = r(\iota_k) \in E^m(\underline{E}_k)$;

(iii) The representing *map* $r \colon \underline{E}_k \to \underline{E}_m$;

and write any of these as $r \colon k \to m$. (We shall retain the parentheses in $r(x)$ whenever r is nonadditive.)

We first deal with the *constant* operations $r \colon k \to m$, those of the form $r(x) = v1_X \in E^m(X)$ for all $x \in E^k(X)$ and all spaces X, where $v \in E^m$.

Lemma 8.1. *Any operation $r \colon k \to m$ decomposes uniquely as the sum of a based operation $s \colon k \to m$ and a constant operation.*

Proof. We set $v = r(0) \in E^m(T) = E^m$ and define the operation s by $s(x) = r(x) - v1_X$ in $E^*(X)$, to make $s(0) = 0$. □

First Answer. Since $E^k(-)$ is represented in Ho by $\underline{E}_k$, we have as in (5.1) the actions

$$\circ \colon E^m(\underline{E}_k) \times E^k(X) \longrightarrow E^m(X), \tag{8.2}$$

except that we cannot write them using tensor products. Instead, we need a Cartan formula for $r(x+y)$ as well as for $r(xy)$.

To find $r(x+y)$, we consider the abelian group object $\underline{E}_k$ of Ho provided by [8, Cor. 7.8], which is equipped with the addition map $\mu_k\colon \underline{E}_k \times \underline{E}_k \to \underline{E}_k$ and zero map $\omega_k\colon T \to \underline{E}_k$. By Lemma 8.1, we may restrict attention to based operations r. The group axioms on $\underline{E}_k$ lead (as in any Hopf algebra) to a formula of the form

$$\mu_k^* r = r\times 1 + \sum_\alpha r'_\alpha \times r''_\alpha + 1\times r \qquad \text{in } E^*(\underline{E}_k \times \underline{E}_k) \cong E^*(\underline{E}_k) \mathbin{\widehat{\otimes}} E^*(\underline{E}_k),$$

where the r'_α and r''_α are also based. The only novelty is that the sum may be infinite. This translates into the desired Cartan formula

$$r(x+y) = r(x) + \sum_\alpha r'_\alpha(x)\, r''_\alpha(y) + r(y) \qquad \text{in } E^*(X) \tag{8.3}$$

for any $x, y \in E^k(X)$.

There is a similar Cartan formula for multiplication, given $x \in E^k(X)$ and $y \in E^m(X)$, of the form

$$r(xy) = \sum_\alpha r'_\alpha(x)\, r''_\alpha(y) \qquad \text{in } E^*(X), \tag{8.4}$$

for certain (other) based operations r'_α and r''_α (which depend on k and m).

This suggests that an unstable algebra should consist of an E^*-algebra M equipped with operations r that compose correctly and satisfy both Cartan formulae. This requires knowing the operations r'_α and r''_α in eqs. (8.3) and (8.4) for all r. In section 10, we shall in effect expand both Cartan formulae explicitly.

Second Answer. We convert the First Answer to adjoint form, corresponding to the Fourth Answer in section 5. (We skip the Second and Third Answers.) Everything becomes far cleaner, more evidence that this is the natural answer.

Any element $x \in E^k(X)$, regarded as a map $x\colon X \to \underline{E}_k$, induces the continuous homomorphism $x^*\colon E^*(\underline{E}_k) \to E^*(X)$ of E^*-algebras. By Thm. 1.18(a), $E^*(\underline{E}_k)$ is Hausdorff and so in FAlg; we may therefore define, for any object M of FAlg,

$$U^k M = \mathit{FAlg}(E^*(\underline{E}_k), M), \tag{8.5}$$

the set of all continuous E^*-algebra homomorphisms $E^*(\underline{E}_k) \to M$. This encodes the set of all possible actions on a typical element of degree k. We convert the action (8.2) to what we continue to call a coaction,

$$\rho_X\colon E^k(X) \longrightarrow U^k(E^*(X)\hat{}) = \mathit{FAlg}(E^*(\underline{E}_k), E^*(X)\hat{}), \tag{8.6}$$

by defining $\rho_X x = x^*$, completing $E^*(X)$ if necessary to get it into FAlg. We assemble the sets $U^k M$ to form the graded set UM, which has the component $(UM)^k = U^k M$ in degree k, and obtain $\rho_X\colon E^*(X) \to U(E^*(X)\hat{})$.

We compare UM with the stable and additive versions. Restriction to $PE^*(\underline{E}_k)$ induces the natural transformation

$$(\tau M)^k\colon U^k M = \mathit{FAlg}(E^*(\underline{E}_k), M) \longrightarrow \mathit{FMod}(PE^*(\underline{E}_k), M) = A^k M\ . \tag{8.7}$$

These form $\tau M\colon UM \to AM$. Composition with $\sigma M\colon AM \to SM$ (see eq. (5.7)) yields

$$U^k M = \mathsf{FAlg}(E^*(\underline{E}_k), M) \longrightarrow \mathsf{FMod}^k(E^*(E,o), M) = (SM)^k,$$

which is induced by the destabilization $\sigma_k^*\colon E^*(E,o) \to PE^*(\underline{E}_k) \subset E^*(\underline{E}_k)$.

Apparently only a morphism of graded sets, ρ_X has far more structure, thanks to the rich structure on the spaces $\underline{E}_k$.

Theorem 8.8. *Assume that $E_*(\underline{E}_k)$ is a free E^*-module for all k* (which is true for $E = H(\mathbb{F}_p)$, BP, MU, KU, or $K(n)$ by Lemma 4.17(a)). *Then:*

(a) *We can make the functor U, defined in eq.* (8.5)*, a comonad in the category* FAlg *of filtered E^*-algebras;*

(b) *If $QE_*(\underline{E}_k)$ is a free E^*-module for all k, $\tau\colon U \to A$* (see (8.7)) *is a morphism of comonads in* FAlg*;*

(c) *If $E_*(E,o)$ is a free E^*-module, $\sigma \circ \tau\colon U \to S$* (see (8.7) and (5.7)) *is a morphism of comonads in* FAlg*.*

Our main definition is now clear.

Definition 8.9. An *unstable (E-cohomology) algebra* is just a U-coalgebra in FAlg, i. e. a complete Hausdorff filtered E^*-algebra M equipped with a continuous morphism $\rho_M\colon M \to UM$ of E^*-algebras that satisfies the coaction axioms [8, (8.7)]. We then *define* the action of $r \in E^*(\underline{E}_k)$ on $x \in M^k$ by $r(x) = \rho_M(x)r \in M$.

A closed ideal $J \subset M$ is called *(unstably) invariant* if the quotient algebra M/J inherits a well-defined unstable algebra structure from M.

It follows that the Cartan formulae (8.3) and (8.4) hold in M. The constant operations behave correctly because $\rho_M(x)$ is required to be a morphism of E^*-algebras. We need to be able to recognize invariant ideals.

Lemma 8.10. *Given an unstable algebra M, a closed ideal $J \subset M$ is unstably invariant if and only if $r(y) \in J$ for all $y \in J$ and all based operations r.*

Proof. To make $\rho_{M/J}$ well defined, we need $r(x+y) \equiv r(x) \bmod J$, for all $x \in M$ and $y \in J$. This is trivial for constant operations r, and so by Lemma 8.1, we need only check for based r. The stated condition is obviously necessary, by taking $x = 0$. It is also sufficient, by eq. (8.3). □

Theorem 8.11. *Assume that $E_*(\underline{E}_k)$ is a free E^*-module for all k* (which is true for $E = H(\mathbb{F}_p)$, BP, MU, KU, or $K(n)$ by Lemma 4.17(a)). *Then:*

(a) *For any space X, the coaction* (8.6) *factors through $E^*(X)\hat{}$ to make $E^*(X)\hat{}$ an unstable E-cohomology algebra;*

(b) *We recover the additively unstable coaction* (5.6) *from ρ_X as*

$$E^*(X) \xrightarrow{\rho_X} U(E^*(X)\hat{}) \xrightarrow{\tau} A(E^*(X)\hat{});$$

(c) *If $E^*(E,o)$ is Hausdorff, we recover the stable coaction* [8, (10.10)] *from ρ_X as*

$$E^*(X) \xrightarrow{\rho_X} U(E^*(X)\hat{}) \xrightarrow{\tau} A(E^*(X)\hat{}) \xrightarrow{\sigma} S(E^*(X)\hat{});$$

(d) *ρ is universal: given an object B of FAlg and an integer k, any natural transformation of sets $\theta X\colon E^k(X) \to \mathsf{FAlg}(B, E^*(X)\hat{})$ (or $\widehat{\theta X}\colon E^k(X)\hat{} \to \mathsf{FAlg}(B, E^*(X)\hat{})$, that is defined for all spaces X, is induced from ρ_X by a unique morphism $f\colon B \to E^*(\underline{E}_k)$ in FAlg as*

$$\theta X\colon E^k(X) \xrightarrow{\rho_X} U(E^*(X)\hat{}) = \mathsf{FAlg}(E^*(\underline{E}_k), E^*(X)\hat{}) \xrightarrow{\mathrm{Mor}(f,1)} \mathsf{FAlg}(B, E^*(X)\hat{})$$

Proof of Thms. 8.8 and 8.11. The proof breaks up into the same seven steps as additively (and stably), in Thms. 5.8 and 5.12. However, it is far simpler than Thms. 7.9 and 7.12 on algebras, because we are able to treat the multiplicative and module structures together. At each step, we also discuss τ and $\tau \circ \sigma$, assuming the extra conditions hold.

Corollary 7.8 of [8] provides the E^*-algebra object $n \mapsto \underline{E}_n$ in Ho. We again write ρ_k for ρ_X when $X = \underline{E}_k$.

Step 1. We endow the functor U with an E^*-algebra structure. For each object M of FAlg, we observe that according to [8, Lemma 6.9], the functor

$$\mathsf{FAlg}(E^*(-)\hat{}, M)\colon \mathsf{Ho} \xrightarrow{E^*(-)\hat{}} \mathsf{FAlg}^{\mathrm{op}} \xrightarrow{\mathrm{Mor}(-,M)} \mathsf{Set}$$

preserves enough products that by [8, Lemmas 7.6(a), 7.7(a)] it takes the E^*-algebra object $n \mapsto \underline{E}_n$ to the E^*-algebra object UM in Set; i. e. UM is an E^*-algebra. It is clear that UM is functorial in M. We shall filter it in Step 3.

To see that τM is a homomorphism of E^*-modules, we apply [8, Lemma 7.6(c)] to the E^*-module object $n \mapsto \underline{E}_n$ in $\mathsf{Gp}(\mathsf{Ho})$, using the natural transformation

$$\mathsf{FAlg}(E^*(-)\hat{}, M) \longrightarrow \mathsf{FMod}(PE^*(-)\hat{}, M)$$

defined by restriction. To see that τ is monoidal, we apply [8, Lemma 7.9(b)]. The monoidal structure of U is simply the multiplicative part of the algebra structure, and diag. (7.3) shows that the inclusions $PE^*(\underline{E}_n) \subset E^*(\underline{E}_n)$ form a morphism of graded monoid objects in $\mathsf{FMod}^{\mathrm{op}}$. The units are correct by definition. For $\tau \circ \sigma$, we bypass $PE^*(\underline{E}_n)$ and use the duals of diags. (6.16) and (6.17) instead.

Step 2. In order to define ρ_X (in (8.6)) as a morphism of E^*-algebras, we consider the Set-valued natural transformation

$$\mathsf{Ho}(X, -) \longrightarrow \mathsf{FAlg}(E^*(-)\hat{}, E^*(X)\hat{})$$

induced by $E^*(-)\hat{}\colon \mathsf{Ho}^{\mathrm{op}} \to \mathsf{FAlg}$. We apply [8, Lemma 7.6(c)] to the E^*-algebra object $n \mapsto \underline{E}_n$, to obtain Thm. 8.11(a). Then Thm. 8.11(b) is clear by comparing with the additive coaction (5.6), and for Thm. 8.11(c), we combine with Thm. 5.12(b).

Step 3. For U to take values in FAlg, we must filter UM. If M is filtered by the ideals F^aM, we filter UM by the ideals

$$F^a(UM) = \mathrm{Ker}\left[UM \longrightarrow U\left(\frac{M}{F^aM}\right)\right]$$

Just as stably, this filtration is complete Hausdorff and makes ρ_X continuous by naturality. This allows us to factor ρ_X through $E^*(X)\hat{}$. Similarly, τM and $\sigma M \circ \tau M$ are also filtered and therefore continuous.

Step 4. We convert the object $E^*(X)\hat{}$ of FAlg to the corepresented functor $F_X = \mathsf{FAlg}(E^*(X)\hat{}, -)\colon \mathsf{FAlg} \to \mathsf{Set}$. For example, when $X = \underline{E}_k$, $F_X = U^k$. As suggested by [8, (8.16)], we also convert the coaction ρ_X to the natural transformation $\rho_X\colon F_X \to F_X U\colon \mathsf{FAlg} \to \mathsf{Set}$. Given M in FAlg, $\rho_X M\colon F_X M \to F_X UM$ is thus defined on $f \in F_X M = \mathsf{FAlg}(E^*(X)\hat{}, M)$ as

$$(\rho_X M)f = Uf \circ \rho_X\colon E^*(X)\hat{} \longrightarrow U(E^*(X)\hat{}) \longrightarrow UM, \tag{8.12}$$

an element of $F_X UM$.

Step 5. We define the natural transformation

$$\psi M\colon U^k M = \mathsf{FAlg}(E^*(\underline{E}_k), M) \longrightarrow \mathsf{FAlg}(E^*(\underline{E}_k), UM) = U^k UM \tag{8.13}$$

by taking $X = \underline{E}_k$ in eq. (8.12). On the element $f\colon E^*(\underline{E}_k) \to M$ of $U^k M$, it is

$$(\psi M)f\colon E^*(\underline{E}_k) \xrightarrow{\rho_k} UE^*(\underline{E}_k) \xrightarrow{Uf} UM\ .$$

(In terms of elements, this is $r \mapsto [s \mapsto f(r^*s) = f(sr)]$.) If we substitute the E^*-algebra object $n \mapsto \underline{E}_n$ for X in eq. (8.12), [8, Lemma 7.6(c)] shows that ψM takes values in Alg. Naturality in M shows that ψM is filtered and so takes values in FAlg as required.

Step 6. The other required natural transformation,

$$\epsilon M\colon U^k M = \mathsf{FAlg}(E^*(\underline{E}_k), M) \longrightarrow M,$$

is defined simply as evaluation on $\iota_k \in E^*(\underline{E}_k)$. As before, naturality in M shows that ϵM is filtered, but we have to calculate that ϵ is an E^*-algebra homomorphism.

Take any binary operation $s(-,-)$ in E^*-algebras (addition, multiplication, or any other), represented in Ho by the map $s\colon \underline{E}_k \times \underline{E}_m \to \underline{E}_q$, which therefore induces $s^*\iota_q = s(p_1^*\iota_k, p_2^*\iota_m)$. We need to show that the square

$$\begin{array}{ccc} U^k M \times U^m M & \xrightarrow{s} & U^q M \\ \downarrow{\scriptstyle \epsilon\times\epsilon} & & \downarrow{\scriptstyle \epsilon} \\ M^k \times M^m & \xrightarrow{s} & M^q \end{array}$$

commutes. We evaluate on $f \in U^k M$ and $g \in U^m M$. Because $E^*(\underline{E}_k \times \underline{E}_m)$ is by [8, Lemma 6.9] the coproduct in FAlg, there is a unique $h\colon E^*(\underline{E}_k \times \underline{E}_m) \to M$ in FAlg such that $h \circ p_1^* = f$ and $h \circ p_2^* = g$. Then by definition of the algebra structure of UM, $s(f,g) = h \circ s^*\colon E^*(\underline{E}_q) \to M$. Since h is an algebra homomorphism,

$$\epsilon s(f,g) = hs^*\iota_n = hs(p_1^*\iota_k, p_2^*\iota_m) = s(hp_1^*\iota_k, hp_2^*\iota_m) = s(f\iota_k, g\iota_m) = s(\epsilon f, \epsilon g).$$

For unary and 0-ary operations, we may adapt the above proof, or simply throw away any unwanted arguments. (For example, given $v \in E^*$, we could define the constant binary operation $s(x,y) = v1$ in any E^*-algebra, to deduce that $\epsilon v = v$.)

Step 7. The proof that $E^*(X)^\wedge$ is a U-coalgebra and that U is a comonad is formally identical to the stable case, except that we need versions of [8, Lemmas 8.20, 8.22] for graded objects.

We use [8, Lemma 8.24] to show that $\tau\colon U \to A$ is a natural transformation of comonads. We take R as $n \mapsto E^*(\underline{E}_n)$, R' as $n \mapsto PE^*(\underline{E}_n)$, $1_R = 1'_R$ as $n \mapsto \iota_n$, and $u\colon PE^*(\underline{E}_k) \subset E^*(\underline{E}_k)$ as the inclusion. The first hypothesis on u is the commutativity of the diagram

$$\begin{array}{ccc} PE^*(\underline{E}_k) & \xrightarrow{\quad\subset\quad} & E^*(\underline{E}_k) \\ \Big\downarrow{\scriptstyle P\rho_k} & & \Big\downarrow{\scriptstyle \rho_k} \\ & & \mathit{FAlg}(E^*(\underline{E}_k), E^*(\underline{E}_k)) \\ & & \Big\downarrow \\ \mathit{FMod}(PE^*(\underline{E}_k), PE^*(\underline{E}_k)) & \xrightarrow{\subset} & \mathit{FMod}(PE^*(\underline{E}_k), E^*(\underline{E}_k)) \end{array}$$

which is obvious by construction, as $r \in PE^*(\underline{E}_k)$ yields $r^*|PE^*(\underline{E}_k)$.

The proof of Thm. 8.11(d) is formally the same as stably. Since $E^k(-)$ is represented by $\iota_k \in E^k(\underline{E}_k)$, θ is classified by $f = (\theta\underline{E}_k)\iota_k \in \mathit{FAlg}(B, E^*(\underline{E}_k))$. □

9. Unstable, additive, and stable objects

In previous sections and [8], we constructed five different kinds of object: stable modules and algebras, additively unstable modules and algebras, and unstable algebras. In this section we compare them all. Unstable modules are conspicuous by their absence; Thm. 9.4 will show that they cannot be defined compatibly with our other objects.

Each kind of object is defined by a comonad. Theorems 8.8(b) and 7.9(b) provide natural transformations

$$U \xrightarrow{\tau} A \xrightarrow{\sigma} S \qquad \text{in } \mathit{FAlg} \tag{9.1}$$

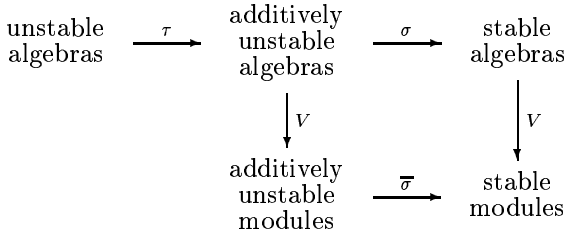
between the comonads that define unstable, additively unstable, and stable algebras. Theorem 5.8(b) provides the natural transformation

$$\overline{A} \xrightarrow{\overline{\sigma}} \overline{S} \qquad \text{in } \mathit{FMod} \tag{9.2}$$

between the comonads that define additively unstable and stable modules (where we temporarily rename the module versions of A and S to $\overline{A}$ and $\overline{S}$). They are related to the algebra versions by the forgetful functor $V\colon \mathit{FAlg} \to \mathit{FMod}$, so that $VA = \overline{A}V$ and $VS = \overline{S}V$.

We have the category, e. g. U-coalgebras, of each kind of object. We consider the diagram of categories and functors in fig. 5. For example, a stable algebra B with coaction $\rho_B\colon B \to SB$ in FAlg yields the stable module VB with coaction $V\rho_B\colon VB \to VSB = \overline{S}VB$ in FMod.

Figure 5. Five kinds of object



Theorem 9.3. *Assume that $E_*(\underline{E}_k)$, $QE_*(\underline{E}_k)$, and $E_*(E,o)$ are free E^*-modules for all k* (which is true for $E = H(\mathbb{F}_p)$, BP, MU, KU, or $K(n)$ by Lemma 4.17(a) and [8, Lemma 9.21]). *Then we have the diagram fig. 5 of categories and functors.*

For any space X, $E^*(X)\hat{}$ is an object in each of the five categories, related by these functors.

Proof. The last assertion combines Thms. 5.12, 7.12, and 8.11 with Thms. 10.16 and 12.13 of [8]. □

There is a glaring gap: we have not defined unstable modules. We now show that this gap cannot be filled, for rather silly reasons. In fact, the three most natural definitions are for stable modules, additively unstable modules, and unstable algebras. We can enrich the two kinds of module with multiplicative structure, but it is not possible to remove the multiplicative structure from the definition of unstable algebra. This is already strongly suggested by the appearance of multiplication in the Cartan formula (8.3) for $r(x+y)$.

We ignore most of the structure and the topology, fix k, and restrict attention to the two functors $U^k\colon \mathsf{FAlg} \to \mathsf{Ab}$ and $A^k\colon \mathsf{FMod} \to \mathsf{Ab}$ and the natural transformation $\tau^k\colon U^kV \to A^k$.

Theorem 9.4. *Even in the classical case $E = H(\mathbb{F}_p)$, unstable modules do not exist in the sense that we cannot insert a suitable comonad $\overline{U}$ into diag.* (9.2). *Specifically, for fixed $k > 0$ there do not exist:*

(i) *a functor $\overline{U}^k\colon \mathsf{FMod} \to \mathsf{Ab}$;*

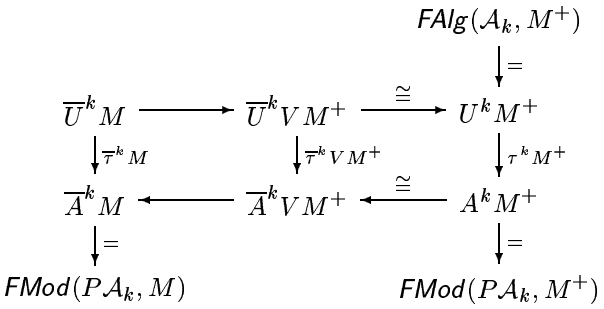
(ii) *a natural isomorphism $\overline{U}^kV \cong U^k\colon \mathsf{FAlg} \to \mathsf{Ab}$;*

(iii) *a natural transformation $\overline{\tau}^k\colon \overline{U}^k \to A^k$ of functors $\mathsf{FMod} \to \mathsf{Ab}$;*

such that on FAlg, $\overline{\tau}^kV\colon \overline{U}^kV \to A^k$ agrees with $\tau^k\colon U^k \to A^k$.

Proof. We assume that $\overline{U}^k$ and $\overline{\tau}^k$ exist as stated and derive a contradiction. Given any (filtered) graded $\mathbb{F}_p$-module M, we construct the $\mathbb{F}_p$-algebra $M^+ = \mathbb{F}_p \oplus M$

with the unit element $1 \in \mathbb{F}_p$ and $xy = 0$ for all $x, y \in M$. Then M is a retract in *FMod* of VM^+ and we can compute $\overline{\tau}^k M$ from the commutative diagram



where $\mathcal{A}_k = H^*(\underline{H}_k)$. Because M^+ has no decomposables, every homomorphism $P\mathcal{A}_k \to M$ in the image of $\overline{\tau}^k M$ kills the decomposable elements of $P\mathcal{A}_k$ (of which there are many).

But for a general algebra B, $\tau^k B \colon U^k B \to A^k B$ does not have this property, e. g. $(\tau^k \mathcal{A}_k)\mathrm{id}_k \in A^k \mathcal{A}_k$ is the inclusion $P\mathcal{A}_k \subset \mathcal{A}_k$. Taking $M = VB$ shows that $\overline{\tau}^k VB$ does not agree with $\tau^k B$. □

Objects in ordinary cohomology. Theorem 9.4 demands an immediate explanation of our terminology even in the case of ordinary cohomology. We give details for $E = H(\mathbb{F}_2)$; the case $E = H(\mathbb{F}_p)$ for odd p is similar, with the usual changes.

The *Steenrod algebra* $\mathcal{A} = E^*(E, o)$ is exactly as expected: it is the $\mathbb{F}_2$-algebra generated by the Steenrod squares Sq^i for $i > 0$, subject to the standard Adem relations. It is useful to write $\mathrm{Sq}^0 = \iota$. We note that for $E = H(\mathbb{F}_2)$:

(i) σ_k^* makes $PE^*(\underline{E}_k)$ a *quotient* of $E^*(E, o)$;

(ii) $E^*(\underline{E}_k)$ is a *primitively generated* Hopf algebra.

Below, M is to be an object of *FMod* (or *FAlg*), i. e. a complete Hausdorff filtered graded $\mathbb{F}_2$-module (or commutative $\mathbb{F}_2$-algebra). Topological conditions apply (which we ignore for now). We list the five kinds of object we have defined, under our names for them:

(i) A *stable module* M is an $\mathcal{A}$-module.

(ii) A *stable algebra* M is both an $\mathbb{F}_2$-algebra and an $\mathcal{A}$-module that satisfies the Cartan formula

$$\mathrm{Sq}^k(xy) = \sum_{i=0}^{k} (\mathrm{Sq}^i x)(\mathrm{Sq}^{k-i} y) \qquad \text{for } k > 0.$$

It follows by induction that $\mathrm{Sq}^k 1_M = 0$ for all $k > 0$.

(iii) An *additively unstable module* M is an $\mathcal{A}$-module that satisfies the extra condition

$$\mathrm{Sq}^i x = 0 \qquad \text{for all } x \in M \text{ and all } i > \deg(x). \tag{9.5}$$

Since $\mathrm{Sq}^0 x = x$, it follows that $M^n = 0$ for all $n < 0$.

(iv) An *additively unstable algebra* is a stable algebra that satisfies (9.5).

(v) An *unstable algebra* M is a stable algebra that satisfies (9.5) as well as the extra condition

$$\mathrm{Sq}^k x = x^2 \qquad \text{for } x \in M \text{ and } k = \deg(x).$$

The objects normally known as unstable modules appear here as *additively* unstable modules (although the word "additively" could well be omitted, there being no danger of confusion with something that does not exist).

However, we do have two kinds of unstable algebra. We emphasize that in (iv), the squaring operation $M^k \to M^{2k}$ given by $x \mapsto x^2$ (which looks additive but from our point of view is not, because it is defined only when M is an algebra) is unrelated to Sq^k.

We have equivalent comodule descriptions in terms of $E_*(E,o) = \mathbb{F}_2[\xi_1, \xi_2, \xi_3, \ldots]$ and the corresponding bigraded algebra $Q(E)^*_* = \mathbb{F}_2[\xi_0, \xi_1, \xi_2, \ldots]$, which has polynomial generators $\xi_i \in Q(E)^1_{2^i}$ (as we shall see in Thm. 16.2):

(i) A *stable comodule* M has a coaction

$$\rho_M \colon M \longrightarrow M \mathbin{\widehat{\otimes}} E_*(E,o) = M \mathbin{\widehat{\otimes}} \mathbb{F}_2[\xi_1, \xi_2, \xi_3, \ldots]$$

that satisfies the usual axioms [8, (8.7)]. Then Sq^k is dual to ξ_1^k.

(ii) A *stable comodule algebra* M is both a stable comodule and a commutative $\mathbb{F}_2$-algebra, in such a way that ρ_M is an algebra homomorphism.

(iii) An *unstable comodule* M has coactions

$$\rho_M \colon M^k \longrightarrow M^i \mathbin{\widehat{\otimes}_i} Q(E)^k_i \subset M \mathbin{\widehat{\otimes}} \mathbb{F}_2[\xi_0, \xi_1, \xi_2, \ldots]$$

that satisfy the coaction axioms (6.33). The unstable operation $\mathrm{Sq}^i \colon k \to k+i$ is now dual to $\xi_0^{k-i}\xi_1^i$ for $i \leqslant k$, or is zero if $i > k$.

(iv) An *unstable comodule algebra* M is an unstable comodule that is also a commutative $\mathbb{F}_2$-algebra, in such a way that ρ_M is an algebra homomorphism.

The special features of $H(\mathbb{F}_2)$ allow us to handle unstable algebras too:

(v) For any $x \in M^k$, $\rho_M x$ contains the term $x^2 \otimes \xi_1^k$.

Remark. There is one candidate for an unstable module, but it does not work. One could try defining $G^k M = \mathit{FMod}(E^*(\underline{E}_k), M)$ for any object M of FMod, with $\rho_X \colon E^k(X) \to G^k E^*(X)$ defined as usual, by $\rho_X x = x^*$. We would like ρ_X to be at least additive, but the standard additive structure on FMod does not give this.

Indeed, it is easy to see that in general *no* abelian group structure on $G^k M$ makes ρ_X additive (not even for $E = H(\mathbb{F}_p)$). By [8, Lemma 7.7(d)], such a structure would have to be induced by some morphism $\psi \colon E^*(\underline{E}_k) \to E^*(\underline{E}_k) \oplus E^*(\underline{E}_k)$ in FMod. Take any $r \in E^*(\underline{E}_k)$ and write $\psi r = (r', r'')$. Then additivity of ρ_X translates into $r(x+y) = r'x + r''y$ for all $x, y \in E^k(X)$, which is absurd unless r happens to be additive.

In fact, these objects appear to be particularly devoid of interest. In the case $E = H(\mathbf{F}_2)$, for example, they are modules equipped not only with Steenrod squares Sq^i that behave as expected, but also operations such as $x \mapsto (\mathrm{Sq}^2 x)(\mathrm{Sq}^3 x)$, without having cup products.

10. Enriched Hopf rings

In Defn. 8.9 we condensed all the structure of an unstable algebra down to the single word *U-coalgebra*. In this section, we unpack the information again to give a complete description of an unstable algebra in the language of Hopf rings, enriched with certain additional structure. This description is summarized in Thm. 10.47, which may be regarded as the unstable analogue of Thm. 11.14 of [8] and Thm. 6.36. Indeed, we find a whole new paradigm for handling unstable operations, making computations with them reasonably practical and efficient. It serves as the true successor to the Second Answer of section 5 and [8, §10].

We assume in this section that $E_*(\underline{E}_k)$ is a free E^*-module for all k, which is true for our five examples by Lemma 4.17(a). Thus all the results of section 8 are available, and by [8, Lemma 6.16(c)], the topological dual $\mathsf{FMod}^*(E^*(\underline{E}_k), E^*)$ of $E^*(\underline{E}_k)$ is $E_*(\underline{E}_k)$.

We shall consistently identify (with some abuse of notation):

(i) the *cohomology operation* $r\colon E^k(-) \to E^m(-)$;
(ii) the *cohomology class* $r(\iota_k) \in E^m(\underline{E}_k)$, which we often write simply as $r \in E^m(\underline{E}_k)$;
(iii) the representing *map of spaces* $r\colon \underline{E}_k \to \underline{E}_m$;
(iv) the *E^*-linear functional* $\langle r, - \rangle\colon E_*(\underline{E}_k) \to E^*$ of degree m.

(10.1)

Remark. In some situations, these identifications can obscure the correct signs in formulae. Considered as a cohomology class or functional, r has degree m, while its degree as an operation is $m-k$, and as a map of spaces, r has no degree at all.

In any unstable algebra M, including $E^*(X)\hat{}$ for any space X, Defn. 8.9 gives, for each $x \in M^k$, the homomorphism $\rho_M(x)\colon E^*(\underline{E}_k) \to M$. Then we defined $r(x) = \rho_M(x)r \in M$ for any operation (i.e. class) $r \in E^*(\underline{E}_k)$. In practice, we find it more convenient to revert to the First Answer $r(x)$ of section 8, although the Second Answer, in terms of ρ_M, will continue to inform us as to what to do, even when only implicit. Classically, one investigates cohomology operations by studying what happens to $r(x)$ when r is fixed and x varies; but it is clear from section 8 that what we should do is fix x and allow r to vary.

Linear functionals. We need to develop a *computational* description of ρ_M in an unstable algebra M. We start from the fact that $\rho_M(x)$ is E^*-linear, i.e. $r(x)$ is E^*-linear in r.

Definition 10.2. Let M be an unstable algebra, and fix an element $x \in M^k$. We say $r(x)$ is written *in standard form* if

$$r(x) = \sum_\alpha \langle r, c_\alpha \rangle\, x_\alpha \qquad \text{in } M \qquad (\text{for all } r), \tag{10.3}$$

for suitable choices $c_\alpha \in E_*(\underline{E}_k)$ and $x_\alpha \in M$, where $\deg(x_\alpha) = -\deg(c_\alpha)$. If the sum is infinite, we require each ideal $F^a M$ in the filtration of M to contain all except finitely many of the x_α.

This is the closest we will come to an unstable replacement for the tensor products and homomorphisms of section 6 and [8, §11]. Our convention here and in all similar formulae is that r runs through all unstable cohomology operations having the correct domain degree (different in nearly every formula, and rarely specified) but arbitrary target degree. The indexing set for α is often left implicit.

It is easy to achieve eq. (10.3) in the universal form

$$r(x) = \sum_\alpha \langle r, c_\alpha \rangle\, r_\alpha(x) \qquad \text{in } M \qquad (\text{for all } r), \tag{10.4}$$

by allowing c_α to run through some basis of $E_*(\underline{E}_k)$, which forces us to take $x_\alpha = r_\alpha(x)$, where r_α denotes the operation (linear functional) dual to c_α. Continuity of $\rho_M(x)\colon E^*(\underline{E}_k) \to M$ assures the finiteness condition in Defn. 10.2. We may therefore always assume that $r(x)$ is written in standard form.

Where we depart from tradition is in *not* picking a definite basis of $E_*(\underline{E}_k)$ in advance. We do not even insist on the c_α being linearly independent. Nor do we require the c_α to span; we may obviously omit zero terms. This does not affect the linearity of eq. (10.3) and allows the flexibility that our formulae require. One consequence is that *most cohomology operations will never acquire names.*

We have the analogue of Prop. 6.44.

Proposition 10.5. *Given $x \in E^k(X)$, regarded as a map of spaces $x\colon X \to \underline{E}_k$, assume that $r(x)$ is given by eq.* (10.3). *Then $x_*\colon E_*(X) \to E_*(\underline{E}_k)$ is given by*

$$x_* z = \sum_\alpha (-1)^{\deg(c_\alpha)(\deg(x_\alpha)+\deg(z))} \langle x_\alpha, z \rangle\, c_\alpha = \sum_\alpha c_\alpha \langle x_\alpha, z \rangle\ . \qquad \square$$

The nonuniqueness in eq. (10.3) is really not a problem because we are using it to *describe,* not *define* the structure on M. The real definitions are all in section 8; here, we are only reinterpreting them. Nevertheless, it is easy to convert one standard form to another.

Lemma 10.6. *Any standard form* (10.3) *can be transformed into the universal form* (10.4), *and hence into any other standard form, by iterating three kinds of replacement (in either direction):*

$$\text{(i)} \qquad \langle r, c + c' \rangle\, x' = \langle r, c \rangle\, x' + \langle r, c' \rangle\, x';$$

$$\text{(ii)} \qquad \langle r, vc \rangle\, x' = (-1)^{\deg(c)\deg(v)} \langle r, c \rangle\, vx';$$

$$\text{(iii)} \qquad \langle r, c \rangle\, x' + \langle r, c \rangle\, x'' = \langle r, c \rangle\, (x' + x'').$$

(Infinitely many replacements may be needed; however, each $F^a M$ contains x' for all except finitely many of them.) □

Stabilization. We need to record how eq. (10.3) behaves when we restrict the operation r to be additive or stable. We recall from [8, Defn. 9.3] the stabilization homomorphism $\sigma_{k*}\colon E_*(\underline{E}_k) \to E_*(E,o)$ and from eq. (6.2) the algebraic homomorphism $q_k\colon E_*(\underline{E}_k) \to Q(E)^k_*$, both of which have degree k under our conventions.

Lemma 10.7. *Let M be an unstable algebra, and assume that $r(x)$ is expressed in the standard form* (10.3)*, where $x \in M^k$. Then:*

(a) *The unstable comodule coaction $\rho_M\colon M^k \to M \mathbin{\widehat{\otimes}} Q(E)^k_*$ is given by*

$$\rho_M x = \sum_\alpha (-1)^{\deg(x_\alpha)(k-\deg(x_\alpha))} x_\alpha \otimes q_k c_\alpha \qquad \text{in } M \mathbin{\widehat{\otimes}} Q(E)^k_*,$$

provided $QE_*(\underline{E}_k)$ is a free E^*-module;

(b) *The stable comodule coaction $\rho_M\colon M \to M \mathbin{\widehat{\otimes}} E_*(E,o)$ is given by*

$$\rho_M x = \sum_\alpha (-1)^{\deg(x_\alpha)(k-\deg(x_\alpha))} x_\alpha \otimes \sigma_{k*} c_\alpha \qquad \text{in } M \mathbin{\widehat{\otimes}} E_*(E,o),$$

provided $E_*(E,o)$ is a free E^*-module.

The signs are as expected, once we remember that if $\deg(x_\alpha) = i$, then $\deg(c_\alpha) = -i$ and $\deg(q_k c_\alpha) = \deg(\sigma_{k*} c_\alpha) = k - i$.

Proof. For additive r, Prop. 6.11 converts eq. (10.3) to $r_A x = \sum_\alpha \langle r_Q, q_k c_\alpha \rangle x_\alpha$. We deduce $\rho_M x$ in (a) by comparing eqs. (6.38) and (6.39). Part (b) is similar, using [8, (11.18), (11.19)] instead. □

Unstable algebra structure. Our task is to convert all the algebraic structure of an unstable algebra M in Defn. 8.9 into the current context. There are in effect four pairs of axioms:

(a) Two axioms to make $\rho_M(x)\colon E^*(\underline{E}_k) \to M$ an E^*-algebra homomorphism, rather than merely E^*-linear: $(r \smile s)(x) = r(x)s(x)$ and $1(x) = 1_M$, which will become eqs. (10.14) and (10.15);

(b) Two axioms to make $\rho_M\colon M \to UM$ E^*-linear: $\rho_M(x+y) = \rho_M(x) + \rho_M(y)$ and $\rho_M(vx) = v\rho_M(x)$, which will become eqs. (10.20) and (10.16);

(c) Two axioms to make $\rho_M\colon M \to UM$ multiplicative: $\rho_M(1_M) = 1_{UM}$ and $\rho_M(xy) = \rho_M(x)\rho_M(y)$, which will become eqs. (10.41) and (10.34);

(d) Two axioms to make M a U-coalgebra: $(sr)(x) = s(r(x))$ and $\iota_k x = x$, which will become eqs. (10.45) and (10.43).

The natural language for expressing the first three pairs is that of Hopf rings, while the last requires some additional structure.

Hopf rings. We recall from [8, Lemma 6.12] that in *Coalg*, tensor products of coalgebras serve as products and E^* is the terminal object. A commutative (graded)

ring object in *Coalg* is called a *Hopf ring over* E^*. (The terminology and some of the notation were suggested by Milgram [17]; see [23, §1] for a detailed exposition.)

We start from the E^*-algebra object $n \mapsto \underline{E}_n$ in *Ho* provided by [8, Cor. 7.8]. We apply [8, Lemma 7.6(a)], using the homology functor $E_*(-)$, which takes values in *Coalg* on the spaces we need and preserves enough products to make $n \mapsto E_*(\underline{E}_n)$ an E^*-algebra object in *Coalg*. In particular, this is an E^*-module object, and each $E_*(\underline{E}_k)$ is an abelian group object in *Coalg* and thus a Hopf algebra.

There are seven parts to the Hopf ring structure of $n \mapsto E_*(\underline{E}_n)$: two from the coalgebra, three from the abelian group object $\underline{E}_k$, and two from the multiplicative monoid object, in addition to the underlying E^*-module structure on E-homology. They are as follows (for each k and m, where relevant):

(i) $\psi\colon E_*(\underline{E}_k) \to E_*(\underline{E}_k) \otimes E_*(\underline{E}_k)$, the *comultiplication* induced by the diagonal map $\Delta\colon \underline{E}_k \to \underline{E}_k \times \underline{E}_k$;

(ii) $\epsilon\colon E_*(\underline{E}_k) \to E^*$, the *counit* for ψ, induced by the map $q\colon \underline{E}_k \to T$;

(iii) $*\colon E_*(\underline{E}_k) \otimes E_*(\underline{E}_k) \to E_*(\underline{E}_k)$, a *multiplication*, induced by the addition map $\mu_k\colon \underline{E}_k \times \underline{E}_k \to \underline{E}_k$;

(iv) $1_k = \omega_{k*}1 \in E_0(\underline{E}_k)$, the $*$-*unit element*, induced by the zero map $\omega_k\colon T \to \underline{E}_k$;

(v) $\chi\colon E_*(\underline{E}_k) \to E_*(\underline{E}_k)$, the *canonical (anti)automorphism* of the Hopf algebra $E_*(\underline{E}_k)$, induced by the inversion map $\nu_k\colon \underline{E}_k \to \underline{E}_k$;

(vi) $\circ\colon E_*(\underline{E}_k) \otimes E_*(\underline{E}_m) \to E_*(\underline{E}_{k+m})$, another *multiplication*, induced by the multiplication map $\phi\colon \underline{E}_k \times \underline{E}_m \to \underline{E}_{k+m}$;

(vii) $[1] = \eta_*1 \in E_0(\underline{E}_0)$, the $\circ$-*unit element*, induced by the algebra unit map $\eta\colon T \to \underline{E}_0$.

Because $n \mapsto E_*(\underline{E}_n)$ is an E^*-algebra object rather than merely a ring object, we have, for each $v \in E^h$, the actions $(\xi v)_*\colon E_*(\underline{E}_k) \to E_*(\underline{E}_{k+h})$. As in section 6, this reduces to a simpler structure.

Definition 10.8. We define the *right unit* function $\eta_R\colon E^* \to E_*(\underline{E}_*)$. We regard $v \in E^h = E^h(T)$ as a map $v\colon T \to \underline{E}_h$, and use the induced homomorphism $v_*\colon E^* \cong E_*(T) \to E_*(\underline{E}_h)$ to define $[v] = v_*1 \in E_0(\underline{E}_h)$ and $\eta_R(v) = [v]$.

In particular, this includes $[1] = \eta_*1$ as in (vii), and $[0_k] = \omega_{k*}1 = 1_k$ as in (iv). It is clear from Defn. 6.19 and [8, Defn. 11.2] that $q_h[v]$ and $\sigma_{h*}[v]$ are the additive and stable versions of $\eta_R v$. The elements $[v]$ determine the E^*-module object structure completely, because when we apply E-homology to [8, (7.5)], we obtain

$$(\xi v)_* c = [v] \circ c \qquad \text{for all } c \in E_*(\underline{E}_*). \tag{10.9}$$

For the sake of completeness, we list all 33 laws that a Hopf ring satisfies, beyond the usual axioms for an E^*-module. (Your count may vary.) Most need no comment. They are as follows, where in several we write $\psi c = \sum_i c'_i \otimes c''_i$:

(i) The five operations are (bi)additive: $\psi(b+c) = \psi b + \psi c$, $\epsilon(b+c) = \epsilon b + \epsilon c$, $(a+b) * c = a*c + b*c$, $\chi(b+c) = \chi b + \chi c$, and $(a+b) \circ c = a \circ c + b \circ c$;

(ii) The five operations are E^*-linear: $\psi(vc) = \sum_i vc_i' \otimes c_i''$, $\epsilon(vc) = v\epsilon c$, $(vb) * c = v(b * c)$, $\chi(vc) = v\chi c$, and $(vb) \circ c = v(b \circ c)$, for all $v \in E^*$;

(iii) Three coalgebra axioms: ψ is coassociative and cocommutative (with the standard sign), and ϵ is a counit: $\sum_i (\epsilon c_i')c_i'' = c$;

(iv) The five parts of the ring object structure respect ψ: $\psi(b*c) = (\psi b) * (\psi c)$ (where we give $E_*(\underline{E}_k) \otimes E_*(\underline{E}_k)$ the obvious $*$-multiplication, with signs), $\psi(b \circ c) = (\psi b) \circ (\psi c)$ (similarly), $\psi 1_k = 1_k \otimes 1_k$, $\psi\chi c = \sum_i \chi c_i' \otimes \chi c_i''$, and $\psi[1] = [1] \otimes [1]$;

(v) The five parts of the ring object structure respect ϵ: $\epsilon(b * c) = (\epsilon b)(\epsilon c)$, $\epsilon 1_k = 1$, $\epsilon\chi c = \epsilon c$, $\epsilon(b \circ c) = (\epsilon b)(\epsilon c)$, and $\epsilon[1] = 1$;

(vi) Four abelian group object axioms: associativity $(a * b) * c = a * (b * c)$, commutativity $b * c = (-1)^{ij} c * b$ (where $i = \deg(b)$, $j = \deg(c)$), unit $1_k * c = c$, and inverse $\sum_i \chi c_i' * c_i'' = (\epsilon c) 1_k$;

(vii) Three axioms for a commutative monoid: associativity $(a \circ b) \circ c = a \circ (b \circ c)$, commutativity, which takes the somewhat complicated form (see [23, Lemma 1.12(c)(v)])

$$b \circ c = (-1)^{ij} \chi^{km} c \circ b \tag{10.10}$$

for $b \in E_i(\underline{E}_k)$ and $c \in E_j(\underline{E}_m)$ (where $\chi^{km} = \chi$ if k and m are odd, and is the identity otherwise, as in Prop. 10.12(b) below), and $[1] \circ c = c$;

(viii) Three ring object axioms to state that $- \circ c$ respects the abelian group object structure: for addition, which yields the distributive law, in the complicated form [ibid. (vi)]

$$(a * b) \circ c = \sum_i (-1)^{\deg(c_i') \deg(b)} a \circ c_i' * b \circ c_i''; \tag{10.11}$$

for the zero, $1_m \circ c = (\epsilon c) 1_{m+k}$ [ibid. (ii)]; and for the inverse, $\chi(b \circ c) = (\chi b) \circ c$.

Many standard laws follow from these axioms. In order to simplify notation in eq. (10.11) and elsewhere, we give $\circ$-multiplication greater binding strength than $*$-multiplication, so that $a * b \circ c$ always means $a * (b \circ c)$, never $(a * b) \circ c$. In all our Hopf rings, Prop. 11.2 will provide the laws relating the added elements $[v]$ and identify the useful element $\chi[1]$ with $[-1]$.

Proposition 10.12. *In any Hopf ring, the operation χ has the following properties:*

(a) $\chi c = \chi[1] \circ c$, *so that* $\chi[1]$ *determines* χ;

(b) $\chi\chi c = c$;

(c) $\chi(a * b) = \chi a * \chi b$;

(d) $\chi[1] \circ \chi[1] = [1]$.

Proof. For (a), $\chi c = \chi([1] \circ c) = \chi[1] \circ c$. Since $\psi[1] = [1] \otimes [1]$ and hence $\psi\chi[1] = \chi[1] \otimes \chi[1]$, the distributive law gives (c), by

$$\chi(a * b) = \chi[1] \circ (a * b) = \chi[1] \circ a * \chi[1] \circ b = \chi a * \chi b \ .$$

Also, we have $\chi[1] * [1] = 1_0$ and similarly $\chi\chi[1] * \chi[1] = 1_0$, which yield

$$\chi\chi[1] = \chi\chi[1] * 1_0 = \chi\chi[1] * \chi[1] * [1] = 1_0 * [1] = [1].$$

But (a) gives $\chi\chi[1] = \chi[1] \circ \chi[1]$, and hence (d) and the general case of (b). □

Generators. We wish to use the laws to reduce any element of a Hopf ring to some standard form. The distributive law (10.11) plays a key role. We shall describe our Hopf rings H by specifying two sets of elements:

(i) the $\circ$-*generators* of H;

(ii) the $*$-*generators* of H, each of which is a $\circ$-product of $\circ$-generators and possibly $\chi[1]$, where we allow the empty $\circ$-product $[1]$.

We require every element of H to be an E^*-linear combination of $*$-products of the $*$-generators of H; in other words, the $*$-generators generate H as an E^*-algebra. For each $\circ$-generator g, we need formulae for ψg (so we can expand eq. (10.11)), ϵg, and χg. Although Hopf rings tend to be huge, each of our examples (see section 17) has a conveniently small set of $\circ$-generators.

Hopf rings over $\mathbb{F}_p$. One can define the *Frobenius* operator $Fc = c^{*p}$ in any algebra with multiplication $*$, and it is multiplicative if $*$ is commutative. It is additive if also the ground ring has characteristic p. It is most useful when the ground ring is $\mathbb{F}_p$, because it is then automatically $\mathbb{F}_p$-linear. Commutativity of $*$-multiplication implies that $Fc = 0$ whenever c has odd degree (unless $p = 2$).

Moreover, in a Hopf ring (or cocommutative Hopf algebra) H over $\mathbb{F}_p$, one has dually the *Verschiebung* operator $V\colon H \to H$, defined so that $DV = F\colon DH \to DH$ in the dual Hopf algebra. It divides degrees by p. Then $Vc = 0$ unless $\deg(c)$ is divisible by $2p$ (if $p \neq 2$). Both F and V preserve all the Hopf algebra structure: $F(a * c) = Fa * Fc$, $F1_k = 1_k$, $\psi Fc = (F \otimes F)\psi c$, $\epsilon Fc = \epsilon c$, and dually $V(a * c) = Va * Vc$, $V1_k = 1_k$, $\psi Vc = (V \otimes V)\psi c$, and $\epsilon Vc = \epsilon c$. For $\circ$-products, we can iterate eq. (10.11) and obtain the identity

$$a \circ (Fc) = F(Va \circ c), \tag{10.13}$$

which is useful for reducing elements of the Hopf ring to standard form. (Normally, a and c both have even degree.)

Multiplication of operations. The first pair of axioms on M we listed earlier, that for fixed $x \in M$, $\rho_M(x)$ is a homomorphism of E^*-algebras, is easily translated into Hopf rings. Because the diagonal map in $\underline{E}_k$ induces both the cup product $r \smile s$ and the comultiplication ψ on $E_*(\underline{E}_k)$, we can write down the cup product from eq. (10.3) as

$$(r \smile s)(x) = \sum_\gamma \langle r \smile s, c_\gamma \rangle\, x_\gamma = \sum_\gamma \langle r \otimes s, \psi c_\gamma \rangle\, x_\gamma \qquad \text{in } M.$$

The product $r(x)s(x)$ becomes, after some shuffling,

$$r(x)s(x) = \sum_\alpha \sum_\beta (-1)^{\deg(x_\alpha)\deg(x_\beta)} \langle r \otimes s, c_\alpha \otimes c_\beta \rangle\, x_\alpha x_\beta.$$

Since $(r \smile s)(x) = r(x)s(x)$ has to hold for all r and s, we deduce the identity

$$\sum_\gamma \psi c_\gamma \otimes x_\gamma = \sum_\alpha \sum_\beta (-1)^{\deg(x_\alpha)\deg(x_\beta)} c_\alpha \otimes c_\beta \otimes x_\alpha x_\beta \tag{10.14}$$

in $(E_*(\underline{E}_*) \otimes E_*(\underline{E}_*)) \mathbin{\widehat{\otimes}} M$, where the tensor products are formed using only the usual left E^*-actions.

The identity element $1_k \in E^0(\underline{E}_k)$ is the constant operation $E^k(X) \to E^0(X)$ that sends everything to 1_X; regarded as a linear functional, it is simply ϵ. In terms of eq. (10.3), the axiom $1_k(x) = 1_M$ becomes

$$\sum_\alpha (\epsilon c_\alpha)\, x_\alpha = 1_M \qquad \text{in } M. \tag{10.15}$$

Linear structure. We next decode the statement that $\rho_M \colon M \to UM$ is linear, namely that $\rho_M(x+y) = \rho_M(x) + \rho_M(y)$ and $\rho_M(vx) = v\rho_M(x)$. Related to the first is the formula for $r_*(b * c)$, which can be shown to be the translation of the statement that $\psi M \colon UM \to UUM$ is additive. We assume that $r(x)$ is given by eq. (10.3), where $x \in M^k$.

The v-action $U^k M \to U^{k+h} M$ was given by composing with $(\xi v)^* \colon E^*(\underline{E}_{k+h}) \to E^*(\underline{E}_k)$; dually, we use eq. (10.9) to translate $\rho_M(vx) = v\rho_M(x)$ into

$$r(vx) = \sum_\alpha \langle r, [v] \circ c_\alpha \rangle\, x_\alpha \qquad \text{in } M \qquad (\text{for all } r). \tag{10.16}$$

For addition, the idea is that $\mu_k \colon \underline{E}_k \times \underline{E}_k \to \underline{E}_k$ induces both the additive structure in UM and the $*$-multiplication in $E_*(\underline{E}_k)$. Of course, $r_* c$ is not additive in r. Given two operations $r, s \colon k \to m$, their sum may be constructed as

$$r + s \colon \underline{E}_k \xrightarrow{\Delta} \underline{E}_k \times \underline{E}_k \xrightarrow{r \times s} \underline{E}_m \times \underline{E}_m \xrightarrow{\mu_m} \underline{E}_m\ ,$$

as we can check by composing with $x \colon X \to \underline{E}_k$. When we apply E-homology, we find

$$(r+s)_* c = \sum_i r_* c_i' * s_* c_i'' \qquad \text{in } E_*(\underline{E}_m), \tag{10.17}$$

if we write $\psi c = \sum_i c_i' \otimes c_i''$ for $c \in E_*(\underline{E}_k)$. (In other words, we add r_* and s_* according to the group structure on $\mathit{Mod}(E_*(\underline{E}_k), E_*(\underline{E}_m))$ described by Milnor and Moore in [19, Defn. 8.1], which makes use of the coalgebra structure of $E_*(\underline{E}_k)$ and the algebra structure of $E_*(\underline{E}_m)$.) To add more than two operations, we need *iterated* coproducts: given any finite indexing set Λ, we write the iterated comultiplication $\Psi \colon E_*(\underline{E}_k) \to \bigotimes_{\alpha \in \Lambda} E_*(\underline{E}_k)$ in the form

$$\Psi c = \sum_i \bigotimes_\alpha c_{i,\alpha} \qquad \text{in } \textstyle\bigotimes_{\alpha \in \Lambda} E_*(\underline{E}_k) \tag{10.18}$$

for suitable elements $c_{i,\alpha} \in E_*(\underline{E}_k)$. We can of course replace $\underline{E}_k$ by any space for which we have the necessary Künneth formulae.

Theorem 10.19. *Let M be an unstable algebra and assume that $E_*(\underline{E}_k)$ is a free E^*-module for all k. Take $x, y \in M^k$ and assume that $r(x)$ is in the standard form* (10.3). *Then:*

(a) *We have the* Cartan formula for addition

$$r(x+y) = \sum_\alpha x_\alpha r''_\alpha(y) \qquad \text{for all } r\colon k \to m, \tag{10.20}$$

where for each α, the operation $r''_\alpha\colon k \to m + \deg(c_\alpha)$ is defined as having the functional

$$\langle r''_\alpha, c\rangle = (-1)^{\deg(c_\alpha)(m+\deg(c_\alpha))}\langle r, c_\alpha * c\rangle \qquad \text{for all } c \in E_*(\underline{E}_k); \tag{10.21}$$

(b) *If, similarly, $r(y)$ has the standard form*

$$r(y) = \sum_\beta \langle r, d_\beta\rangle\, y_\beta, \tag{10.22}$$

then we have the full Cartan formula for addition,

$$r(x+y) = \sum_\alpha \sum_\beta (-1)^{\deg(x_\alpha)\deg(y_\beta)}\langle r, c_\alpha * d_\beta\rangle\, x_\alpha y_\beta \tag{10.23}$$

for all $r\colon k \to m$;

(c) *Assume $a, b \in E_*(\underline{E}_k)$. Let c_α run through a basis of $E_*(\underline{E}_k)$, and denote by r'_α the operation dual to c_α. Let $\Psi a = \sum_i \otimes_\alpha a_{i,\alpha}$ and $\Psi b = \sum_j \otimes_\alpha b_{j,\alpha}$ be the iterated coproducts of a and b as in eq.* (10.18), *where in both cases, we ignore those α for which*

$$r'_{\alpha *} a_{i,\alpha} = (\epsilon a_{i,\alpha})1 \qquad \text{for all } i \tag{10.24}$$

(see the Remark following). *Then the homology homomorphism $r_*\colon E_*(\underline{E}_k) \to E_*(\underline{E}_m)$ satisfies*

$$r_*(a * b) = \sum_i \sum_j \pm \mathop{\ast}_\alpha r'_{\alpha *} a_{i,\alpha} \circ r''_{\alpha *} b_{j,\alpha} \qquad \text{in } E_*(\underline{E}_m), \tag{10.25}$$

where r''_α is defined by eq. (10.21) *and the only signs come from shuffling the factors to form $\Psi(a \times b)$.*

Remark. The formula (10.25) demands some explanation. The proof will show that the relevant set of α is in fact finite, so that the iterated coproducts Ψa and Ψb are defined.

If α satisfies eq. (10.24), we have

$$r'_{\alpha *} a_{i,\alpha} \circ r''_{\alpha *} b_{j,\alpha} = (\epsilon a_{i,\alpha})1 \circ r''_{\alpha *} b_{j,\alpha} = (\epsilon a_{i,\alpha}) \epsilon r''_{\alpha *} b_{j,\alpha} = (\epsilon a_{i,\alpha})(\epsilon b_{j,\alpha}) 1_m\ .$$

In the usual (and sufficient) case when $\epsilon a = \epsilon b = 0$, we can easily arrange for each $a_{i,\alpha}$ and $b_{j,\alpha}$ to be 1 or lie in $\operatorname{Ker}\epsilon$, by breaking up terms and shuffling as necessary. Then the ij-term contributes nothing to $r_*(a*b)$ unless $a_{i,\alpha} = 1$ and $b_{j,\alpha} = 1$ for all $\alpha \in \Lambda$ that satisfy eq. (10.24). Such an index α may be omitted from the $*$-product in eq. (10.25) and the iterated coproducts Ψa and Ψb.

Proof. We first assume that the c_α form a basis of $E_*(\underline{E}_k)$, so that $x_\alpha = r'_\alpha(x)$ as in eq. (10.4). By the Künneth homeomorphism, we can write

$$\mu_k^* r = \sum_\alpha r'_\alpha \times r''_\alpha \qquad \text{in } E^*(\underline{E}_k \times \underline{E}_k), \tag{10.26}$$

for uniquely determined elements $r''_\alpha \in E^*(\underline{E}_k)$. In other words, in the diagram

$$\begin{array}{ccccc} X & \xrightarrow{u} & \underline{E}_k \times \underline{E}_k & \xrightarrow{r'_\alpha \times r''_\alpha} & \underline{E}_? \times \underline{E}_? \\ & \searrow^{x+y} & \downarrow{\scriptstyle \mu_k} & & \downarrow{\scriptstyle \phi} \\ & & \underline{E}_k & \xrightarrow{r} & \underline{E}_m \end{array} \tag{10.27}$$

the map $r \circ \mu_k$ is expressed as the sum of the maps $g_\alpha = \phi \circ (r'_\alpha \times r''_\alpha)$, and is the universal example for computing $r(x+y)$, where $u\colon X \to \underline{E}_k \times \underline{E}_k$ has coordinates $x\colon X \to \underline{E}_k$ and $y\colon X \to \underline{E}_k$. Evaluation on $c_\alpha \times c$ identifies r''_α as in eq. (10.21), with the help of

$$\langle \mu_k^* r, c_\alpha \times c \rangle = \langle r, \mu_{k*}(c_\alpha \times c) \rangle = \langle r, c_\alpha * c \rangle \ .$$

Then eq. (10.20) is induced from eq. (10.26). To deduce (b), we substitute eq. (10.22) in eq. (10.20) and watch the signs.

To remove the requirement that the c_α form a basis, we note that by linearity, eq. (10.20) is preserved by each of the replacements listed in Lemma 10.6. (The operation r'_α is no longer defined, but appears only in (c).)

For (c), we apply homology everywhere. We have to add the homomorphisms $g_{\alpha *}$ in the sense of eq. (10.17), using the iterated coproduct $\Psi(a \times b)$, which is obtained from $\Psi a \times \Psi b$ by shuffling. We note that any $a \in E_*(\underline{E}_k)$ comes from some finite subcomplex Y of $\underline{E}_k$. All but finitely many of the r'_α vanish on Y, by the strong duality for $\underline{E}_k$; these α satisfy eq. (10.24), as we see by computing the iterated coproduct Ψa first in Y, since the zero operation $0\colon k \to m$ induces $0_* c = (\epsilon c) 1_m$. □

Similarly, the zero map $\omega_k\colon T \to \underline{E}_k$ and inversion map $\nu_k\colon \underline{E}_k \to \underline{E}_k$ of $\underline{E}_k$ yield the useful formulae

$$r(0_k) = \langle r, 1_k \rangle \, 1_M \qquad \text{in } M \qquad (\text{for all } r) \tag{10.28}$$

and

$$r(-x) = \sum_\alpha \langle r, \chi c_\alpha \rangle \, x_\alpha \qquad \text{in } M \qquad (\text{for all } r). \tag{10.29}$$

For some applications, it is useful to cut out the finiteness argument in the proof of Thm. 10.19(c) and work directly in a finite space Y.

Proposition 10.30. *Let $f\colon Y \to \underline{E}_k$ be a map, where $E_*(Y)$ is a free E^*-module of finite rank, with basis elements z_α. Let $y_\alpha \in E^*(Y)$ be dual to z_α. Then for any $a \in E_*(Y)$, $b \in E_*(\underline{E}_k)$, and operation $r\colon k \to m$,*

$$r_*(f_* a * b) = \sum_i \sum_j \pm \mathop{\ast}_\alpha y_{\alpha *} a_{i,\alpha} \circ r''_{\alpha *} b_{j,\alpha} \qquad \text{in } E_*(\underline{E}_m),$$

where $r''_\alpha\colon k \to m + \deg(z_\alpha)$ *denotes the operation having the functional*

$$\langle r''_\alpha, c\rangle = (-1)^{\deg(z_\alpha)(m+\deg(z_\alpha))}\langle r, f_* z_\alpha * c\rangle,$$

Ψa *and* Ψb *are computed as in eq.* (10.18)*, and we use* $y_{\alpha *}\colon E_*(Y) \to E_*(\underline{E}_?)$.

Proof. By Thm. 1.18(a), $E^*(Y)$ is dual to $E_*(Y)$ and y_α is defined. We modify the proof of the Theorem by composing the square in diag. (10.27) with $f\times 1\colon Y\times \underline{E}_k \to \underline{E}_k \times \underline{E}_k$. We work in $E^*(Y\times \underline{E}_k)$ instead of $E^*(\underline{E}_k\times\underline{E}_k)$ and write $(f\times 1)^*\mu_k^* r = \sum_\alpha y_\alpha \times r''_\alpha$. We evaluate this on $z_\alpha \times c$ to determine r''_α. □

Remark. The commutativity of $*$-multiplication ensures that $r(x+y) = r(y+x)$. Conversely, one could say that $x + y = y + x$ in M *requires* $*$-multiplication to be commutative. The universal example has $M = E^*(\underline{E}_k\times\underline{E}_k)$, $x = \iota_k \times 1$, and $y = 1 \times \iota_k$, and c_α and d_β run through bases of $E_*(\underline{E}_k)$. Then $r(x+y) = r(y+x)$ for all r implies that $c_\alpha * d_\beta = \pm\, d_\beta * c_\alpha$ for all α and β. The commutativity of $*$ in general follows by linearity.

Similar discussions hold for other laws in a ring. In particular, $x + 0 = x$ corresponds in this way to $c * 1_k = c$, $-(-x) = x$ to $\chi\chi c = c$, $-(x + y) = (-x) + (-y)$ to $\chi(a * b) = \chi a * \chi b$, and the associativity of $+$ to the associativity of $*$.

Given a prime p, we can iterate eq. (10.23) to get

$$r(px) = r(x+x+\ldots+x) = \sum \pm\langle r, c_{\alpha_1} * c_{\alpha_2} * \ldots * c_{\alpha_p}\rangle\, x_{\alpha_1} x_{\alpha_2}\ldots x_{\alpha_p}\ .$$

If the indices α_i are not all the same, we can permute them cyclically and obtain p distinct terms which by commutativity are all equal, with the same sign. This leaves only the terms with $\alpha_i = \alpha$ for all i, and we find

$$r(px) \equiv \sum_\alpha \langle r, Fc_\alpha\rangle\, Fx_\alpha \bmod p. \tag{10.31}$$

This is particularly useful when E^* has characterisitic p, so that $px = 0$, because comparison with eq. (10.28) then yields

$$\sum_\alpha \langle r, Fc_\alpha\rangle\, Fx_\alpha = \langle r, 1_k\rangle\, 1_M \qquad \text{in } M \qquad (\text{for all } r). \tag{10.32}$$

Multiplicative structure. The multiplication maps $\phi\colon \underline{E}_k\times\underline{E}_m \to \underline{E}_{k+m}$ induce both the multiplication in UM and the $\circ$-multiplication in $E_*(\underline{E}_*)$. This allows us to translate the axiom that ρ_M is multiplicative, $\rho_M(xy) = \rho_M(x)\rho_M(y)$ in UM.

Theorem 10.33. *Let* M *be an unstable algebra, and assume that* $E_*(\underline{E}_k)$ *is a free* E^**-module for all* k*. Take* $x \in M^k$ *and* $y \in M^m$ *and assume that* $r(x)$ *is in the standard form* (10.3)*. Then:*

(a) *We have the* Cartan formula for multiplication

$$r(xy) = \sum_\alpha x_\alpha r''_\alpha(y) \qquad \text{for all } r\colon k+m \to h, \tag{10.34}$$

where for each α, the operation $r''_\alpha: m \to h + \deg(c_\alpha)$ is defined as having the functional

$$\langle r''_\alpha, c\rangle = (-1)^{\deg(c_\alpha)(h+\deg(c_\alpha))}\langle r, c_\alpha \circ c\rangle \qquad \text{for all } c \in E_*(\underline{E}_m); \tag{10.35}$$

(b) *If, similarly, $r(y)$ is given by eq.* (10.22), *we have the* full Cartan formula for multiplication,

$$r(xy) = \sum_\alpha \sum_\beta (-1)^{\deg(x_\alpha)\deg(y_\beta)}\langle r, c_\alpha \circ d_\beta\rangle\, x_\alpha y_\beta \tag{10.36}$$

for all $r: k+m \to h$;

(c) *Take $a \in E_*(\underline{E}_k)$ and $b \in E_*(\underline{E}_m)$. Assume that c_α runs through a basis of $E_*(\underline{E}_k)$, and denote by r'_α the operation dual to c_α. Let $\Psi a = \sum_i \otimes_\alpha a_{i,\alpha}$ and $\Psi b = \sum_j \otimes_\alpha b_{j,\alpha}$ be the iterated coproducts of a and b as in eq.* (10.18), *where in both cases, we ignore all α that satisfy eq.* (10.24). *Then the homology homomorphism $r_*: E_*(\underline{E}_{k+m}) \to E_*(\underline{E}_h)$ satisfies*

$$r_*(a \circ b) = \sum_i \sum_j \pm \mathop{\ast}_\alpha r'_{\alpha *} a_{i,\alpha} \circ r''_{\alpha *} b_{j,\alpha} \qquad \text{in } E_*(\underline{E}_m), \tag{10.37}$$

where r''_α is defined by eq. (10.35) *and the only signs come from shuffling the factors to form $\Psi(a \times b)$.*

The Remark following Thm. 10.19 applies.

Proof. This is formally identical to the proof of Thm. 10.19, with $\mu_k: \underline{E}_k \times \underline{E}_k \to \underline{E}_k$ replaced everywhere by $\phi: \underline{E}_k \times \underline{E}_m \to \underline{E}_{k+m}$. □

By naturality, we can adapt eq. (10.36) to $\times$-products.

Corollary 10.38. *Given spaces X and Y and elements $x \in E^k(X)$ and $y \in E^m(Y)$, assume that $r(x)$ and $r(y)$ are given by eqs.* (10.3) *and* (10.22). *Then we have the Cartan formula*

$$r(x \times y) = \sum_\alpha \sum_\beta (-1)^{\deg(x_\alpha)\deg(y_\beta)}\langle r, c_\alpha \circ d_\beta\rangle\, x_\alpha \times y_\beta \tag{10.39}$$

in $E^*(X \times Y)$, for any operation $r: k+m \to h$. □

We have also the analogue of Prop. 10.30.

Proposition 10.40. *Let $f: Y \to \underline{E}_k$ be a map as in Prop. 10.30. Then for any $a \in E_*(Y)$, $b \in E_*(\underline{E}_k)$, and operation $r: k+m \to h$,*

$$r_*(f_* a \circ b) = \sum_i \sum_j \pm \mathop{\ast}_\alpha y_{\alpha *} a_{i,\alpha} \circ r''_{\alpha *} b_{j,\alpha} \qquad \text{in } E_*(\underline{E}_h),$$

where $r''_\alpha: m \to h + \deg(z_\alpha)$ denotes the operation having the functional

$$\langle r''_\alpha, c\rangle = (-1)^{\deg(z_\alpha)(m+\deg(z_\alpha))}\langle r, f_* z_\alpha \circ c\rangle$$

and Ψa and Ψb are computed as in eq. (10.18). □

Since the unit element of UM is $\eta_M \circ \eta^*: E^*(\underline{E}_0) \to E^* \to M$, the axiom $\rho_M(1_M) = 1_{UM}$ translates easily into

$$r(1_M) = \langle \eta^* r, 1\rangle\, 1_M = \langle r, \eta_* 1\rangle\, 1_M = \langle r, [1]\rangle\, 1_M \qquad \text{in } M \tag{10.41}$$

for all r.

Just as with addition, certain laws in the Hopf ring correspond to laws in an E^*-algebra M. For example, associativity of $\circ$-multiplication corresponds to associativity of multiplication in M. Commutativity is slightly trickier: given $x \in M^k$ and $y \in M^m$, $r(yx) = r((-1)^{km}xy)$ leads to the identity (10.10), thus explaining the signs and the appearance of χ.

The comonad structure. Finally, we translate the two axioms which state that ρ_M makes M a U-coalgebra. Since we have in effect returned to the First Answer of section 8, these are the usual axioms for an action, $(sr)(x) = s(r(x))$ and $\iota_k x = x$.

The second is easily handled. From Prop. 6.11, we can use (6.41) to express the identity operation ι_k as the functional

$$\langle \iota_k, -\rangle = Q(\epsilon) \circ q_k: E_*(\underline{E}_k) \longrightarrow Q(E)^k_* \longrightarrow E_*(E, o) \longrightarrow E^*. \tag{10.42}$$

When we put $r = \iota_k$, eq. (10.3) expands easily to yield the axiom

$$\sum_\alpha (Q(\epsilon) q_k c_\alpha)\, x_\alpha = x \qquad \text{in } M \tag{10.43}$$

for $x \in M^k$. We have thus interpreted the counit natural transformation $\epsilon M: UM \to M$ of the comonad U, which was defined by $(\epsilon M)f = f\iota_k$. The functional $\epsilon_S \circ \sigma_{k*}$ is *not* part of the Hopf ring structure as given so far, so we add it. (It is unrelated to the counit $\epsilon: E_*(\underline{E}_k) \to E^*$ of the Hopf algebra $E_*(\underline{E}_k)$.)

It is easy to recover the functional $\langle r, -\rangle$ from r_*, as in eq. (6.50), in the form

$$\langle r, -\rangle: E_*(\underline{E}_k) \xrightarrow{r_*} E_*(\underline{E}_m) \xrightarrow{\sigma_{m*}} E_*(E, o) \xrightarrow{\epsilon_S} E^*, \tag{10.44}$$

by writing $\langle r, c\rangle = \langle r^* \iota_m, c\rangle = \langle \iota_m, r_* c\rangle$ and using eq. (10.42). In the additive context, the reverse construction of r_* from $\langle r, -\rangle$ was neatly encoded in the comultiplication $Q(\psi)$ on $Q(E)^*_*$. Here, we have no such map and must rely on the definition of ψM, which explicitly uses r^*. In effect, we dualize and use r_* instead.

The first is the most complicated of all the axioms. When we substitute sr and r in eq. (10.3) and use $\langle sr, c_\alpha\rangle = \langle r^* s, c_\alpha\rangle = \langle s, r_* c_\alpha\rangle$, the axiom $(sr)(x) = s(r(x))$ expands to

$$\sum_\alpha \langle s, r_* c_\alpha\rangle\, x_\alpha = s(r(x)) = s\left(\sum_\alpha \langle r, c_\alpha\rangle\, x_\alpha\right) \qquad \text{in } M, \tag{10.45}$$

for all r, s. The right side is to be expanded using eqs. (10.20) and (10.16), and in general is extremely complicated.

Our conclusion is that we need to know the induced homology homomorphism $r_*: E_*(\underline{E}_k) \to E_*(\underline{E}_m)$ for every operation $r: E^k(-) \to E^m(-)$. This is the final piece of structure to add to the Hopf ring. To compute it successfully, we need $r_* c$ for each $\circ$-generator c of $E_*(\underline{E}_*)$, and then use formulae (10.25) and (10.37) for $r_*(a * b)$ and $r_*(a \circ b)$.

Summary. We collect the various formulae to form the main theorem of this section. In addition to the Hopf ring structure on $E_*(\underline{E}_*)$, we need:

(i) The element $[v] \in E_0(\underline{E}_*)$ for each $v \in E^*$ (see Defn. 10.8);

(ii) The augmentation (see eq. (10.42))

$$Q(\epsilon) \circ q_k \colon E_*(\underline{E}_k) \longrightarrow Q(E)^k_* \longrightarrow E_*(E, o) \longrightarrow E^* \tag{10.46}$$

which may be written $\epsilon_S \circ \sigma_{k*}$;

(iii) The homomorphism $r_* \colon E_*(\underline{E}_k) \to E_*(\underline{E}_m)$ induced by each operation $r \colon k \to m$.

These constitute what we mean by an *enriched* Hopf ring structure.

Theorem 10.47. *Let M be an object of* FAlg*, i. e. a complete Hausdorff filtered E^*-algebra, and assume that $E_*(\underline{E}_k)$ is a free E^*-module for all k* (which is true for $E = H(\mathbb{F}_p)$, BP, MU, KU, or $K(n)$ by Lemma 4.17(a)). *Then an unstable algebra structure on M consists of a value $r(x) \in M$ for all $x \in M$ and all $r \in E^*(\underline{E}_k)$ (where $k = \deg(x)$ and $r(x) \in M^m$ if $r \in E^m(\underline{E}_k)$), which is E^*-linear in r and therefore (for fixed x) expressible in the standard form* (10.3)

$$r(x) = \sum_\alpha \langle r, c_\alpha \rangle x_\alpha \qquad \text{in } M \qquad (\text{for all } r).$$

These values are subject to the following axioms:

(a) *For fixed $x \in M^k$, $r(x)$ satisfies the three consistency conditions:*

(i) $$\sum_\gamma \psi c_\gamma \otimes x_\gamma = \sum_\alpha \sum_\beta (-1)^{\deg(x_\alpha)\deg(x_\beta)} c_\alpha \otimes c_\beta \otimes x_\alpha x_\beta$$
$$\text{in } (E_*(\underline{E}_k) \otimes E_*(\underline{E}_k)) \mathbin{\widehat{\otimes}} M;$$

(ii) $$\sum_\alpha (\epsilon c_\alpha)\, x_\alpha = 1_M \qquad \text{in } M;$$

(iii) $$\sum_\alpha (\epsilon_S \sigma_{k*} c_\alpha)\, x_\alpha = x \qquad \text{in } M;$$

(b) *As x varies, $r(x)$ satisfies the following identities in M for all r, where we assume similarly* (as in eq. (10.22)) *that $r(y) = \sum_\beta \langle r, d_\beta \rangle\, y_\beta$:*

(i) $$r(x+y) = \sum_\alpha \sum_\beta (-1)^{\deg(x_\alpha)\deg(y_\beta)} \langle r, c_\alpha * d_\beta \rangle\, x_\alpha y_\beta;$$

(ii) $$r(vx) = \sum_\alpha \langle r, [v] \circ c_\alpha \rangle\, x_\alpha;$$

(iii) $$r(xy) = \sum_\alpha \sum_\beta (-1)^{\deg(x_\alpha)\deg(y_\beta)} \langle r, c_\alpha \circ d_\beta \rangle\, x_\alpha y_\beta;$$

(iv) $$r(1_M) = \langle r, [1] \rangle\, 1_M;$$

(c) *The composition law*

$$\sum_\alpha \langle s, r_* c_\alpha \rangle\, x_\alpha = s(r(x)) = s\left(\sum_\alpha \langle r, c_\alpha \rangle\, x_\alpha\right) \qquad \text{in } M$$

holds for all r, s, and all $x \in M$;

(d) *For each of the ideals $F^a M$ in the filtration of M:*

(i) *For fixed $x \in M$, all except finitely many of the x_α lie in $F^a M$;*

(ii) *There exists $F^b M$ such that $r(x) \in F^a M$ for all $x \in F^b M$ and all based operations r.*

Proof. The equations in (a) are (10.14), (10.15), and (10.43). Those in (b) are (10.23), (10.16), (10.36), and (10.41). The equation in (c) is (10.45). In (d), (i) states that $\rho_M(x)\colon E^*(\underline{E}_k) \to M$ is continuous for each x, while (ii) states that $\rho_M\colon M \to UM$ is continuous. □

Remark. By (b), an unstable algebra structure on M is determined by the values $r(x)$ on a set of (topological) E^*-algebra generators x of M. Moreover, the Hopf ring laws imply that it is sufficient to verify axioms (a) and (d)(i) on these generators. In practice, the topological conditions (d) rarely cause us any distress.

11. The E-cohomology of a point

In this section, we study the first of our test spaces, the one-point space T, for which $E^*(T)$ is by definition the coefficient ring E^*. Its unstable structure is completely determined by eqs. (10.41) and (10.16) as

$$r(v) = \langle r, [v] \rangle \qquad \text{in } E^* = E^*(T) \qquad (\text{for all } r), \tag{11.1}$$

which may be taken as an alternate definition of the Hopf ring elements $[v]$, instead of Defn. 10.8.

It is easy to deduce how $[v]$ interacts with each piece of the structure on $E_*(\underline{E}_*)$. Much of this can be read off from the Hopf ring structure in section 10. In particular, η_R is still in some sense a ring homomorphism.

Proposition 11.2. *The Hopf ring elements $[v] \in E_0(\underline{E}_h)$ for each $v \in E^h$ have the properties:*

(a) $\psi[v] = [v] \otimes [v]$;

(b) $\epsilon[v] = 1$;

(c) $[v + v'] = [v] * [v']$ for $v' \in E^h$;

(d) $[-v] = \chi[v]$;

(e) $[vv'] = [v] \circ [v']$ for $v' \in E^k$;

(f) $1_m \circ [v] = 1_{m+h}$;

(g) $r_*[v] = [\langle r, [v] \rangle]$ (for all r);

(h) $r_* 1_h = [\langle r, 1_h \rangle]$;

(i) $q_h[v] = \eta_R v$ *in* $Q(E)_0^h$;

(j) $\sigma_{h*}[v] = \eta_R v$ *in* $E_{-h}(E, o)$, *under stabilization.*

Proof. For (a) and (b) we substitute eq. (11.1) in eqs. (10.14) and (10.15). For (c) and (e), we write down the Cartan formulae (10.23) and (10.36) for $r(v+v')$ and $r(vv')$ and compare with eq. (11.1). For (d), we write down $r(-v)$ from eq. (10.29) and compare with eq. (11.1). For (g), we use eq. (11.1) to compute $s(r(v)) = \langle s, [\langle r, [v]\rangle]\rangle$; by eq. (10.45), this must agree with $\langle s, r_*[v]\rangle$ for all s. Since $[0_n] = 1_n$, (f) and (h) are special cases of (e) and (g). For (i) and (j), we compare eq. (11.1) with eq. (6.43) and [8, (11.23)], and use eqs. (6.14) and (6.13). □

Constant operations. Constant operations were introduced briefly in section 8. Although they are of no real interest and contain no information, they are undeniably natural and we have to be able to recognize them in their several disguises.

Proposition 11.3. *Let* $r\colon k \to m$ *be the constant operation defined by* $r(x) = v1_X$ *for all* $x \in E^k(X)$, *where* $v \in E^m$. *Then:*

(a) *As a class,* $r = v1_k \in E^*(\underline{E}_k)$;

(b) *As a map,* r *is the composite* $v \circ q\colon \underline{E}_k \to T \to \underline{E}_m$;

(c) *As a functional,* $\langle r, c\rangle = (\epsilon c)v$ *in* E^* *for all* $c \in E_*(\underline{E}_k)$;

(d) $r_*\colon E_*(\underline{E}_k) \to E_*(\underline{E}_m)$ *is given by* $r_*c = (\epsilon c)[v]$ *for all* $c \in E_*(\underline{E}_k)$. □

Based operations. Given a based space (X, o), we consider the naturality of an operation $r\colon k \to m$ with respect to the inclusion of the basepoint. We augment Lemma 2.3.

Proposition 11.4. *The following conditions on an operation* $r\colon k \to m$ *are equivalent:*

(a) $r(0) = 0$ *in* $E^*(T) = E^*$, *i. e.* r *is* based *in the sense of Defn. 2.2;*

(b) $r(0) = 0$ *in* $E^*(X)$ *for all spaces* X;

(c) *The operation* r *induces* $r\colon E^k(X, o) \to E^m(X, o)$ *for all* X;

(d) *The class* r *lies in* $E^m(\underline{E}_k, o) \subset E^m(\underline{E}_k)$;

(e) *The map* $r\colon \underline{E}_k \to \underline{E}_m$ *is based (up to homotopy);*

(f) *The linear functional* $\langle r, -\rangle$ *satisfies* $\langle r, 1_k\rangle = 0$;

(g) *The homomorphism* $r_*\colon E_*(\underline{E}_k) \to E_*(\underline{E}_m)$ *satisfies* $r_*1_k = 1_m$.

Proof. Part (b) is equivalent to (a) by naturality. Because $r(0) = \langle r, 1_k\rangle 1_X$ by eq. (10.28), (f) is equivalent to (b), and with the help of Prop. 11.2(h), to (g). □

We can generalize (f).

Lemma 11.5. *Let* (X, o) *be a based space. Then for any* $x \in E^k(X, o)$ *and any operation* $r\colon k \to m$, *we have* $r(x) \equiv \langle r, 1_k\rangle 1_X \bmod E^*(X, o)$.

Proof. We use eq. (10.28) and the naturality of r in diag. [8, (3.2)]. □

It is sometimes useful to be more specific. If we choose a basis of $E_*(\underline{E}_k)$ consisting of 1_k and elements $c_\alpha \in \operatorname{Ker}\epsilon$, then for any $x \in E^k(X,o)$, eq. (10.4) takes the form

$$r(x) = \langle r, 1_k\rangle\, 1_X + \sum_\alpha \langle r, c_\alpha\rangle\, x_\alpha \qquad \text{in } E^*(X)\hat{} \qquad \text{(for all } r\text{)}, \tag{11.6}$$

where the elements $x_\alpha \in E^*(X,o)\hat{}$.

Formulae are often simpler for based operations, but the case of general r can be recovered easily enough by decomposing as in Lemma 8.1.

Lemma 11.7. *If we write $r(x) = s(x) + v1_X$, where s is a based operation and $v \in E^m$, the homology homomorphism $r_*\colon E_*(\underline{E}_k) \to E_*(\underline{E}_m)$ is given by $r_*c = s_*c * [v]$, where we recognize $v = r(0) = \langle r, 1_k\rangle$.*

Proof. We write r as the composite

$$\underline{E}_k \xrightarrow{\Delta} \underline{E}_k \times \underline{E}_k \xrightarrow{1\times q} \underline{E}_k \times T \xrightarrow{s\times v} \underline{E}_m \times \underline{E}_m \xrightarrow{\mu_m} \underline{E}_m$$

and take E-homology. The first two maps just give $\underline{E}_k \cong \underline{E}_k \times T$. □

12. Spheres, suspensions, and additive operations

So far, except for adding an extra grading, our additive results are formally very similar to the stable case discussed in [8]. What is new is that suspension is no longer an isomorphism, but defines a new element e. The stable results can all be recovered by stabilizing, which consists merely of setting $e = 1$.

We assume throughout that $E_*(\underline{E}_k)$, $QE_*(\underline{E}_k)$, and $E_*(E,o)$ are free E^*-modules, so that we have available the machinery of comodule algebras of sections 6 and 7 as well as the stable results of [8]. In particular, the coaction $\rho_X\colon E^*(X) \to E^*(X) \mathbin{\widehat{\otimes}} Q(E)^*_*$ is a homomorphism of E^*-algebras for any X.

Spheres. Our second test space, after the one-point space T, is the circle S^1. Its cohomology $E^*(S^1,o)$ is a free E^*-module with the basis $\{1_S, u_1\}$, where the canonical generator $u_1 \in E^1(S^1,o)$ is provided by [8, Defn. 3.23]. Thus $\rho_S\colon E^*(S^1) \to E^*(S^1) \otimes Q(E)^*_*$ is determined by $\rho_S u_1$.

Definition 12.1. We define the *suspension element* $e = e_Q \in Q(E)^1_1$ by the identity

$$\rho_S u_1 = u_1 \otimes e \qquad \text{in } E^*(S^1,o) \otimes Q(E)^1_* \cong Q(E)^1_*\,. \tag{12.2}$$

It has degree zero.

More generally, for the k-sphere S^k, $E^*(S^k)$ is free on the basis $\{u_k, 1_S\}$, where $u_k \in E^k(S^k,o)$.

Proposition 12.3. *The suspension element $e \in Q(E)^1_1$ has the following properties, where $k \geqslant 0$:*

(a) $\rho_S u_k = u_k \otimes e^k$ *in* $E^*(S^k) \otimes Q(E)^k_*$;

(b) $ru_k = \langle r, e^k\rangle\, u_k$ *in* $E^*(S^k)$ *for any additive operation* $r\colon k \to m$;

(c) *The class* $u_k \in E^k(S^k)$, *regarded as a map* $u_k\colon S^k \to \underline{E}_k$, *induces* $q_k u_{k*} z = e^k \in Q(E)^k_k$, *where* $z \in E_k(S^k)$ *is dual to* u_k;

(d) *In the coalgebra structure on* $Q(E)^*_*$, $Q(\psi)e = e \otimes e$ *and* $Q(\epsilon)e = 1$;

(e) $Q(\psi)(ve^kw) = ve^k \otimes e^k w$ *in* $Q(E)^*_* \otimes Q(E)^*_*$, *for any* $v \in E^*$ *and* $w \in \eta_R E^*$;

(f) *Given* $v \in E^*$ *and* $w \in \eta_R E^h$, *the homomorphism* $Q(r)\colon Q(E)^{k+h}_* \to Q(E)^m_*$ *induced on homology by any operation* $r\colon k+h \to m$ *satisfies*

$$Q(r)(ve^kw) = ve^k\eta_R\langle r, e^k w\rangle \qquad \text{in } Q(E)^m_*;$$

(g) *Under stabilization,* $Q(\sigma)e = 1$ *in* $E_*(E, o)$.

Proof. We prove (a) for $k > 0$ by induction on k, starting from eq. (12.2). If it holds for k and m, the multiplicativity of ρ gives

$$\rho(u_k \times u_m) = (u_k \times u_m) \otimes e^{k+m} \qquad \text{in } E^*(S^k \times S^m).$$

The projection map $q\colon S^k \times S^m \to S^{k+m}$ induces $q^* u_{k+m} = u_k \times u_m$, which gives (a) for $k+m$. The result is true also for $k = 0$, if we make the obvious identification $e^0 = 1$. Then (b) follows by eq. (6.39) and (c) is an application of Prop. 6.44.

To prove (d), we evaluate both axioms (6.33) for $M = E^*(S^1)$ on u_1. Part (e) follows immediately from (d) and the fact that $Q(\psi)$ is a homomorphism of algebras and of E^*-bimodules. Then (f) follows from (e) and Lemma 6.51(c). For (g), we apply $1 \otimes Q(\sigma)$ to eq. (12.2) and compare with the stable coaction $\rho_S u_1 = u_1 \otimes 1$ in [8, (11.24)]. □

Remark. As v, k, and w vary, the elements ve^kw span $Q(E)^*_* \otimes \mathbb{Q}$ as a $\mathbb{Q}$-module. (In fact, $Q(\sigma)$ induces $Q(E)^k_* \otimes \mathbb{Q} \cong E_*(E, o) \otimes \mathbb{Q}$ if E is $(-k-1)$-connected.) Thus in the important case when $Q(E)^k_*$ has no torsion, the innocuous formulae in (e) and (f) are powerful enough to determine $Q(\psi)$ and $Q(r)$ completely.

Corollary 12.4. *Let* $r\colon k \to m$ *be an additive operation, regarded as a map of H-spaces* $r\colon \underline{E}_k \to \underline{E}_m$. *Then the induced homomorphism on homotopy groups*

$$E^* \cong \pi_*(\underline{E}_k, o) \xrightarrow{r_*} \pi_*(\underline{E}_m, o) \cong E^*$$

is given on $v \in E^{-h}$ *by* $r_* v = \langle r, e^{k+h}\eta_R v\rangle$.

Proof. We reinterpret r_* as the action of the operation r on $E^k(S^{k+h}, o)$. The element v corresponds to the class vu_{k+h}. From Prop. 12.3(b) and eq. (6.40),

$$r(vu_{k+h}) = \langle r, e^{k+h}\eta_R v\rangle\, u_{k+h} \qquad \text{in } E^*(S^{k+h}, o)\ .$$ □

Suspensions. More generally, the action of the operations on the suspension ΣX of a based space X is easily deduced from the action on X.

Lemma 12.5. *Given a based space* (X, o) *and* $x \in E^k(X, o)$, *the coaction* $\rho_{\Sigma X}\Sigma x$ *is the image of* $\rho_X x$ *under*

$$\Sigma \otimes e\colon E^*(X, o)\, \widehat{\otimes}\, Q(E)^k_* \longrightarrow E^*(\Sigma X, o)\, \widehat{\otimes}\, Q(E)^{k+1}_*,$$

where e *denotes multiplication by the element* $e \in Q(E)^1_*$.

Proof. The projection map $S^1 \times X \to \Sigma X$ embeds $E^*(\Sigma X, o)$ in $E^*(S^1 \times X, S^1 \times o)$. Here, Σx corresponds to $u_1 \times x$, whose coaction is known. □

We can mimic this algebraically. We defined the formal suspension ΣM of any E^*-module M in [8, Defn. 6.6], merely by shifting all the degrees up one.

Definition 12.6. Given any unstable comodule M, we make the *suspension* ΣM of M an unstable comodule by equipping it with the coaction $\rho_{\Sigma M}$ defined by the commutative square

$$\begin{array}{ccc} M^k & \xrightarrow{\rho_M} & M \mathbin{\widehat{\otimes}} Q(E)^k_* \\ \cong \big\downarrow \Sigma & & \big\downarrow \Sigma \otimes e \\ (\Sigma M)^{k+1} & \xrightarrow{\rho_{\Sigma M}} & \Sigma M \mathbin{\widehat{\otimes}} Q(E)^{k+1}_* \end{array}$$

The axioms on $\rho_{\Sigma M}$ are readily verified.

13. Spheres, suspensions, and unstable operations

In this section, we continue section 12 by computing all the unstable operations on $E^*(S^k)$ for the spheres S^k, which requires one new Hopf ring element, the suspension element e. This leads to the unstable structure of $E^*(\Sigma X)$ in terms of $E^*(X)$.

We recall that $E^*(S^k)$ is a free E^*-module with basis $\{1_S, u_k\}$, where u_k is the standard generator. The algebra structure is given by $u_k^2 = 0$, except that of course $u_0^2 = u_0$. By the Remark after Thm. 10.47, we have only to find $r(u_k)$. Lemma 11.5 gives partial information.

We assume that $E_*(\underline{E}_k)$ is a free E^*-module for all k.

Definition 13.1. We define the *suspension element* $e = e_U \in E_1(\underline{E}_1)$ by the identity

$$r(u_1) = \langle r, 1_1 \rangle \, 1_S + \langle r, e \rangle \, u_1 \qquad \text{in } E^*(S^1) \qquad (\text{for all } r). \tag{13.2}$$

Here and in similar definitions, we use the freeness of $E^*(S^1)$ and the duality $\mathit{FMod}^*(E^*(\underline{E}_k), E^*) \cong E_*(\underline{E}_k)$ to ensure that e exists and is well defined. We note that $\epsilon e = 0$ from eq. (10.15). Rather than develop all the properties of e now, we include them below in Prop. 13.7 as the special case $e_1 = e$.

Suspensions. We deduce from eq. (13.2) the behavior of unstable operations under the suspension isomorphism $\Sigma\colon E^*(X, o) \cong E^*(\Sigma X, o)$. We take an element $x \in E^k(X, o) \subset E^k(X)$ and assume that $r(x)$ is given by eq. (11.6), so that $\epsilon c_\alpha = 0$. The quotient map $q\colon S^1 \times X \to \Sigma X$ embeds $E^*(\Sigma X)$ in $E^*(S^1 \times X) \cong E^*(S^1) \otimes E^*(X)$; under this embedding, Σx corresponds to $u_1 \times x$. We compute $r(u_1 \times x)$ from the Cartan formula (10.39) and find

$$r(\Sigma x) = \langle r, 1_{k+1} \rangle \, 1_{\Sigma X} + \sum_\alpha (-1)^{\deg(x_\alpha)} \langle r, e \circ c_\alpha \rangle \, \Sigma x_\alpha \tag{13.3}$$

for all r. The other terms drop out because $1_1 \circ c_\alpha = \epsilon c_\alpha = 0$ and $e \circ 1_k = \epsilon e = 0$.

This suggests how the suspension of an unstable algebra should be defined. The treatment is slightly different from the additive version in section 12. First, we need a basepoint.

Definition 13.4. We call the unstable algebra M *based* if we are given an augmentation $M \to E^*$ of unstable algebras. Then the kernel $\overline{M}$ is an invariant ideal, and we have the splitting $M = E^* \oplus \overline{M}$ as E^*-modules.

We define the *unstable suspension* $\Sigma_U M$ of M as the subalgebra

$$\Sigma_U M = (1_S \otimes E^*) \oplus (u_1 \otimes \overline{M}) \subset E^*(S^1) \otimes M. \tag{13.5}$$

The action of r is given on $u_1 \otimes \overline{M}$ by eq. (13.3) and on $1_S \otimes E^*$ by eq. (11.1).

For example, if (X, o) is a based space, we have the augmentation $E^*(X) \to E^*(o) = E^*$, with kernel $E^*(X, o)$ (as in [8, (3.2)]). Inspection shows that much of the structure on M is not used. The multiplication on M is totally ignored. Indeed, we do not need an unstable structure on M at all.

Theorem 13.6. *Given an additively unstable module $\overline{M}$, we can make $E^* \oplus \Sigma\overline{M}$ an unstable algebra, with $1 \in E^*$ as the unit element and trivial multiplication on $\Sigma\overline{M}$, as follows. If $x \in \overline{M}^k$ and $r(x) = \sum_\alpha \langle r_Q, c_\alpha \rangle x_\alpha$ for additive operations r, where $c_\alpha \in Q(E)^k_*$, we lift each c_α to $\tilde{c}_\alpha \in E_*(\underline{E}_k)$ such that $q_k \tilde{c}_\alpha = c_\alpha$, and define the action of unstable operations r on Σx by*

$$r(\Sigma x) = \langle r_U, 1_{k+1} \rangle \, 1 + \sum_\alpha (-1)^{\deg(x_\alpha)} \langle r_U, e \circ \tilde{c}_\alpha \rangle \, \Sigma x_\alpha .$$

Proof. Because $e \circ 1 = 0$ and $e \circ (b * c) = 0$ whenever $\epsilon b = 0$ and $\epsilon c = 0$, $r(\Sigma x)$ is independent of the choices of the $\tilde{c}_\alpha$. The definition (with signs) has been chosen so that: (a) the additive unstable structure on $E^* \oplus \Sigma\overline{M}$ restricts to that on $\Sigma\overline{M}$ given by Defn. 12.6, and (b) it includes eq. (13.5) for a based unstable algebra M. (For (a), we note that diag. (6.16) gives $q_{k+1}(e_U \circ \tilde{c}_\alpha) = (-1)^k e_Q c_\alpha$.) Verification of the axioms of Thm. 10.47 is tedious but routine. □

The elements e_k. It is convenient to use eq. (13.3) to find the structure of $E^*(S^k)$. We deduce the fundamental properties of the Hopf ring element e.

Proposition 13.7. *We define the Hopf ring elements $e_k \in E_k(\underline{E}_k)$ for $k \geqslant 0$ in terms of $e \in E_1(\underline{E}_1)$ by $e_k = (-1)^{k(k-1)/2} e^{\circ k}$ for $k > 0$ (so that $e_1 = e$) and $e_0 = [1] - 1_0$. They have the following properties:*

(a) *In $E^*(S^k)$ we have, for any $k \geqslant 0$:*

$$r(u_k) = \langle r, 1_k \rangle \, 1_S + \langle r, e_k \rangle \, u_k \qquad \text{(for all } r\text{)}; \tag{13.8}$$

(b) *The class u_k, regarded as a map $u_k \colon S^k \to \underline{E}_k$, induces $u_{k*} z = e_k \in E_k(\underline{E}_k)$ in homology, where $z \in E_k(S^k)$ is dual to u_k;*

(c) *$e_k \circ e_m = (-1)^{km} e_{k+m}$ if $k > 0$ or $m > 0$;*

(d) *$\psi e_k = e_k \otimes 1 + 1 \otimes e_k$ for all $k > 0$;*

(e) *$\epsilon e_k = 0 \in E^*$ for all $k \geqslant 0$;*

(f) $\chi e_k = -e_k$ *for all* $k > 0$;

(g) $e_k \circ [\lambda] = \lambda e_k$ *for all rational numbers* $\lambda \in E^0$ *and all* $k > 0$;

(h) $r_* e_k = [\langle r, 1_k \rangle] * [\langle r, e_k \rangle] \circ e_k$ *for all* $k \geqslant 0$ *and all* $r\colon k \to m$;

(i) $q_k e_k = e_Q^k = e^k$ *in* $Q(E)_*^k$, *for all* $k \geqslant 0$, *for additive operations;*

(j) $\sigma_{k*} e_k = 1$ *in* $E_*(E, o)$, *for all* $k \geqslant 0$, *under stabilization.*

Remark. The results make it clear that the correct interpretation of $e^{\circ 0}$ is $[1] - 1_0 = [1] - [0_0]$, as in [28] and elsewhere, rather than just the element [1].

Proof. We give extensive details of this proof (only), as a good example of our machinery in action.

We establish eq. (13.8) for $k > 0$, and thus (a), by induction on k. It holds for $k = 1$ by definition. We recognize ΣS^k as S^{k+1} and Σu_k as u_{k+1}; then by eq. (13.3), eq. (13.8) holds for $k+1$ if it holds for k, provided that $e_{k+1} = (-1)^k e \circ e_k$. Our definition of e_k is designed to do exactly this. More generally, we have (c).

For $k = 0$, we write $E^*(S^0) = E^* \oplus E^*$. In *Alg*, this is a product of algebras, with the projections induced respectively by the inclusions of the basepoint and the other point. In this presentation, $u_0 = (0, 1)$, and of course $1_S = (1, 1)$. By eq. (11.1), the action on u_0 is

$$r(u_0) = r((0,1)) = (\langle r, 1_0 \rangle, \langle r, [1] \rangle) = \langle r, 1_0 \rangle (1_S - u_0) + \langle r, [1] \rangle u_0,$$

which gives (a) if we define $e_0 = [1] - 1_0$.

Then (b) is an application of Prop. 10.5. When we substitute eq. (13.8) into eq. (10.14), we find, for $k > 0$,

$$\psi 1_k \otimes 1_S + \psi e_k \otimes u_k = 1_k \otimes 1_k \otimes 1_S + 1_k \otimes e_k \otimes u_k + e_k \otimes 1_k \otimes u_k,$$

since $u_k^2 = 0$. This gives (d). (But ψe_0 acquires the extra term $e_0 \otimes e_0$, because $u_0^2 \neq 0$; this is obvious anyway from Prop. 11.2. Also, (c), (d), and (g) are clearly false for $k = m = 0$.) Similarly, eq. (10.15) yields $1_S + (\epsilon e_k) u_k = 1_S$ (even for $k = 0$), which gives (e).

For (g), which includes (f) as the special case $\lambda = -1$ (by Prop. 10.12(a) and Prop. 11.2(d)), the distributive law (10.11) and (d) yield $e_k \circ [\lambda + \mu] = e_k \circ [\lambda] + e_k \circ [\mu]$ for all $\lambda, \mu \in E^0$. Since $e_k \circ [1] = e_k$, (g) follows. (We are in effect expanding $r(\lambda u_k)$.)

For (h), we substitute eq. (13.8) into eq. (10.45). On the left, we have

$$(sr)(u_k) = \langle s, r_* 1_k \rangle \, 1_S + \langle s, r_* e_k \rangle \, u_k,$$

while on the right, iteration of eq. (13.8) yields, after simplification,

$$s(r(u_k)) = \langle s, [\langle r, 1_k \rangle] \rangle \, 1_S + \langle s, [\langle r, 1_k \rangle] * [\langle r, e_k \rangle] \circ e_k \rangle \, u_k,$$

with the help of eqs. (10.16) and (10.23). Comparison of these gives $r_* e_k$.

For $k = 1$ in (i) and (j), we stabilize eq. (13.2) by Lemma 10.7 and compare with Defn. 12.1 and [8, (11.24)]. For general k, we use the multiplicative properties in diag. (6.16) of q_k and σ_{k*}. □

We have the analogue of Cor. 12.4. By Lemma 2.3(d), a based operation $r\colon k \to m$ is represented by a based map $r\colon (\underline{E}_k, o) \to (\underline{E}_m, o)$. We need to know its effect on homotopy groups.

Lemma 13.9. *Given a based operation $r\colon k \to m$, the induced homomorphism on homotopy groups*

$$E^{k-h} \cong \pi_h(\underline{E}_k, o) \xrightarrow{r_*} \pi_h(\underline{E}_m, o) \cong E^{m-h}$$

is given on $v \in E^{k-h}$ for any $h \geqslant 0$ by

$$r_* v = \langle r, [v] \circ e_h \rangle \qquad \text{in } E^{m-h} \ .$$

Proof. Viewed cohomologically, the element $v \in E^{k-h}$ or map $v\colon S^h \to \underline{E}_k$ corresponds to $vu_h \in E^k(S^h, o)$. From eqs. (10.16) and (13.8), we compute $r(vu_h) = \langle r, [v] \circ e_h \rangle\, u_h$, which simplified because r is based, so that $\langle r, 1_k \rangle = 0$. □

14. Complex orientation and additive operations

In this section, we study the effect of a complex orientation on additive operations. The relevant test space is $\mathbb{C}P^\infty$, for which $E^*(\mathbb{C}P^\infty) = E^*[[x]]$ by [8, Lemma 5.4], where $x = x(\xi)$ is the Chern class of the Hopf line bundle ξ. All the stable results carry over, almost without change, except that now $b_1 = e^2$ instead of 1.

We assume that $E_*(\underline{E}_k)$, $Q(E)^k_*$, and $E_*(E, o)$ are free E^*-modules.

Definition 14.1. We define elements $b_i \in Q(E)^2_{2i}$ for all $i \geqslant 0$ by the identity

$$\rho x = b(x) = \sum_{i=0}^{\infty} x^i \otimes b_i \qquad \text{in } E^*(\mathbb{C}P^\infty)\, \widehat{\otimes}\, Q(E)^2_* \cong Q(E)^2_*[[x]], \tag{14.2}$$

where $b(x)$ is a convenient formal abbreviation that rapidly becomes essential.

We use eq. (6.39) to convert eq. (14.2) to the equivalent form

$$rx = \sum_{i=0}^{\infty} \langle r, b_i \rangle\, x^i \qquad \text{in } E^*(\mathbb{C}P^\infty) = E^*[[x]], \text{ for all } r \ . \tag{14.3}$$

Since the Hopf bundle is universal, eqs. (14.2) and (14.3) hold for the Chern class $x = x(\theta)$ of any complex line bundle θ over any space X (after completion, if necessary).

Proposition 14.4. *The elements $b_i \in Q(E)^2_{2i}$ have the following properties:*

(a) *$b_0 = 0$ and $b_1 = e^2$, so that $b(x) = x \otimes e^2 + x^2 \otimes b_2 + x^3 \otimes b_3 + \dots$;*

(b) *The Chern class $x \in E^2(\mathbb{C}P^\infty)$, regarded as a map of spaces $x\colon \mathbb{C}P^\infty \to \underline{E}_2$, induces $q_2 x_* \beta_i = b_i \in Q(E)^2_{2i}$, where $\beta_i \in E_{2i}(\mathbb{C}P^\infty)$ is dual to x^i ;*

(c) *$Q(\psi) b_k$ is given by*

$$Q(\psi) b_k = \sum_{i=1}^{k} B(i,k) \otimes b_i \qquad \text{in } Q(E)^*_* \otimes Q(E)^2_* \ ,$$

where $B(i,k)$ denotes the coefficient of x^k in $b(x)^i$, or formally,

$$Q(\psi)b(x) = \sum_{i=1}^{\infty} b(x)^i \otimes b_i;$$

(d) $Q(\epsilon)b_k = 0$ *for* $k > 1$, *or formally,* $Q(\epsilon)b(x) = x$;

(e) *The stabilization* $Q(\sigma)\colon Q(E)^2_* \to E_*(E,o)$ *sends the element* $b_i \in Q(E)^2_{2i}$ *to the stable element* $b_i \in E_{2i-2}(E,o)$ *of* [8, Defn. 13.1].

Proof. For (a), we restrict eq. (14.2) to $\mathbb{C}P^1 \cong S^2$ and compare with eq. (12.2). For (b), we apply Prop. 6.44. For (c) and (d), we substitute ρ into diags. (6.33) and evaluate on x. For (e), we compare Defn. 14.1 with [8, Defn. 13.1]. □

Still following the stable strategy, we next apply ρ to the multiplication map $\mu\colon \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$, to obtain the formal identity

$$b(F(x,y)) = F_R(b(x),b(y)) = b(x) + b(y) + \sum_{i,j} b(x)^i\, b(y)^j\, \eta_R a_{i,j} \qquad (14.5)$$

in $Q(E)^*_*[[x,y]]$, which looks exactly like the stable version [8, (13.6)]. Again, $F_R(X,Y)$ is a convenient abbreviation. The consequences are the same.

The p-local case.

Lemma 14.6. *Assume that E^* is a p-local ring. Then the generator b_k of $Q(E)^*_*$ is redundant unless k is a power of p.*

Proof. The proof of [8, Lemma 13.7] applies without change. □

We therefore reindex the b's.

Definition 14.7. When E^* is a p-local ring, we define $b_{(i)} = b_{p^i}$ for each $i \geqslant 0$.

As in [8, §13], we obtain

$$b([p](x)) = [p]_R(b(x)) = pb(x) + \sum_{i>0} b(x)^{i+1}\eta_R g_i \qquad (14.8)$$

in $Q(E)^2_*[[x]]$, which looks exactly like the stable version [8, (13.11)] but is in a different place. Again, we extract the coefficient of x^{p^k}.

Definition 14.9. For each $k > 0$, we define the kth *main (additively unstable) relation* as

$$(\mathcal{R}_k): \qquad L(k) = R(k) \qquad \text{in } Q(E)^2_*\,, \qquad (14.10)$$

where $L(k)$ and $R(k)$ denote the coefficient of x^{p^k} in the left and right sides of eq. (14.8) respectively.

15. Complex orientation and unstable operations

In this section, we extend our study of the test space $\mathbb{C}P^\infty$ to all unstable operations. Everything we did in section 14 carries over, with a lot more complication but no essential difficulty. Again, it is enough to know $r(x)$ for all operations r, where $x = x(\xi) \in E^2(\mathbb{C}P^\infty)$ is the Chern class.

We assume that $E_*(\underline{E}_k)$ is a free E^*-module for all k.

Definition 15.1. We define elements $b_i \in E_{2i}(\underline{E}_2)$ for $i \geqslant 0$ by the identity

$$r(x) = \sum_{i=0}^{\infty} \langle r, b_i \rangle\, x^i = \langle r, b(x) \rangle \qquad \text{in } E^*(\mathbb{C}P^\infty) = E^*[[x]] \tag{15.2}$$

for all r, where we take x^i inside the $\langle\ ,\ \rangle$ and write formally $b(x) = \sum_i b_i x^i$.

We first determine how the elements b_k interact with the Hopf ring structure.

Proposition 15.3. *The elements $b_k \in E_{2k}(\underline{E}_2)$ of the Hopf ring $E_*(\underline{E}_*)$ have the properties:*

(a) *$b_0 = 1_2$ and $b_1 = e_2 = -e^{\circ 2}$, so that $b(x) = 1_2 + \overline{b}(x)$ if we define*

$$\overline{b}(x) = \sum_{i=1}^{\infty} b_i x^i \qquad \text{in } E_*(\underline{E}_2)[[x]]; \tag{15.4}$$

(b) *The universal Chern class $x \in E^2(\mathbb{C}P^\infty)$, regarded as a map $x\colon \mathbb{C}P^\infty \to \underline{E}_2$, induces $x_*\beta_k = b_k \in E_{2k}(\underline{E}_2)$, where $\beta_k \in E_{2k}(\mathbb{C}P^\infty)$ is dual to x^k* (as in [8, Lemma 5.4]);

(c) *$\psi b_k = \sum_{i+j=k} b_i \otimes b_j$, or formally, $\psi b(x) = b(x) \otimes b(x)$;*

(d) *$\epsilon b_k = 0$ if $k > 0$, and $\epsilon b_0 = 1$, or formally, $\epsilon b(x) = 1$;*

(e) *$\chi b(x) = (1_2 + \overline{b}(x))^{*(-1)} = 1_2 - \overline{b}(x) + \overline{b}(x)^{*2} - \overline{b}(x)^{*3} + \dots$;*

(f) *For all rational numbers $\lambda \in E^0$,*

$$b(x) \circ [\lambda] = (1_2 + \overline{b}(x))^{*\lambda} = 1_2 + \lambda\overline{b}(x) + \frac{\lambda(\lambda-1)}{2}\,\overline{b}(x)^{*2} + \dots; \tag{15.5}$$

(g) *For all r, $r_* b_k$ is given as the coefficient of x^k in the formal identity*

$$r_* b(x) = [\langle r, 1_2 \rangle] * \mathop{\ast}_{j=1}^{\infty} b(x)^{\circ j} \circ [\langle r, b_j \rangle] \qquad \text{in } E_*(\underline{E}_*)[[x]];$$

(h) *$q_2 b_k = b_k \in Q(E)^2_{2k}$, the additively unstable element in Defn. 14.1;*

(i) *$\sigma_{2*} b_k = b_k \in E_{2k-2}(E, o)$, the stable element in* [8, Defn. 13.1].

Remark. The sign in (a) is absent from [23, Prop. 2.4]. The commutativity of diag. (6.16) requires

$$Q(\phi)(q_1 \otimes q_1)(e \otimes e) = -(q_1 e)(q_1 e) = -q_2 b_{(0)} = -q_2 e_2 = q_2(e \circ e),$$

bearing in mind that $\deg(q_1) = 1$. The unexpected sign first appeared in Prop. 13.7(c).

Proof. Naturality for the inclusion $S^2 \cong \mathbb{C}P^1 \subset \mathbb{C}P^\infty$ gives (a), by comparison with Prop. 13.7. Part (b) comes from Prop. 10.5. We read off (c) and (d) from eqs. (10.14) and (10.15). Part (e) is the special case $\lambda = -1$ of (f). For (f), eq. (10.11) and (c) give $b(x) \circ [\lambda+\mu] = b(x) \circ [\lambda] * b(x) \circ [\mu]$ for all $\lambda, \mu \in E^*$. Since $b(x) \circ [1] = b(x)$ and we are working in the $*$-multiplicative group of formal power series over $E_*(\underline{E}_2)$ of the form $1 + \ldots$, which has no n-torsion if $1/n \in E^*$, the result follows. (We are in effect expanding $r(\lambda x)$; cf. eq. (10.16).) For (g), we apply eq. (10.45) to $x \in E^2(\mathbb{C}P^\infty)$ and expand. For (h) and (i), we stabilize eq. (15.2) by Prop. 6.11 and compare with the additive and stable versions, eq. (14.3) and [8, (13.3)]. □

From (c) and eq. (10.11), we deduce the convenient distributive law

$$(a * c) \circ b(x) = a \circ b(x) * c \circ b(x), \tag{15.6}$$

where a and c are allowed to involve x. This formal device will prove extremely useful for computations in Hopf rings. We have one immediate application to the Frobenius operator F defined by $Fc = c^{*p}$.

Corollary 15.7. *For any element c in the Hopf ring $E_*(\underline{E}_*)$,*

$$(Fc) \circ b_k \equiv \begin{cases} F(c \circ b_n) \bmod p, & \text{if } k = pn; \\ 0 \bmod p, & \text{if } k \text{ is not divisible by } p. \end{cases}$$

Proof. By iterating eq. (15.6) we have $(Fc) \circ b(x) = F(c \circ b(x))$. We pick out the coefficient of x^k, working mod p. □

We next study the naturality of operations with respect to the multiplication $\mu\colon \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$. We expand $\mu^* r(x) = r(\mu^* x)$ by the formal group law [8, (13.5)] and the Cartan formulae, to obtain the analogue of eq. (14.5). The complicated result is best expressed formally as

$$b(F(x,y)) = F_R(b(x), b(y)) = b(x) * b(y) * \mathop{\Large *}_{i,j} \left(b(x)^{\circ i} \circ b(y)^{\circ j} \circ [a_{i,j}] \right) \tag{15.8}$$

as in [23, Thm. 3.8(i)], where $F_R(X,Y) = X * Y * *_{i,j} X^{\circ i} \circ Y^{\circ j} \circ [a_{i,j}]$, in the sense that the $\circ$- and $*$-multiplications apply only to Hopf ring elements, not to x and y.

The p-local case. Lemma 14.6 carries over.

Lemma 15.9. *Assume that E^* is a p-local ring. Then the $\circ$-generator b_k of the Hopf ring $E_*(\underline{E}_*)$ is redundant unless k is a power of p.*

Proof. As before, we take the coefficient of $x^i y^j$ in eq. (15.8), where $i + j = k$. On the left, there is a term $\binom{k}{i} b_k$, from $b_k(x+y)^k$, and this is the highest b that occurs; on the right, no b beyond b_i or b_j occurs. We choose i and j as in [8, Lemma 13.7], to make $\binom{k}{i}$ not divisible by p and therefore invertible, which shows that b_k is redundant. □

We therefore reindex the b's as usual.

Definition 15.10. When E^* is a p-local ring, we define $b_{(i)} = b_{p^i}$ for each $i \geqslant 0$.

We extend standard multi-index notation slightly by defining

$$b^{\circ I} = b_{(0)}^{\circ i_0} \circ b_{(1)}^{\circ i_1} \circ b_{(2)}^{\circ i_2} \circ b_{(3)}^{\circ i_3} \circ \dots \tag{15.11}$$

for any multi-index $I = (i_0, i_1, i_2, \dots)$. We also need a shift operation.

Definition 15.12. Given a multi-index $I = (i_0, i_1, i_2, \dots)$, we define the *shifted* multi-index $s(I) = (0, i_0, i_1, i_2, \dots)$. We iterate this process h times, for any $h \geqslant 0$, to form $s^h(I) = (0, \dots, 0, i_0, i_1, i_2, \dots)$. We even undo it, by defining $s^{-1}(I) = (i_1, i_2, i_3, \dots)$, provided $i_0 = 0$; our convention is that this is *undefined* if $i_0 \neq 0$.

This notation allows us to iterate Cor. 15.7 neatly in the form

$$(Fc) \circ b^{\circ I} \equiv \begin{cases} F(c \circ b^{\circ s^{-1}(I)}) \bmod p & \text{if } i_0 = 0; \\ 0 \bmod p & \text{if } i_0 \neq 0. \end{cases} \tag{15.13}$$

We follow the stable plan and study instead of μ the much simpler p-th power map $\zeta\colon \mathbb{C}P^\infty \to \mathbb{C}P^\infty$. Naturality of the general operation r is expressed by $\zeta^* r(x) = r(\zeta^* x)$. When we substitute the p-series [8, (13.9)] and expand, we obtain, as in [23, Thm. 3.8(ii)],

$$b\left(px + \sum_i g_i x^{i+1}\right) = b(x)^{*p} * \mathop{\ast}_i b(x)^{\circ i+1} \circ [g_i] \tag{15.14}$$

in $E_*(\underline{E}_*)[[x]]$, or, in condensed notation, $b([p](x)) = [p]_R(b(x))$.

Definition 15.15. For each $k > 0$, we define the kth *main unstable relation* as

$$(\mathcal{R}_k): \qquad L(k) = R(k) \qquad \text{in } E_*(\underline{E}_2), \tag{15.16}$$

where $L(k)$ and $R(k)$ denote respectively the coefficient of x^{p^k} in the left and right sides of eq. (15.14).

Thus $L(k)$ is the coefficient of x^{p^k} in $b([p](x))$, exactly as in Defn. 14.9. However, $R(k)$ is vastly more complicated than before, and we study it in more detail in section 19 in the case $E = BP$. The work of Ravenel-Wilson [23], which we review in section 17, implies that, despite appearances, the relations $(\mathcal{R}_n)$ contain all the information present in eq. (15.8), with the understanding that we use eq. (15.8), by way of Lemma 15.9, only to express the redundant b_j's (which still appear in $\psi b_{(k)}$, $b_{(k)} \circ [\lambda]$, and $r_* b_{(k)}$) in terms of the $b_{(i)}$.

16. Examples for additive operations

In section 5, we developed a comonad to express all the structure of additive unstable E-cohomology operations, for favorable E. In section 6, we developed a bigraded algebra $Q(E)^*_*$ that contains equivalent information, where $Q(E)^k_i$ has degree $k - i$. In this section, we describe $Q(E)^*_*$ for each of our five cohomology theories $E^*(-)$, namely $E = H(\mathbb{F}_p)$, MU, BP, KU, and $K(n)$. (The first example splits into two,

and we break out the degenerate special case $K(0) = H(\mathbb{Q})$.) As stably in [8], our purpose is to exhibit the structure of the results, not to derive them.

All the results here are formally very close to the stable results. By Prop. 12.3(g), $Q(\sigma)e = 1$. As $E_*(E,o) = \operatorname{colim}_k Q(E)^k_*$ by eq. (4.8), where the suspensions $Q(E)^k_* \to Q(E)^{k+1}_*$ have been revealed in Lemma 12.5 as simply multiplication by e, we stabilize everything merely by setting the suspension element $e = 1$. In this way, we recover all the corresponding stable results. Indeed, in the case $E = KU$, we have to obtain the stable structure this way.

All four answers of section 5 are of course available, but the Second Answer remains the most practical, consisting as in Thm. 7.7 of the coactions

$$\rho_X \colon E^k(X) \longrightarrow E^*(X) \mathbin{\widehat{\otimes}} Q(E)^k_*.$$

These coactions are automatically additive, multiplicative (for cup products and $\times$-products), and unital ($\rho_X 1_X = 1_X \otimes 1$). (We simplify notation by suppressing redundant completions and suffixes.)

We use exactly the same test spaces and test maps as we did stably. The point remains that complete knowledge of the behavior of $E^*(-)$ on these is sufficient to suggest the correct structure of $Q(E)^*_*$ (except that the case $E = K(n)$ requires some extra work). By Prop. 6.42(b), the one-point space T in effect defines the right unit η_R, and the circle S^1 defines $e \in Q(E)^1_1$ by eq. (12.2). As all our examples have complex orientation, we have available the elements b_i of Defn. 14.1.

In each case, we list the generators and relations for the bigraded E^*-algebra $Q(E)^*_*$, describe the right unit η_R, and give the values of the algebra homomorphisms $\psi = Q(\psi) \colon Q(E)^*_* \to Q(E)^*_* \otimes Q(E)^*_*$ and $\epsilon = Q(\epsilon) \colon Q(E)^*_* \to E^*$ on each generator. In some cases, we can express the universal property of $Q(E)^*_*$ very simply. The stabilization $Q(\sigma)$ maps each generator to its stable namesake, except that of course $Q(\sigma)e = 1$.

Example: $H(\mathbb{F}_2)$. We take $E = H = H(\mathbb{F}_2)$, the Eilenberg-MacLane spectrum. Our test space is $\mathbb{R}P^\infty$, for which $H^*(\mathbb{R}P^\infty) = \mathbb{F}_2[t]$, a polynomial algebra on the generator $t \in H^1(\mathbb{R}P^\infty)$. We define elements $c_i \in Q(H)^1_*$ by the identity

$$\rho t = \sum_{i=0}^{\infty} t^i \otimes c_i \qquad \text{in } H^*(\mathbb{R}P^\infty) \mathbin{\widehat{\otimes}} Q(H)^1_* \cong Q(H)^1_*[[t]].$$

Restriction to $S^1 = \mathbb{R}P^1$ shows that $c_0 = 0$ and $c_1 = e$. As stably, the multiplication $\mu \colon \mathbb{R}P^\infty \times \mathbb{R}P^\infty \to \mathbb{R}P^\infty$ implies that $c_i = 0$ unless i is a power of 2. We therefore write $\xi_i = c_{2^i} \in Q(H)^1_{2^i}$ for each $i \geqslant 0$, so that

$$\rho t = \sum_{i=0}^{\infty} t^{2^i} \otimes \xi_i \qquad \text{in } H^*(\mathbb{R}P^\infty) \mathbin{\widehat{\otimes}} Q(H)^1_* \cong Q(H)^1_*[[t]], \tag{16.1}$$

which looks just like the stable version [8, (14.1)], except that now $\xi_0 = e$.

Theorem 16.2. *For the Eilenberg-MacLane ring spectrum $H = H(\mathbb{F}_2)$:*

(a) *$Q(H)^*_* = \mathbb{F}_2[\xi_0, \xi_1, \xi_2, \xi_3, \ldots]$, a polynomial algebra over $\mathbb{F}_2$ on generators $\xi_i \in Q(H)^1_{2^i}$ for $i \geqslant 0$, where $\xi_0 = e$;*

(b) *In the complex orientation for* $H(\mathbb{F}_2)$, $b_{(i)} = \xi_i^2$ *for all* $i \geqslant 0$, *and* $b_j = 0$ *if* j *is not a power of 2;*

(c) ψ *is given by*

$$\psi\xi_n = \sum_{i=0}^{n} \xi_{n-i}^{2^i} \otimes \xi_i \qquad \text{in } Q(H)_*^* \otimes Q(H)_*^1;$$

(d) ϵ *is given by* $\epsilon\xi_n = 0$ *for* $n > 0$ *and* $\epsilon\xi_0 = 1$.

Proof. Part (a) is of course a reformulation of classical results. For fixed k, the stabilization $Q(\sigma)\colon Q(H)_*^k \to H_*(H, o)$ is the monomorphism that is dual (with a shift in degree) to the well-known epimorphism $\sigma_k^*\colon H^*(H, o) \to PH^*(\underline{H}_k)$ that tells which Steenrod operations can act nontrivially on $H^k(-)$. The proof of (b) is the same as stably. We prove (c) and (d) by taking $M = H^*(\mathbb{R}P^\infty)$ in diags. (6.33) and evaluating on t. □

As stably in [8, §14], we combine the universal property of the polynomial algebra $\mathbb{F}_2[\xi_0, \xi_1, \xi_2, \ldots]$ with Thm. 7.7(b).

Corollary 16.3. *Let* B *be a discrete commutative graded* $\mathbb{F}_2$*-algebra. Assume that the ring homomorphism* $\theta\colon H^*(X) \to H^*(X) \mathbin{\widehat{\otimes}} B$ *is natural for spaces* X. *Then on* $t \in H^1(\mathbb{R}P^\infty)$, θ *has the form*

$$\theta t = \sum_{i=0}^{\infty} t^{2^i} \otimes \xi_i' \qquad \text{in } H^*(\mathbb{R}P^\infty) \mathbin{\widehat{\otimes}} B \cong B[[t]],$$

where the elements $\xi_i' \in B^{-(2^i-1)}$ *determine* θ *uniquely for all* X *and may be chosen arbitrarily.* □

Example: $H(\mathbb{F}_p)$ (for p odd)**.** We write $H = H(\mathbb{F}_p)$, the Eilenberg-MacLane spectrum. The complex orientation defines elements $\xi_i = b_{(i)}$ for $i \geqslant 0$, and, just as stably, $b_j = 0$ whenever j is not a power of p. The only difference now is that $\xi_0 = b_1 = e^2$ instead of 1.

The other test space is the lens space $L = K(\mathbb{F}_p, 1)$, for which $H^*(L) = \mathbb{F}_p[x] \otimes \Lambda(u)$. As x is a Chern class, $\rho_L x$ is given by eq. (14.2). This leaves only $\rho_L u$, which reduces (as stably) to

$$\rho_L u = u \otimes e + \sum_{i=0}^{\infty} x^{p^i} \otimes \tau_i \qquad \text{in } H^*(L) \mathbin{\widehat{\otimes}} Q(H)_*^1, \tag{16.4}$$

for certain elements τ_i that it defines.

Theorem 16.5. *For the Eilenberg-MacLane ring spectrum* $H = H(\mathbb{F}_p)$, *with* p *odd:*

(a) $Q(H)_*^*$ *is the commutative algebra over* $\mathbb{F}_p$ *with generators:*

$e \in Q(H)_1^1$, *a polynomial generator;*

$\xi_i \in Q(H)_{2p^i}^2$ *for all* $i \geqslant 0$, *a polynomial generator for* $i > 0$;

$\tau_i \in Q(H)_{2p^i}^1$ *for all* $i \geqslant 0$, *an exterior generator;*

subject to the relation $\xi_0 = e^2$*;*

(b) ψ *is given by* $\psi e = e \otimes e$*,*

$$\psi\xi_k = \sum_{i=0}^{k} \xi_{k-i}^{p^i} \otimes \xi_i \qquad \text{in } Q(H)_*^* \otimes Q(H)_*^2, \tag{16.6}$$

and

$$\psi\tau_k = \tau_k \otimes e + \sum_{i=0}^{k} \xi_{k-i}^{p^i} \otimes \tau_i \qquad \text{in } Q(H)_*^* \mathbin{\widehat{\otimes}} Q(H)_*^1;$$

(c) ϵ *is given by* $\epsilon e = 1$*,* $\epsilon\xi_i = 0$ *for* $i > 0$*, and* $\epsilon\tau_i = 0$ *for all* i*.*

Proof. Part (a) is again a reformulation of classical results, which may be recovered in this form from [27, Thm. 8.5], in somewhat different notation, by taking the indecomposables. We obtain (b) and (c) by substituting ρ_L in diags. (6.33) and evaluating on x and u. □

We have the analogue of Cor. 16.3.

Corollary 16.7. *Let* B *be a discrete commutative graded* $\mathbb{F}_p$*-algebra. Assume that the ring homomorphism* $\theta\colon H^*(X) \to H^*(X) \mathbin{\widehat{\otimes}} B$ *is natural for spaces* X*. Then on* $H^*(L) = \mathbb{F}_p[x] \otimes \Lambda(u)$*,* θ *has the form*

$$\theta x = x \otimes e'^2 + \sum_{i=1}^{\infty} x^{p^i} \otimes \xi_i'$$

$$\theta u = u \otimes e' + \sum_{i=0}^{\infty} x^{p^i} \otimes \tau_i'$$

where the elements $e' \in B^0$*,* $\xi_i' \in B^{-2(p^i-1)}$*, and* $\tau_i' \in B^{-(2p^i-1)}$ *determine* θ *uniquely for all* X *and may be chosen arbitrarily.* □

Example: $H(\mathbb{Q})$**.** We write $E = H = H(\mathbb{Q})$, the Eilenberg-MacLane spectrum. As always, there is the suspension element $e \in Q(H(\mathbb{Q}))_1^1$, whose properties we know from Prop. 12.3. There is nothing else.

Theorem 16.8. *For the ring spectrum* $H = H(\mathbb{Q})$*:*

(a) $Q(H)_*^* = \mathbb{Q}[e]$*, a polynomial algebra on* $e \in Q(H)_1^1$*;*

(b) *The coalgebra structure is given by* $\psi e = e \otimes e$ *and* $\epsilon e = 1$*.* □

Example: MU**.** The coefficient ring is $MU^* = \mathbb{Z}[x_1, x_2, x_3, \ldots]$, with a polynomial generator x_i in degree $-2i$ for each i. These give rise to the elements $\eta_R x_i \in Q(MU)_0^{-2i}$. We have complex orientation, almost by definition, and therefore the elements $b_i \in Q(MU)_{2i}^2$, with $b_0 = 0$ and $b_1 = e^2$. We have the relations (14.5) between the b's and the $\eta_R v$, but unlike the stable case, because e is no longer invertible, they do *not* render the generators $\eta_R x_i$ redundant. Implicit in [23, Cor. 4.6(a)] is that this is the whole story.

Theorem 16.9 (Ravenel-Wilson)**.** *For the unitary Thom ring spectrum MU:*

(a) *$Q(MU)^*_*$ is the commutative algebra over MU^* with generators:*

$\eta_R x_i \in Q(MU)_0^{-2i}$ *(for $i > 0$);*
$e \in Q(MU)_1^1$;
$b_i \in Q(MU)_{2i}^2$ *(for $i \geqslant 1$);*

all of even degree, subject to the relations (14.5) *and $b_1 = e^2$;*

(b) *ψ is given by $\psi e = e \otimes e$ and*

$$\psi b_k = \sum_{i=1}^{k} B(i,k) \otimes b_i \qquad \text{in } Q(MU)^*_* \otimes Q(MU)^2_* \ ,$$

where $B(i,k)$ denotes the coefficient of x^k in $b(x)^i$;

(c) *ϵ is given by $\epsilon e = 1$ and $\epsilon b_k = 0$ for $k > 1$.* □

Although we no longer have a polynomial algebra, part of Cor. 16.3 carries over. It applies equally well to the two following cases, which we include here.

Corollary 16.10. *Let B be a discrete commutative E^*-algebra, where $E = MU$, BP, or KU. Then a ring homomorphism $\theta\colon E^*(X) \to E^*(X) \mathbin{\widehat{\otimes}} B$ that is natural for spaces X is uniquely determined by its values on $E^*(S^1)$ and $E^*(\mathbb{C}P^\infty)$.* □

Example: *BP.* The coefficient ring is now $BP^* = \mathbb{Z}_{(p)}[v_1, v_2, v_3, ...]$, with polynomial generators v_n in degree $-2(p^n-1)$. We have complex orientation, but because BP^* is p-local, we need only the generators $b_{(i)} \in Q(BP)^2_{2p^i}$, where $b_{(0)} = e^2$. Again, [23, Cor. 4.6(b)] implies that this is all there is; in particular, eq. (14.5) is redundant, except to express the other b_j in terms of the $b_{(i)}$ and the elements v_i and $w_i = \eta_R v_i$.

Theorem 16.11 (Ravenel-Wilson)**.** *For the Brown-Peterson ring spectrum BP:*

(a) *$Q(BP)^*_*$ is the commutative algebra over BP^* with generators:*

$w_i = \eta_R v_i \in Q(BP)_0^{-2(p^i-1)}$ *(for $i > 0$);*
$e \in Q(BP)_1^1$;
$b_{(i)} \in Q(BP)_{2p^i}^2$ *(for $i \geqslant 0$);*

subject to the main relations $(\mathcal{R}_k)$ (from eq. (14.10)*) for $k > 0$ and $b_{(0)} = e^2$;*

(b) *ψ is given by $\psi e = e \otimes e$ and*

$$\psi b_{(k)} = \sum_{i=1}^{p^k} B(i,p^k) \otimes b_i \qquad \text{in } Q(BP)^*_* \otimes Q(BP)^2_*,$$

where $B(i,p^k)$ denotes the coefficient of x^{p^k} in $b(x)^i$;

(c) *ϵ is given by $\epsilon e = 1$ and $\epsilon b_{(k)} = 0$ (for $k > 0$).* □

We discuss the structure of $Q(BP)^*_*$ in more detail in section 18.

Remark. Alternatively, we could use the generator h_i instead of $b_{(i)}$ as in [6]; however, Quillen's element t_i (see [21] or Adams [1, II.16]) *does not exist* in this context for $i > 1$, for lack of conjugation in $Q(BP)^*_*$.

Example: KU. We take $E = KU$, the complex Bott spectrum, with the coefficient ring $KU^* = \mathbb{Z}[u, u^{-1}]$ (where $u \in KU^{-2}$), right unit $\eta_R\colon KU^* \to Q(KU)^*_*$ given by $\eta_R u = v$, and Chern class x given by [8, (5.2)]. The simple form [8, (5.16)] of the formal group law reduces eq. (14.5) to

$$b(x + y + uxy) = b(x) + b(y) + b(x)b(y)v, \tag{16.12}$$

which looks like the stable version [8, (14.13)], with $b(x) = b_1 x + b_2 x^2 + b_3 x^3 + \dots$, except that now $b_1 = e^2 \neq 1$. The coefficient of $x^i y^j$ yields the relation

$$b_i b_j = \sum_{k=0}^{\min(i,j)} \binom{i+j-k}{i} \binom{i}{k} u^k b_{i+j-k} v^{-1}, \tag{16.13}$$

like [8, (14.15)], except that the case $i = 1$ now gives the reduction formula

$$b_1 b_i = (i+1) b_{i+1} v^{-1} + iu b_i v^{-1} \qquad \text{for } i > 0. \tag{16.14}$$

The results here are much clearer than in the stable case, and there is some overlap with the work of tom Dieck [10].

Theorem 16.15. *For the complex Bott spectrum KU:*

(a) *$Q(KU)^*_*$ is generated as an algebra over $KU^* = \mathbb{Z}[u, u^{-1}]$ by the elements:*

$v = \eta_R u \in Q(KU)^{-2}_0$;

$v^{-1} = \eta_R u^{-1} \in Q(KU)^2_0$;

$e \in Q(KU)^1_1$, *the suspension element;*

$b_i \in Q(KU)^2_{2i}$ *for $i > 0$;*

subject to the relations $b_1 = e^2$ and (16.13) for $i > 0$, $j > 0$;

(b) *$Q(KU)^*_*$ is a free KU^*-module, with a basis consisting of all monomials of the forms v^n, $b_i v^n$, ev^n, and $eb_i v^n$, for $i > 0$ and $n \in \mathbb{Z}$;*

(c) *ψ is given by $\psi e = e \otimes e$ and*

$$\psi b_k = \sum_{i=1}^{k} B(i,k) \otimes b_i \qquad \text{in } Q(KU)^*_* \otimes Q(KU)^2_*,$$

where $B(i,k)$ denotes the coefficient of x^k in $b(x)^i$;

(d) *ϵ is given by $\epsilon e = 1$ and $\epsilon b_k = 0$ for all $k > 1$.*

Proof. We start with (b). We take the Hopf line bundle ξ over $\mathbb{C}P^\infty$ and regard the element $u^{-1}[\xi] \in KU^2(\mathbb{C}P^\infty)$ as a map $f\colon \mathbb{C}P^\infty \to \underline{KU}_2 = \mathbb{Z} \times BU$. By Lemma 4.6, f induces an isomorphism of KU^*-modules

$$KU_*(\mathbb{C}P^\infty) \longrightarrow QKU_*(\mathbb{Z} \times BU) \cong KU^* \oplus QKU_*(BU),$$

which we compute. By the definition [8, (5.2)] of the Chern class x, $u^{-1}[\xi] = u^{-1}+x$ in $KU^2(\mathbb{C}P^\infty)$; geometrically, the components of f are the map $\mathbb{C}P^\infty \to \mathbb{Z}$ with image 1, and $x\colon \mathbb{C}P^\infty \to BU$.

Thus $q_2 f_*\beta_0 = v^{-1}$ and $q_2 f_*\beta_i = q_2 x_*\beta_i = b_i$ for $i > 0$, with the help of Prop. 14.4(b); we have the desired basis of $Q(KU)^2_*$. For $Q(KU)^{2n}_*$, we multiply by v^{-n+1}, an isomorphism.

For the odd case, the description of $KU_*(U)$ in [8, Cor. 5.12] in terms of the Bott map $b\colon \Sigma(\mathbb{Z}\times BU) \to U$ shows that multiplication by e induces an isomorphism $Q(KU)^{2n}_* \cong Q(KU)^{2n+1}_*$.

We have specified enough relations to reduce any monomial in the b's, e, v, and v^{-1} to a linear combination of the elements in (b), which proves (a). Parts (c) and (d) are included in Props. 14.4 and 12.3. □

Now that we know the additive situation, we return to finish off the stable case. We may discard the odd spaces in eq. (4.8) and write

$$KU_*(KU, o) = \operatorname*{colim}_n Q(KU)^{2n}_*.$$

Corollary 16.16. *In the stable algebra $KU_*(KU, o)$:*

(a) *Every element of $KU_*(KU, o)$ of even degree can be written in the form*

$$c = u^q(\lambda_1 u^{-1} + \lambda_2 u^{-2} b_2 + \ldots + \lambda_n u^{-n} b_n)v^{-m}$$

for some integers q, m, n, and λ_i;

(b) *This element $c = 0$ if and only if $\lambda_i = 0$ for all i.*

Proof. By Thm. 16.15(b), we can write the general element of $Q(KU)^{2m+2}_{2q}$ uniquely in the form

$$c = u^q\left(\lambda_0 v^{-1} + \sum_{i=1}^{n} \lambda_i u^{-i} b_i\right)v^{-m}$$

with integer coefficients. Since $e^2 = b_1$, eq. (16.14) yields

$$e^2 c = u^{q+1}\left(\lambda_0 u^{-1} b_1 + \sum_{i=1}^{n}(i+1)\lambda_i u^{-i-1} b_{i+1} + \sum_{i=1}^{n} i\lambda_i u^{-i} b_i\right)v^{-m-1}$$

in $Q(KU)^{2m+4}_{2q+2}$, which gives (a). Further, $e^2 c = 0$ only if $c = 0$, which implies (b). □

Example: $K(n)$. The coefficient ring is $K(n)^* = \mathbb{F}_p[v_n, v_n^{-1}]$, where $v_n \in K(n)^{-2(p^n-1)}$. We write $w_n = \eta_R v_n$, as we did for BP. Obviously, w_n and v_n are no longer equal as they were stably, because they lie in different groups.

We have a complex orientation, and therefore the usual elements b_j. Because $K(n)^*$ is p-local, we need only the $b_{(i)}$ for $i \geqslant 0$. (In fact, $b_j = 0$ if j is not a power of p and $j < p^n$, for dimensional reasons, but not in general if $j > p^n$.) When we apply ρ to the p-th power map $\zeta\colon \mathbb{C}P^\infty \to \mathbb{C}P^\infty$, which induces $\zeta^* x = v_n x^{p^n}$ as in [8, (14.26)], we obtain $b_j^{p^n} w_n = v_n^j b_j$, and therefore

$$b_{(i)}^{p^n} = v_n^{p^i} b_{(i)} w_n^{-1} \qquad \text{in } Q(K(n))^{2p^n}_* \tag{16.17}$$

for $i \geqslant 0$. This stabilizes to [8, (14.27)].

In particular, $b_{(0)}^{p^n} = v_n b_{(0)} w_n^{-1}$. As always, $b_{(0)} = e^2$. A more sophisticated analysis, involving other cohomology theories as in [28, Prop. 1.1(j)], shows that this relation can be desuspended once to give

$$eb_{(0)}^{p^n-1} = v_n e w_n^{-1} \qquad \text{in } Q(K(n))_*^{2p^n-1}. \tag{16.18}$$

The other test space is the skeleton $Y = L^{2p^n-1}$ of the lens space L, for which $K(n)^*(Y) = \Lambda(u) \otimes K(n)^*[x : x^{p^n} = 0]$. We know $\rho_Y x$, because x is inherited from $\mathbb{C}P^\infty$. As stably, we define elements $a_i, c_i \in Q(K(n))_*^1$ by the coaction

$$\rho_Y u = \sum_{i=0}^{p^n-1} x^i \otimes a_i + \sum_{i=0}^{p^n-1} ux^i \otimes c_i \ . \tag{16.19}$$

By restriction to $S^1 \subset Y$, we see that $a_0 = 0$ and $c_0 = e$. Then eq. (16.18) is equivalent to the statement $\rho_Y y = y \otimes e$, where $y = v_n u x^{p^n-1} \in K(n)^*(Y)$; in other words, y behaves like $u_1 \in K(n)^1(S^1)$. The same partial multiplications $\mu : L^{2k+1} \times L^{2m} \to Y$ as in [8, §14] show that $c_i = 0$ for all $i > 0$ and that $a_i = 0$ for i not a power of p. We therefore reindex, as usual.

Definition 16.20. We define $a_{(i)} = a_{p^i} \in Q(K)_{2p^i}^1$, for $0 \leqslant i < n$.

In the new notation,

$$\rho_Y u = u \otimes e + \sum_{i=0}^{n-1} x^{p^i} \otimes a_{(i)} \qquad \text{in } K(n)^*(Y) \otimes Q(K(n))_*^1 \ . \tag{16.21}$$

Having odd degree, the $a_{(i)}$ are exterior generators of $Q(K(n))_*^*$. This is not all; we again appeal to [28, Prop. 1.1(i)] to find that one more factor e can be squeezed out of eq. (16.18) if we first multiply by $a_{(0)}$, to give the relation

$$a_{(0)} b_{(0)}^{p^n-1} = v_n a_{(0)} w_n^{-1} \qquad \text{in } Q(K(n))_*^{2p^n-1}. \tag{16.22}$$

Theorem 16.23. *For the Morava K-theory ring spectrum $K(n)$:*

(a) *$Q(K(n))_*^*$ is the commutative bigraded algebra over $K(n)^* = \mathbb{F}_p[v_n, v_n^{-1}]$, where $v_n \in K(n)^{-2(p^n-1)}$, with generators:*

$w_n = \eta_R v_n \in Q(K(n))_0^{-2(p^n-1)}$;
$w_n^{-1} = \eta_R v_n^{-1}$;
$e \in Q(K(n))_1^1$;
$a_{(i)} \in Q(K(n))_{2p^i}^1$ *(for $0 \leqslant i < n$);*
$b_{(i)} \in Q(K(n))_{2p^i}^2$ *(for $i \geqslant 0$);*

subject to the relations $b_{(0)} = e^2$, (16.17), (16.18), and (16.22);

(b) *ψ is given by $\psi e = e \otimes e$,*

$$\psi a_{(k)} = a_{(k)} \otimes e + \sum_{i=0}^{k} b_{(k-i)}^{p^i} \otimes a_{(i)} \qquad \text{in } Q(K(n))_*^* \otimes Q(K(n))_*^1, \tag{16.24}$$

and

$$\psi b_{(k)} = \sum_{i=1}^{p^k} B(i,p^k) \otimes b_i \qquad \text{in } Q(K(n))^*_* \otimes Q(K(n))^2_*, \tag{16.25}$$

where $B(i,p^k)$ denotes the coefficient of x^{p^k} in $b(x)^i$ (and Lemma 14.6 is used to express $b(x)$ in terms of the $b_{(j)}$, v_n, and w_n);

(c) *ϵ is given by $\epsilon e = 1$, $\epsilon a_{(k)} = 0$ (for $k \geqslant 0$), and $\epsilon b_{(k)} = 0$ (for $k > 0$).*

Proof. The algebra structure (a) is implicit in the main theorem of [28], by taking indecomposables. As always, we obtain $\psi a_{(i)}$ and $\epsilon a_{(i)}$ by evaluating the coaction axioms (6.33) on $u \in K(n)^*(Y)$. The rest of (b) and (c) can be obtained similarly, or by appealing to Props. 12.3 and 14.4. □

Corollary 16.26. *Let B be a discrete commutative $K(n)^*$-algebra. Then a ring homomorphism $\theta\colon K(n)^*(X) \to K(n)^*(X) \mathbin{\widehat{\otimes}} B$ that is natural for spaces X is uniquely determined by its values on $K(n)^*(\mathbb{C}P^\infty)$ and $K(n)^*(Y)$.* □

Remark. If $k \leqslant n$, eq. (16.25) simplifies just as in [8, Thm. 14.32] to

$$\psi b_{(k)} = \sum_{i=1}^{k} b_{(k-i)}^{p^i} \otimes b_{(i)} \qquad \text{in } Q(K(n))^*_* \otimes Q(K(n))^2_* \ ,$$

which resembles eq. (16.6).

17. Examples for unstable operations

In this section, we discuss the enriched Hopf ring for each of our five cohomology theories $E^*(-)$, namely for $E = H(\mathbb{F}_p)$, MU, BP, KU, and $K(n)$. According to section 10, this is what we need to handle general unstable operations. As in section 16, we divide the case $H(\mathbb{F}_p)$ in two and treat $K(0) = H(\mathbb{Q})$ separately. Even more than before, our intent is to exhibit the structure of the results, not to reestablish them.

Our strategy is the same as in the stable and additive contexts, using exactly the same test spaces and test maps. Each E has a complex orientation, which provides by Defn. 15.1 the elements b_i of the Hopf ring, in addition to e and the $[v]$. We have $\chi[1] = [-1]$ by Prop. 11.2(d), and its properties were listed in Prop. 10.12.

As pointed out in (10.46), we need more than just the Hopf ring and the elements $[v]$. The elements $Q(\epsilon)q_k c = \epsilon_S \sigma_{k*} c$ are given by section 16. We also need $r_* c$ for each operation r; by Thms. 10.19(c) and 10.33(c), it is in principle enough to know these for each $\circ$-generator c.

Our presentation changes somewhat from section 16. Each family of $\circ$-generators has its own Proposition, which lists all the pertinent information. It is therefore sufficient to describe each Hopf ring by listing its $\circ$-generators and the defining relations, and to refer to these propositions for further details. We recover all the results for additive operations merely by taking the indecomposables.

Example: MU**.** We recall that $MU^* = \mathbb{Z}[x_1, x_2, x_3, \ldots]$, where $\deg(x_i) = -2i$, is better described as generated by the elements $a_{i,j}$, as in [8, §14]. We have the elements b_i, as well as e and $[v] = \eta_R(v)$. Stably, [8, (13.6)] gave an inductive formula for $\eta_R a_{i,j}$ in terms of MU^* and the b_i. Unstably, eq. (15.8) is only a relation between these elements. Corollary 4.6(a) of [23] says in effect that this is all there is.

Theorem 17.1. (Ravenel-Wilson) *For the unitary cobordism ring spectrum* MU, $MU_*(\underline{MU}_*)$ *is the Hopf ring over* $MU^* = \mathbb{Z}[x_1, x_2, x_3, ...]$ *with* $\circ$*-generators:*

$[x_i] \in MU_0(\underline{MU}_{-2i})$ *for each* $i > 0$ (see Prop. 11.2)*;*
$e \in MU_1(\underline{MU}_1)$ (see Prop. 13.7)*;*
$b_i \in MU_{2i}(\underline{MU}_2)$ *for* $i \geqslant 1$ (see Prop. 15.3)*;*

subject to the relations $e^{\circ 2} = -b_1$ *and eq.* (15.8). □

Example: BP**.** The main reference is still [23]. As BP^* is p-local, Lemma 15.9 and Defn. 15.10 apply, to define the elements $b_{(i)}$ of the Hopf ring. We have as always e and the elements $[v]$ for each $v \in BP^*$.

Theorem 17.2. (Ravenel-Wilson) *For the Brown-Peterson ring spectrum* BP, $BP_*(\underline{BP}_*)$ *is the Hopf ring over* $BP^* = \mathbb{Z}_{(p)}[v_1, v_2, v_3, \ldots]$ *with the* $\circ$*-generators:*

$[\lambda] \in BP_0(\underline{BP}_0)$*, for each* $\lambda \in \mathbb{Z}_{(p)}$ (see Prop. 11.2)*;*
$[v_i] \in BP_0(\underline{BP}_{-2(p^n-1)})$*, for* $i > 0$ (see Prop. 11.2)*;*
$e \in BP_1(\underline{BP}_1)$ (see Prop. 13.7)*;*
$b_{(i)} \in BP_{2p^i}(\underline{BP}_2)$ *for* $i \geqslant 0$ (see Prop. 15.3)*;*

subject to the relations $[\lambda] \circ [\lambda'] = [\lambda\lambda']$, $[\lambda] * [\lambda'] = [\lambda + \lambda']$, $e \circ [\lambda] = \lambda e$, $b_{(i)} \circ [\lambda] = \ldots$ (see Prop. 15.3(f)), $e^{\circ 2} = -b_{(0)}$*, and the main relations* $(\mathcal{R}_n)$ *for* $n > 0$ *as in eq.* (15.16).

We implicitly use eq. (15.8), but only to express inductively the b_j, for j not a power of p, in terms of the $b_{(i)}$, v, and $[v]$; this is needed for computing $\psi b_{(i)}$, $\chi b_{(i)}$, $b_{(i)} \circ [\lambda]$, and $r_* b_{(i)}$.

Proof. This is the content of [23, Cor. 4.6(b)]. By Prop. 11.2, each $[v]$ for $v \in BP^*$ can be expressed in terms of the $[\lambda]$ and $[v_i]$; we have enough generators. The listed relations come from Props. 11.2, 13.7, and 15.3, and eq. (15.16). This reduces the $*$-generators (see section 10) to three types:

$$\begin{aligned} &\text{(i)} \quad b^{\circ I} \circ [v^J]; \\ &\text{(ii)} \quad e \circ b^{\circ I} \circ [v^J]; \\ &\text{(iii)} \quad [\lambda v^J]; \end{aligned} \tag{17.3}$$

in terms of the multi-index notation $b^{\circ I}$ introduced in eq. (15.11).

For each k, the $*$-generators that lie in $BP_*(\underline{BP}_k)$ generate it as a BP^*-algebra. Assume first that k is even, so that we have only types (i) and (iii). We write $\underline{BP}_k = BP^k \times \underline{BP}'_k$ as in Lemma 4.17; then

$$BP_*(\underline{BP}_k) \cong BP_*(BP^k) \otimes BP_*(\underline{BP}'_k), \tag{17.4}$$

where we recognize the first factor as the group ring over BP^* of the abelian group BP^k with basis elements $[v]$ for $v \in BP^k$. The type (i) generators lie in $BP_*(\underline{BP'}_k)$ and the type (iii) in $BP_*(BP^k)$, which is described by Lemma 4.4. Because $[\lambda v^J] * [\lambda' v^J] = [(\lambda+\lambda')v^J]$, we have enough relations for the type (iii) generators. The work of [23] reduces the type (i) generators to certain *allowable* generators $b^{\circ I} \circ [v^J]$, which form a system of polynomial generators $BP_*(\underline{BP'}_k)$. Since this reduction (see section 19) uses only the relations $(\mathcal{R}_n)$, we have enough relations.

If k is odd, only generators of type (ii) occur. These reduce similarly to the allowable generators of type (ii), which are exterior generators of $BP_*(\underline{BP}_k)$. □

Example: $H(\mathbb{Q})$. This example is of course classical.

Theorem 17.5. *For the ring spectrum $H = H(\mathbb{Q})$, $H_*(\underline{H}_*)$ is the Hopf ring over $\mathbb{Q}$ with generators:*

$[\lambda] \in H_0(\underline{H}_0)$ *for each* $\lambda \in \mathbb{Q}$ (see Prop. 11.2);
$e \in H_1(\underline{H}_1)$ (see Prop. 13.7);

subject to the relations $[\lambda] \circ [\lambda'] = [\lambda\lambda']$, $[\lambda] * [\lambda'] = [\lambda+\lambda']$, *and* $e \circ [\lambda] = \lambda e$.

Proof. For $k < 0$, $\underline{H}_k = T$, and we have only the $\mathbb{Q}$-basis element 1_k.

For $k = 0$, $\underline{H}_0 = \mathbb{Q}$, regarded as a discrete group, and the group ring $H_*(\underline{H}_0) = \mathbb{Q}[\mathbb{Q}]$ has a basis consisting of the elements $[\lambda]$. The first two relations, from Prop. 11.2, show how these multiply.

For $k > 0$, the third relation, from Prop. 13.7(g), reduces us to the single $*$-generator $e^{\circ k} \in H_k(\underline{H}_k)$ of $H_*(\underline{H}_k)$. We have the polynomial algebra $\mathbb{Q}[e^{\circ k}]$ if k is even, or the exterior algebra $\Lambda(e^{\circ k})$ if k is odd. □

Example: $H(\mathbb{F}_2)$. We write $H = H(\mathbb{F}_2)$. As $H_*(\underline{H}_*)$ is a Hopf ring over $\mathbb{F}_2$, we have the Frobenius operator F and the Verschiebung V.

We imitate Defns. 15.1 and 15.10 in a mod 2 version, using the same test space $\mathbb{R}P^\infty = K(\mathbb{F}_2, 1)$ as before, for which $H^*(\mathbb{R}P^\infty) = \mathbb{F}_2[t]$. We define $c_i \in H_i(\underline{H}_1) = H_i(\mathbb{R}P^\infty)$ for $i \geqslant 0$ by the identity

$$r(t) = \sum_{i=0}^{\infty} \langle r, c_i \rangle\, t^i = \langle r, c(t) \rangle \qquad \text{in } H^*(\mathbb{R}P^\infty) \qquad (\text{for all } r), \tag{17.6}$$

where we write formally $c(t) = \sum_i c_i t^i$ as in Defn. 15.1. In other words, c_i is dual to t^i and the elements c_i form an $\mathbb{F}_2$-basis of $H_*(\underline{H}_1)$.

We are primarily interested in the accelerated elements $c_{(i)} = c_{2^i}$. As before, we have the suspension element e. The complex orientation provides elements b_i which are redundant, as in section 16.

Proposition 17.7. *The Hopf ring elements $c_i \in H_i(\underline{H}_1)$ (for $i \geqslant 0$) and $c_{(i)} = c_{2^i} \in H_{2^i}(\underline{H}_1)$ (for $i \geqslant 0$) have the following properties:*

(a) $c_0 = 1_1$ *and* $c_{(0)} = c_1 = e$;
(b) $\psi c_k = \sum_{i+j=k} c_i \otimes c_j$, *or formally,* $\psi c(t) = c(t) \otimes c(t)$;
(c) $V c_{(i)} = c_{(i-1)}$ *for* $i > 0$, *and* $V c_{(0)} = 0$;

(d) $\epsilon c_k = 0$ *if* $k > 0$, *and* $\epsilon c_0 = 1$, *or formally,* $\epsilon c(t) = 1$;

(e) $\chi c(t) = c(t)^{*(-1)}$, *expanded as in Prop. 15.3(e);*

(f) $c_i * c_j = \binom{i+j}{i} c_{i+j}$;

(g) $Fc_{(i)} = c_{(i)} * c_{(i)} = 0$;

(h) $b_i = c_i \circ c_i$ *in* $H_{2i}(\underline{H}_2)$;

(i) *For all* r, $r_* c_k$ *is the coefficient of* t^k *in the formal identity*

$$r_* c(t) = \mathop{\ast}_{j=0}^{\infty} c(t)^{\circ j} \circ [\langle r, c_j \rangle] \qquad \text{in } H_*(\underline{H}_*)[[t]];$$

(j) $q_1 c_{(i)} = \xi_i$ *in* $Q(H)^1_*$, *and* $q_1 c_j = 0$ *if* j *is not a power of 2;*

(k) $\sigma_{1*} c_{(i)} = \xi_i$ *in* $H_*(H, o)$, *and* $\sigma_{1*} c_j = 0$ *if* j *is not a power of 2.*

Proof. The naturality of r for the multiplication $\mu\colon \mathbb{R}P^\infty \times \mathbb{R}P^\infty \to \mathbb{R}P^\infty$, which induces $\mu^* t = t \times 1 + 1 \times t$, yields the identity

$$\sum_k \langle r, c_k \rangle \, (t \times 1 + 1 \times t)^k = \sum_i \sum_j \langle r, c_i * c_j \rangle \, t^i \times t^j$$

in $H^*(\mathbb{R}P^\infty \times \mathbb{R}P^\infty) = \mathbb{F}_2[t \times 1, 1 \times t]$, with the help of the Cartan formula (10.23). The coefficient of $t^i \times t^j$ gives (f). The special case (g) of (f) also follows from eq. (10.32). We expand $r(t^2)$ for the Chern class t^2 by eqs. (17.6) and (10.36) and compare with eq. (15.2); most terms cancel, to give (h).

The other parts are formally as in Prop. 15.3, with all degrees halved, except that (c) is immediate from (b). □

Just as in Lemma 15.9, except that everything is now explicit in (f), c_j is redundant unless j is a power of 2. This leads to the following elegant description of the Hopf ring, which is a reformulation of classical results.

Theorem 17.8. *For the Eilenberg-MacLane ring spectrum* $H = H(\mathbb{F}_2)$, $H_*(\underline{H}_*)$ *is the Hopf ring over* $\mathbb{F}_2$ *with generators* $c_{(i)} \in H_{2^i}(\underline{H}_1)$ *for* $i \geqslant 0$ (see Prop. 17.7), *subject to the relation* $[1]^{*2} = 1_0$.

Proof. By Prop. 17.7(c), we can write $c^{\circ I} = Vc^{\circ s(I)}$ for any multi-index $I = (i_0, i_1, i_2, \ldots)$. Then $Fc^{\circ I} = F([1] \circ c^{\circ I}) = F[1] \circ c^{\circ s(I)} = 0$ by eq. (10.13), as in eq. (15.13), and $H_*(\underline{H}_k)$ is an exterior algebra on those generators $c^{\circ I}$ for which $\sum_t i_t = k$. Here, $c^{\circ I}$ is dual to the primitive element $\mathrm{Sq}^{i_1, i_2, \cdots} \iota_k$ in cohomology (in terms of the Milnor basis [18] of $H^*(H, o)$). (The index i_0 serves only as padding, to ensure that $i_1 + i_2 + i_3 + \ldots \leqslant k$.) □

Example: $H(\mathbb{F}_p)$ (for p odd)**.** We write $H = H(\mathbb{F}_p)$. We have, as always, the suspension element e. The complex orientation defines elements b_i for all $i \geqslant 0$; but Lemma 15.9 shows that only the $b_{(i)} = b_{p^i}$ for $i \geqslant 0$ are needed. Also, $b_0 = 1_2$ and $b_1 = e_2 = -e^{\circ 2}$. However, the b_j for j not a power of p do not vanish, but satisfy $b_i * b_j = \binom{i+j}{i} b_{i+j}$, which is all that survives from eq. (15.8). In particular, $b^{*p}_{(i)} = 0$ for all $i > 0$, as is also clear from eq. (10.32) applied to x.

For the other test space $L = K(\mathbb{F}_p, 1)$, we have $H^*(L) = \mathbb{F}_p[x] \otimes \Lambda(u)$. We only need to know $r(u)$. We define elements $a_i \in H_{2i}(\underline{H}_1)$ and $c_i \in H_{2i+1}(\underline{H}_1)$ by

$$r(u) = \sum_{i=0}^{\infty} \langle r, a_i \rangle\, x^i + \sum_{i=0}^{\infty} \langle r, c_i \rangle\, ux^i \qquad \text{in } H^*(L),$$

which we condense formally to $\langle r, a(x) \rangle + \langle r, c(x) \rangle\, u$ by writing $a(x) = \sum_i a_i x^i$ and $c(x) = \sum_i c_i x^i$. Thus a_i is dual to x^i, c_i is dual to ux^i, and the a_i and c_i form a basis of $H_*(\underline{H}_1)$.

Again, we accelerate the indexing by defining $a_{(i)} = a_{p^i}$ for $i \geqslant 0$.

Proposition 17.9. *The Hopf ring elements $a_i \in H_{2i}(\underline{H}_1)$, $a_{(i)} = a_{p^i} \in H_{2p^i}(\underline{H}_1)$, and $c_i \in H_{2i+1}(\underline{H}_1)$, (for $i \geqslant 0$), have the following properties:*

(a) $a_0 = 1_1$ *and* $c_0 = e$;

(b) $\psi a_k = \sum_{i+j=k} a_i \otimes a_j$;

(c) $Va_{(i)} = a_{(i-1)}$ *for* $i > 0$, *and* $Va_{(0)} = 0$;

(d) $\epsilon a_k = 0$ *for all* $k > 0$;

(e) $\chi a(x) = a(x)^{*(-1)}$, *expanded as in Prop. 15.3(e);*

(f) $a_i * a_j = \binom{i+j}{i} a_{i+j}$;

(g) $Fa_{(i)} = a_{(i)}^{*p} = 0$;

(h) $c_i = e * a_i$;

(i) *For all* r, $r_* a_k$ *is the coefficient of* x^k *in the formal identity*

$$r_* a(x) = \mathop{\ast}_{i=0}^{\infty} b(x)^{\circ\, i} \circ [\langle r, a_i \rangle] * \mathop{\ast}_{i=0}^{\infty} a(x) \circ b(x)^{\circ\, i} \circ [\langle r, c_i \rangle] \qquad \text{in } H_*(\underline{H}_*)[[x]];$$

(j) $q_1 a_{(i)} = \tau_i$ *in* $Q(H)^1_*$, *and* $q_1 a_j = 0$ *if* j *is not a power of* p;

(k) $\sigma_{1*} a_{(i)} = \tau_i$ *in* $H_*(H, o)$, *and* $\sigma_{1*} a_j = 0$ *if* j *is not a power of* p.

Proof. We consider naturality of operations with respect to the multiplication $\mu\colon L \times L \to L$, for which $\mu^* u = u \times 1 + 1 \times u$. In condensed notation, we compare

$$\mu^* r(u) = \langle r, a(x \times 1 + 1 \times x) \rangle + \langle r, c(x \times 1 + 1 \times x) \rangle\, (u \times 1 + 1 \times u)$$

with $r(\mu^* u)$, which we expand by eq. (10.23) as

$$\begin{aligned} r(\mu^* u) = {} & \langle r, a(x \times 1) * a(1 \times x) \rangle + \langle r, c(x \times 1) * a(1 \times x) \rangle\, u \times 1 \\ & + \langle r, a(x \times 1) * c(1 \times x) \rangle\, 1 \times u + \langle r, c(x \times 1) * c(1 \times x) \rangle\, u \times u. \end{aligned}$$

The coefficient of $x^i \times x^j$ gives (f), which implies (g). (Alternatively, (g) follows from eq. (10.32) applied to u.) The coefficient of $u \times x^i$ gives (h). The other parts require no new ideas. □

In particular, all the c_i and most of the a_i are redundant. We trivially have the relation $[1]^{*p} = [p] = [0] = 1$, from which it follows, as in the previous example, that $(a^{\circ\, I})^{*p} = 0$ and $(b^{\circ\, I})^{*p} = 0$ for all I. Once again, this is the whole story.

A detailed exposition by Ravenel and Wilson from this point of view is presented in [27, Thm. 8.5] (with slightly different notation: $a_{(i)}$ is written $\alpha_{(i)}$, and $b_{(i)}$ is written $\beta_{(i)}$).

Theorem 17.10. (Ravenel-Wilson) *For the Eilenberg-MacLane ring spectrum $H = H(\mathbb{F}_p)$, $H_*(\underline{H}_*)$ is the Hopf ring over $\mathbb{F}_p$ with the $\circ$-generators:*

$e \in H_1(\underline{H}_1)$ (see Prop. 13.7);
$a_{(i)} \in H_{2p^i}(\underline{H}_1)$, *for* $i \geqslant 0$ (see Prop. 17.9);
$b_{(i)} \in H_{2p^i}(\underline{H}_2)$, *for* $i \geqslant 0$ (see Prop. 15.3);

subject to the relations $[1]^{*p} = 1_0$ *and* $e^{\circ 2} = -b_{(0)}$. □

Example: KU. We recall that $KU^* = \mathbb{Z}[u, u^{-1}]$. The complex orientation defines elements b_i for $i > 0$. As before, these, along with elements $[\lambda u^n] = [\lambda] \circ [u^n]$ and e, are all we need.

In view of the formal group law $F(x, y) = x + y + uxy$, the relation (15.8) becomes

$$1 + \overline{b}(x+y+uxy) = \big(1 + \overline{b}(x)\big) * \big(1 + \overline{b}(y)\big) * \big(1 + \overline{b}(x) \circ \overline{b}(y) \circ [u]\big) \tag{17.11}$$

which is more complicated than the additive analogue (16.12), but still manageable. Again, we take the coefficient of $x^i y^j$. The left side is the same as before. On the right, we may choose $x^s y^t$ with $s > 0$ and $t > 0$ from the third factor, which forces us to take x^{i-s} from the first factor and y^{j-t} from the second; or we can take all of $x^i y^j$ from the first two factors. The result, after some rearranging, is

$$\begin{aligned} b_i \circ b_j = &\sum_{k=0}^{\min(i,j)} \binom{i+j-k}{i} \binom{i}{k} u^k b_{i+j-k} \circ [u^{-1}] \\ &- \sum_{s=1}^{i-1} \sum_{t=1}^{j-1} b_{i-s} \circ [u^{-1}] * b_{j-t} \circ [u^{-1}] * b_s \circ b_t \\ &- \sum_{s=1}^{i-1} b_{i-s} \circ [u^{-1}] * b_s \circ b_j - \sum_{t=1}^{j-1} b_{j-t} \circ [u^{-1}] * b_i \circ b_t \\ &- b_i \circ [u^{-1}] * b_j \circ [u^{-1}] \end{aligned} \tag{17.12}$$

This serves as an inductive reduction formula for $b_i \circ b_j$, for any $i > 0$ and $j > 0$. In particular, the suspension formula becomes

$$\begin{aligned} b_1 \circ b_j = &(j+1) b_{j+1} \circ [u^{-1}] + j u b_j \circ [u^{-1}] \\ &- \sum_{k=1}^{j-1} b_{j-k} \circ [u^{-1}] * b_1 \circ b_k - b_1 \circ [u^{-1}] * b_j \circ [u^{-1}] \end{aligned} \tag{17.13}$$

Theorem 17.14. *For the complex K-theory ring spectrum KU, $KU_*(\underline{KU}_*)$ is the Hopf ring over $KU^* = \mathbb{Z}[u, u^{-1}]$ with the $\circ$-generators:*

$[u] \in KU_0(\underline{KU}_{-2})$ (see Prop. 11.2);
$[u^{-1}] \in KU_0(\underline{KU}_2)$ (see Prop. 11.2);
$e \in KU_1(\underline{KU}_1)$ (see Prop. 13.7);
$b_i \in KU_{2i}(\underline{KU}_2)$ *for* $i > 0$ (see Prop. 15.3);

subject to the relations $[u] \circ [u^{-1}] = [1]$, $\chi e = -e$, $\chi b_i = \ldots$ (see Prop. 15.3(e)), $e^{\circ 2} = -b_1$, *and eq.* (17.12).

Explicitly, for the even spaces we have

$$KU_*(\underline{KU}_{2n}) = \bigoplus_{m \in \mathbb{Z}} [mu^{-n}] * KU^*[b_1 \circ [u^{-n+1}], b_2 \circ [u^{-n+1}], b_3 \circ [u^{-n+1}], \ldots],$$

a direct sum (over m*) of polynomial algebras, and for the odd spaces*

$$KU_*(\underline{KU}_{2n+1}) = \Lambda(e \circ [u^{-n}], e \circ b_1 \circ [u^{-n+1}], e \circ b_2 \circ [u^{-n+1}], \ldots),$$

an exterior algebra over KU^* *(where we use* $[mu^{-n}] = [m] \circ [u^{-n}]$, $[u^n] = [u]^{\circ n}$, $[u^{-n}] = [u^{-1}]^{\circ n}$, $[u^0] = [1]$, $[n] = [1]^{*n}$, *and* $[-n] = [-1]^{*n} = (\chi[1])^{*n}$*).*

Proof. We computed $KU_*(BU)$ in [8, Lemma 5.6]. By Prop. 15.3(b), the Chern class $x\colon \mathbb{C}P^\infty \to \underline{KU}_2$ induces $x_*\beta_i = b_i$, so that we may write $KU_*(0 \times BU) = KU^*[b_1, b_2, \ldots]$. For the copy $KU_*(m \times BU)$, we $*$-multiply this by $[m]$. This gives $KU_*(\underline{KU}_2)$. For other even spaces, we apply the $*$-isomorphism $- \circ [u^n]$.

For the odd spaces, we quote [8, Cor. 5.12].

To see that we have specified enough relations, we note that every $*$-generator reduces to $e \circ [u^n]$ or $e \circ b_i \circ [u^n]$ on the odd spaces, or $b_i \circ [u^n]$ or $[\lambda] \circ [u^n]$ on the even spaces, where $\lambda \in \mathbb{Z}$. We allow $n = 0$ and $\lambda = 1$ and use $[u^m] \circ [u^n] = [u^{m+n}]$ and $[\lambda] \circ [\lambda'] = [\lambda\lambda']$. In the even case, we need at most one $*$-factor of the form $[\lambda] \circ [u^n]$, and we may always insert the redundant factor $[0] \circ [u^n] = 1$. Thus we can reduce any expression in the generators to standard form. □

Example: $K(n)$**.** We use the same test spaces as before, $\mathbb{C}P^\infty$ and the finite lens space L^{2p^n-1}, and follow the same strategy. The main reference is [28]. Some of the algebra resembles the case $E = H(\mathbb{F}_p)$.

As usual, the complex orientation determines Hopf ring elements b_i, where $b_0 = 1_2$ and $b_1 = e_2 = -e^{\circ 2}$. As $K(n)$ is p-local, Lemma 15.9 shows that the b_j other than the $b_{(i)} = b_{p^i}$ are redundant. If we apply eq. (10.32) to the Chern class x, we obtain the identity $\sum_j \langle r, Fb_j \rangle\, x^{pj} = \langle r, 1_2 \rangle\, 1$. This shows that $Fb_j = 0$ for all $j > 0$; in particular, $b_{(i)}^{*p} = 0$.

Next, we apply the general operation r to $\zeta^* x = v_n x^{p^n}$ by eq. (15.2) to obtain $b(v_n x^{p^n}) = b(x)^{\circ p^n} \circ [v_n]$. Equating coefficients of $x^{p^{n+i}}$ yields the relation

$$b_{(i)}^{\circ p^n} = v_n^{p^i} b_{(i)} \circ [v_n^{-1}], \tag{17.15}$$

the obvious analogue of eq. (16.17).

For the other test space $Y = L^{2p^n-1}$, we have $K(n)^*(Y) = \Lambda(u) \otimes K(n)^*[x\colon x^{p^n} = 0]$. The class x is a Chern class, which we know all about. Parallel to eq. (16.19),

we use $u \in K(n)_1(Y)$ to define elements $a_i, c_i \in K(n)_*(\underline{K(n)}_1)$ for $0 \leqslant i < p^n$ by the identity

$$r(u) = \sum_{i=0}^{p^n-1} \langle r, a_i \rangle\, x^i + \sum_{i=0}^{p^n-1} \langle r, c_i \rangle\, u x^i \qquad \text{in } K(n)^*(Y) \qquad (\text{for all } r).$$

Proposition 17.16. *The Hopf ring elements $a_i \in K(n)_{2i}(\underline{K(n)}_1)$ (for $0 \leqslant i < p^n$), $a_{(i)} = a_{p^i} \in K(n)_{2p^i}(\underline{K(n)}_1)$ (for $0 \leqslant i < n$), and $c_i \in K(n)_{2i+1}(\underline{K(n)}_1)$ (for $0 \leqslant i < p^n$) have the following properties:*

(a) $a_0 = 1_1$ *and* $c_0 = e$;

(b) $\psi a_k = \sum_{i+j=k} a_i \otimes a_j$;

(c) $Va_{(i)} = a_{(i-1)}$ *for* $0 < i < n$, *and* $Va_{(0)} = 0$;

(d) $\epsilon a_k = 0$ *for all* $k > 0$;

(e) χa_k *is the coefficient of* x^k *in* $a(x)^{*(-1)}$, *expanded as in Prop. 15.3(e);*

(f) $a_i * a_j = \binom{i+j}{i} a_{i+j}$ *if* $i + j < p^n$;

(g) $Fa_{(i)} = a_{(i)}^{*p} = 0$ *for* $0 \leqslant i < n-1$;

(h) $c_i = e * a_i$;

(i) *For all* r, $r_* a_k$ *is the coefficient of* x^k *in the formal identity*

$$r_* a(x) = \mathop{\ast}_{i=0}^{p^n-1} b(x)^{\circ i} \circ [\langle r, a_i \rangle] * \mathop{\ast}_{i=0}^{p^n-1} a(x) \circ b(x)^{\circ i} \circ [\langle r, c_i \rangle]$$

in $K(n)_*(\underline{K(n)}_*)[x : x^{p^n} = 0]$;

(j) $q_1 a_{(i)} = a_{(i)} \in Q(K(n))^1_*$, *and* $q_1 a_j = 0$ *if* j *is not a power of* p;

(k) $\sigma_{1*} a_{(i)} = a_{(i)} \in K(n)_*(K(n), o)$, *and* $\sigma_{1*} a_j = 0$ *if* j *is not a power of* p.

Proof. All the proofs are formally identical to those of Prop. 17.9, except that we use the space Y instead of L. As in section 16, the partial multiplications $\mu\colon L^{2k+1} \times L^{2m} \to Y$ yield (f) and (h).

For (g), we apply eq. (10.32) to u and obtain $\sum_{i>0} \langle r, Fa_i \rangle\, x^{pi} = 0$. But because $x^{p^n} = 0$ already, we are able to deduce that $a_i^{*p} = 0$ only for $0 < i < p^{n-1}$. (We shall see in a moment that $a_{(n-1)}^{*p} \neq 0$.) □

We have to rely on [28, Prop. 1.1] for two facts, just as in section 16. The first is that when $i = 0$, eq. (17.15) desuspends once, exactly as eq. (16.18) suggests, to

$$e \circ b_{(0)}^{\circ p^n - 1} = v_n e \circ [v_n^{-1}]. \tag{17.17}$$

In other words, the class $y = v_n u x^{p^n-1} \in K(n)^1(Y)$ still behaves like $u_1 \in K(n)^1(S^1)$ and satisfies eq. (13.2). The second is that when we take account of decomposables, eq. (16.22) acquires an extra term,

$$a_{(n-1)}^{*p} = v_n a_{(0)} - a_{(0)} \circ b_{(0)}^{\circ p^n - 1} \circ [v_n]. \tag{17.18}$$

This complements (g). We have the material for the main theorem of [28].

Theorem 17.19. *For the Morava K-theory ring spectrum $K(n)$, $K(n)_*(\underline{K(n)}_*)$ is the Hopf ring over $K(n)^* = \mathbb{F}_p[v_n, v_n^{-1}]$ with the $\circ$-generators:*

$[v_n] \in K(n)_0(\underline{K(n)}_{-2(p^n-1)})$ (see Prop. 11.2);
$[v_n^{-1}] \in K(n)_0(\underline{K(n)}_{2(p^n-1)})$ (see Prop. 11.2);
$e \in K(n)_1(\underline{K(n)}_1)$ (see Prop. 13.7);
$a_{(i)} \in K(n)_{2p^i}(\underline{K(n)}_1)$, *for* $0 \leqslant i < n$ (see Prop. 17.16);
$b_{(i)} \in K(n)_{2p^i}(\underline{K(n)}_2)$, *for* $i \geqslant 0$ (see Prop. 15.3);

subject to the relations $[1]^{*p} = 1_0$, $[v_n] \circ [v_n^{-1}] = [1]$, $e^{\circ 2} = -b_{(1)}$, (17.15), (17.17), *and* (17.18). □

Thus we have the $*$-generators:

(i) $a^{\circ I} \circ b^{\circ J} \circ [v_n^k]$ in even degrees;
(ii) $e \circ a^{\circ I} \circ b^{\circ J} \circ [v_n^k]$ in odd degrees;

where $I = (i_0, i_1, \ldots, i_{n-1})$, with each $i_r = 0$ or 1, and $J = (j_0, j_1, j_2, \ldots)$, with $0 \leqslant j < p^n$, and $k \in \mathbb{Z}$. In (ii), we may assume $j_0 < p^n - 1$ by eq. (17.17). The relations $a_{(i)}^{*p} = 0$ (for $i < n-1$) and $b_{(i)}^{*p} = 0$ (for all i) follow from $[1]^{*p} = 1_0$ by eq. (10.13), as in Thm. 17.10.

18. Relations for additive *BP*-operations

In this section, we discuss relations in the bigraded algebra $Q_*^* = Q(BP)_*^*$, following [23], in preparation for discussing additive unstable operations in *BP*-cohomology. In view of Thm. 16.11(a), Q_*^* is spanned as a BP^*-module by the monomials

$$e^\epsilon b^I w^J = e^\epsilon b_{(0)}^{i_0} b_{(1)}^{i_1} \ldots w_1^{j_1} w_2^{j_2} \ldots \,, \tag{18.1}$$

where $\epsilon \leqslant 1$ and we use standard notation with multi-indices $I = (i_0, i_1, i_2, \ldots)$ and $J = (j_1, j_2, \ldots)$. We define the *length* of I as $|I| = \sum_t i_t$, and similarly $|J| = \sum_t j_t$. We also need the special multi-index $\Delta_0 = (1, 0, 0, \ldots)$.

The main relations. For $E = BP$, we easily compute the first main relation from Defn. 14.9 and [8, (15.4)] (or equivalently from eqs. (14.5) and [8, (15.3)]) as

$$(\mathcal{R}_1): \qquad v_1 b_{(0)} = p b_{(1)} + b_{(0)}^p w_1 \qquad \text{in } Q(BP)_*^2 \,. \tag{18.2}$$

(Indeed, this is the only candidate that stabilizes correctly to [8, (15.6)].) We still have $b_i = 0$ whenever $i-1$ is not a multiple of $p-1$. We can use the p-series [8, (15.5)], just as stably, to simplify the higher relations $(\mathcal{R}_k)$ by neglecting enough. Denote by $\mathfrak{V}$ and $\mathfrak{W}$ respectively the ideals $(p, v_1, v_2, \ldots)$ and $(p, w_1, w_2, \ldots)$ in Q_*^*, which correspond to the left and right actions of the ideal I_∞. We also need the ideal $\mathfrak{M} = (e, b_{(0)}, b_{(1)}, b_{(2)}, \ldots) \subset Q_*^*$, so that $\mathfrak{M} + \mathfrak{V}$ is the obvious augmentation

ideal consisting of all the Q_i^k for $i > 0$. In particular, $b_i \in \mathfrak{M} + \mathfrak{V}$ for all i. From Defn. 14.9 and [8, (15.7)], the right side of $(\mathcal{R}_k)$ has the form

$$R(k) \equiv \sum_{i=1}^{k-1} b_{(k-i)}^{p^i} w_i + b_{(0)}^{p^k} w_k \qquad \bmod \mathfrak{V} + \mathfrak{M}\,\mathfrak{W}^2, \tag{18.3}$$

while the left side $L(k) \in \mathfrak{V}$ and will not much concern us here. The new feature is that because w_k appears in the form $b_{(0)}^{p^k} w_k$, where $b_{(0)} = e^2$ is no longer 1, $(\mathcal{R}_k)$ fails to express w_k in terms of the other generators, and $\mathfrak{W} \neq \mathfrak{V}$; this made it necessary to add w_k as a new generator of Q_*^* in Thm. 16.11.

The Ravenel-Wilson basis. The relations $(\mathcal{R}_k)$ show that many of the monomials (18.1) are redundant. In defining the basis, it is easier to specify which monomials are not wanted.

Definition 18.4. We *disallow* all monomials of the form

$$b_{(i_1)}^{p} b_{(i_2)}^{p^2} \dots b_{(i_n)}^{p^n} w_n c \qquad (i_1 \leqslant i_2 \leqslant \dots \leqslant i_n,\ n > 0), \tag{18.5}$$

where c stands for any monomial in the $b_{(i)}$, w_i, and e ($c = 1$ is permitted). All monomials (18.1) *not* of this form are declared to be *allowable*.

Nevertheless, we need a positive construction of the allowable monomials, and we need to know how they behave under suspension. Given any indices

$$0 = k_0 \leqslant k_1 \leqslant k_2 \leqslant \dots \leqslant k_n, \qquad \text{where } n \geqslant 0, \tag{18.6}$$

we define the monomial

$$b^L = b_{(k_0)} b_{(k_1)}^{p} b_{(k_2)}^{p^2} \dots b_{(k_n)}^{p^n} = b_{(0)} b^{L - \Delta_0}. \tag{18.7}$$

It is easy to see that every allowable monomial can be written *uniquely* in the canonical form

$$c = e^{\epsilon} b^{L-\Delta_0} b^M w^J = e^{\epsilon} b_{(k_1)}^{p} b_{(k_2)}^{p^2} \dots b_{(k_n)}^{p^n} b^M w^J, \tag{18.8}$$

where $\epsilon = 0$ or 1 and M and J satisfy the conditions:

$$\begin{aligned} &\text{(i)}\ \ t < k_u \text{ implies } m_t < p^u, \text{ for } 0 < u \leqslant n; \\ &\text{(ii)}\ \ t \geqslant k_n \text{ implies } m_t < p^{n+1}; \\ &\text{(iii)}\ \ j_t = 0 \text{ for all } t \leqslant n; \end{aligned} \tag{18.9}$$

as well as (18.6). In detail, we choose, by induction on u, the smallest k_u such that $b_{(k_1)}^{p} b_{(k_2)}^{p^2} \dots b_{(k_u)}^{p^u}$ divides c, to make (i) hold for u. If no such k_u exists, we set $n = u - 1$ and have (ii). Since c is allowable, it can have no factor w_u, which gives (iii) for $t = u$. (In case $n = 0$, we have merely $c = e^{\epsilon} b^M w^J$, (i) and (iii) are vacuous, and (ii) says only that $m_t < p$ for all t.)

The main technical result is that there is only one way the suspension ec of c can fail to be allowable. (This is in effect equivalent to the discussion in [23, §5].) We recall from Defn. 15.12 the shifted multi-index $s(I)$.

Lemma 18.10. *Assume that the monomial* $b^H = b_{(i_0)}^{p} b_{(i_1)}^{p^2} \dots b_{(i_n)}^{p^{n+1}}$ *divides* $b^L b^M$, *where* b^L *(with the same* n*) is as in eq.* (18.7), $i_0 \leqslant i_1 \leqslant \dots \leqslant i_n$, *and* M *satisfies conditions (i) and (ii) of* (18.9). *Then:*

(a) $i_u = k_u$ *for* $0 \leqslant u \leqslant n$, *so that* $H = pL$;

(b) *We can write* $M = (p-1)L + s(M')$, *where* M' *again satisfies (i) and (ii).*

Proof. We show first that $i_u \geqslant k_u$ for all u. For any $t < k_u$, we have $m_t < p^u$ by (i). Then the exponent of $b_{(t)}$ in $b^L b^M$ is at most

$$(1 + p + p^2 + \dots + p^{u-1}) + (p^u - 1) < p^{u+1},$$

which shows that $t \neq i_u$.

We proceed by induction on n. For $n = 0$, $b_{(i_0)}^{p}$ divides $b_{(0)} b^M$, where $m_t < p$ for all t. We must have $i_0 = 0$ and $m_0 = p - 1$, which gives M the required form.

For $n > 0$ we must have $i_n = k_n$, since $i_n > k_n$ is forbidden by (ii). Let $\alpha \geqslant 0$ be the smallest index such that $k_\alpha = k_n$; then we must have $i_\alpha = i_{\alpha+1} = \dots = i_n = k_n$. From $l_{k_n} = p^\alpha + p^{\alpha+1} + \dots + p^n$ and $m_{k_n} < p^{n+1}$ we deduce

$$m_{k_n} - (p-1) l_{k_n} < p^{n+1} - (p-1)(p^\alpha + \dots + p^n) = p^{n+1} - (p^{n+1} - p^\alpha) = p^\alpha. \quad (18.11)$$

If $k_n = 0$ we clearly have $\alpha = 0$ and hence $m_0 = (p-1) l_0$, and can write $M = (p-1)L + s(M')$. If $k_n > 0$, we have $\alpha > 0$. We delete the factors $b_{(i_u)}^{p^{u+1}}$ for $\alpha \leqslant u \leqslant n$ from both sides of our hypothesis, as well as any factors $b_{(t)}$ for $t > k_n$, to deduce that $b_{(i_0)}^{p} b_{(i_1)}^{p^2} \dots b_{(i_\alpha - 1)}^{p^\alpha}$ divides $b_{(k_0)} b_{(k_1)}^{p} \dots b_{(k_{\alpha-1})}^{p^{\alpha-1}} b^{M''}$, where M'' satisfies (i) and (ii) for the sequence $(k_0, k_1, \dots, k_{\alpha-1})$. By induction, we deduce that $H = pL$ and that M has the form $(p-1)L + s(M')$.

If $t \geqslant k_n$, we have $m'_t = m_{t+1} < p^{n+1}$, which gives (ii) for M'. To establish (i), assume that $t < k_u$. If also $t + 1 < k_u$, we have $m'_t \leqslant m_{t+1} < p^u$, as desired. Otherwise, $k_u = t + 1$. Let β be the smallest index such that $k_\beta > t + 1$, so that $k_u = k_{u+1} = \dots = k_{\beta-1} = t+1$. Then $m_{t+1} < p^\beta$ and $l_{t+1} \geqslant p^u + p^{u+1} + \dots + p^{\beta-1}$. As in eq. (18.11), we find $m'_t = m_{t+1} - (p-1) l_{t+1} < p^u$. □

Lemma 18.12. *In the bigraded algebra* $Q_*^* = Q(BP)_*^*$:

(a) *Every allowable monomial can be written uniquely in the form* (18.8), *subject to the conditions* (18.6) *and* (18.9), *and conversely, every monomial of this form is allowable;*

(b) *The suspended monomial from eq.* (18.8)

$$b_{(0)} c = e^\epsilon b^L b^M w^J = e^\epsilon b_{(0)} b_{(k_1)}^{p} \dots b_{(k_n)}^{p^n} b^M w^J$$

is disallowed if and only if $j_{n+1} > 0$ *and* $b^{(p-1)L}$ *divides* b^M, *in which case we can write* $w^J = w_{n+1} w^{J'}$ *and* $b^M = b^{(p-1)L} b^{s(M')}$, *with* $b^{L - \Delta_0} b^{M'} w^{J'}$ *allowable;*

(c) *Every allowable monomial can be written uniquely in the* extended canonical form

$$c = e^\epsilon \, b^{L - \Delta_0} \, b^{(p-1)(L + s(L) + s^2(L) + \dots + s^{h-1}(L))} \, b^{s^h(M)} \, w_{n+1}^h w^J \quad (18.13)$$

with L as in eq. (18.7)*, where $h \geqslant 0$, either $b^{(p-1)L}$ does not divide b^M or $j_{n+1} = 0$ (or both), and conditions* (18.6) *and* (18.9) *hold;*

(d) *In (c), the monomial $b^L b^M w^J$ is allowable.*

Proof. In (a), we need to establish the converse. If c is disallowed, so is $b_{(0)}c$. By Lemma 18.10(a), $b_{(0)}c$ can be disallowed only if $H = pL$; but by Lemma 18.10(b), the necessary factors $b_{(0)}$ are not present in c.

Moreover, $b_{(0)}c$ is disallowed if and only if it contains $b^{pL}w_{n+1}$ as a factor (using the same n). If so, we write $b^M = b^{(p-1)L}b^{s(M')}$, where $b^{L-\Delta_0}b^{M'}w^{J'}$ is allowable by (a). This proves (b).

Parts (c) and (d) follow by induction on h. We take h maximal. □

Lemma 18.14. *In the stable range defined by $i \leqslant pk$, every allowable monomial in $Q_i^k = Q(BP)_i^k$ has the form $e^\epsilon b_{(0)}^{i_0} b_{(1)}^{i_1} \dots$, with no factors of the form w_t^j .*

Proof. For each monomial $c \in Q_*^*$, we define $g(c) = i - pk$, where $c \in Q_i^k$. We compute g from $g(b_{(n)}) \geqslant 0$ if $n > 0$, $g(w_n) = 2p(p^n-1)$, $g(e) = -(p-1)$, and $g(b_{(0)}) = -2(p-1)$, using $g(ac) = g(a) + g(c)$. Thus if c contains w_n as a factor, $g(c) > 0$ unless c contains at least $\{2p(p^n-1) - (p-1)\}/2(p-1)$ factors $b_{(0)}$, which disallows it. □

Theorem 18.15. (Ravenel-Wilson) *The allowable monomials* (18.1) *form a basis of the free BP*-module $Q_*^* = Q(BP)_*^*$.*

This is proved in [23, Thm. 5.3, Prop. 5.1]. We content ourselves with showing, as part of Thm. 18.16, that the allowable monomials *span* Q_*^*, assuming that it is spanned by all the monomials (18.1). We shall obtain for each disallowed monomial (18.5) a reduction formula that expresses it in terms of other monomials. A finiteness argument then implies that the allowable monomials must span. A counting argument is needed to show they in fact form a basis. As only the relations $(\mathcal{R}_k)$ and $e^2 = b_{(0)}$ are used in the reduction, they constitute sufficient relations in Thm. 16.11.

Knowing that the allowable monomials form a basis of Q_*^* is not enough. In order to work with this basis, we need to know how the ideal $\mathfrak{W}$ looks in terms of the basis. We therefore define $\mathfrak{A}_m$ for any $m \geqslant 0$ as the *BP*-submodule* of Q_*^* spanned by all the *allowable* monomials $e^\epsilon b^I w^J$ that have $I \neq 0$ and $|J| \geqslant m$. Although $\mathfrak{A}_m$ is not an ideal for $m > 0$, it is convenient for computation, because when an element $c \in Q_*^*$ is expressed in terms of the basis, it is obvious whether or not it lies in $\mathfrak{A}_m$. We shall prove the following parts of the structure of Q_*^*, after developing the necessary reduction formula.

Theorem 18.16. *In the bigraded algebra $Q_*^* = Q(BP)_*^*$:*

(a) $\mathfrak{A}_m + \mathfrak{V} = \mathfrak{M}\mathfrak{W}^m + \mathfrak{V}$ *for any $m > 0$ (or $\mathfrak{A}_0 + \mathfrak{V} = \mathfrak{M} + \mathfrak{V}$ if $m = 0$), so that the image of $\mathfrak{A}_m$ in the quotient algebra $\overline{Q}_*^*$* (see eq. (18.17)) *is an ideal;*

(b) *The allowable monomials span $Q_*^* = Q(BP)_*^*$ as a BP*-module.*

Lemma 15.2 of [8] allows us to work mod $\mathfrak{V}$, in the quotient $\mathbb{F}_p$-algebra

$$\overline{Q}_*^* = Q_*^*/\mathfrak{V} \cong QH_*(\underline{BP}_*; \mathbb{F}_p) \ . \tag{18.17}$$

Better yet, we may ignore e and work in the subalgebra $\overline{Q}_*^{\text{even}}$.

Higher order relations. As they stand, the relations $(\mathcal{R}_k)$ are not very practical. We derive a more useful relation by eliminating the terms that come from $b(x)^{p^j} w_j$ for $j < n$ in eq. (18.3) from the n relations $(\mathcal{R}_{k_1})$, $(\mathcal{R}_{k_2})$, ..., $(\mathcal{R}_{k_n})$, as in [23, Lemma 5.13]. The result is of course a determinant. For ulterior purposes, we make the elimination totally explicit.

Definition 18.18. Given any positive integers $i_1, i_2, \ldots, i_n$, where $n \geqslant 1$, we define $L(i_1, i_2, \ldots, i_n)$ and $R(i_1, i_2, \ldots, i_n)$ as the coefficient of $x_1^{p^{i_1}} x_2^{p^{i_2}} \ldots x_{n-1}^{p^{i_{n-1}}} x_n^{p^{i_n}}$ in

$$b(x_1)^p b(x_2)^{p^2} \ldots b(x_{n-1})^{p^{n-1}} b([p](x_n))$$

and

$$b(x_1)^p b(x_2)^{p^2} \ldots b(x_{n-1})^{p^{n-1}} [p]_R(b(x_n))$$

respectively. By eq. (14.8), these are equal in Q_*^*.

Then given any integers $0 < k_1 < k_2 < \ldots < k_n$, where $n > 1$, we deduce the *nth order derived relation*

$$(\mathcal{R}_{k_1,k_2,\ldots,k_n}): \qquad \sum_\pi \epsilon_\pi L(i_1, i_2, \ldots, i_n) = \sum_\pi \epsilon_\pi R(i_1, i_2, \ldots, i_n) \tag{18.19}$$

in Q_*^* by summing over all permutations $\pi \in \Sigma_n$, where ϵ_π denotes the sign of π and we permute the n entries in $(i_1, i_2, \ldots, i_n) = \pi(k_1, k_2, \ldots, k_n)$. (For $n = 1$, it reduces to $L(k_1) = R(k_1)$, which is just $(\mathcal{R}_{k_1})$.)

We note that this relation lies in $Q_*^{f(n)}$, where the numerical function

$$f(n) = 2(1 + p + p^2 + \ldots + p^{n-1}) = \frac{2(p^n - 1)}{p - 1} = \frac{|\deg(v_n)|}{p - 1}$$

was introduced in eq. (1.4).

The left side of $(\mathcal{R}_{k_1,k_2,\ldots,k_n})$ lies in $\mathfrak{V}$ and will be of little interest here. By eq. (18.3), the right side reduces to

$$\sum_{\pi,j} \epsilon_\pi b_{(i_1-1)}^p b_{(i_2-2)}^{p^2} \ldots b_{(i_{n-1}-n+1)}^{p^{n-1}} b_{(i_n-j)}^{p^j} w_j \mod \mathfrak{V} + \mathfrak{M}\mathfrak{W}^2, \tag{18.20}$$

where we sum over all permutations π and all $j > 0$, and adopt the convention that $b_{(i)} = 0$ for $i < 0$. However, we have arranged matters so that no (explicit) terms in w_j with $j < n$ survive; when we interchange i_j and i_n, we find identical terms having opposite signs. The term of most interest is the *leading* term with $\pi = \mathrm{id}$,

$$b^L w_n = b_{(k_1-1)}^p b_{(k_2-2)}^{p^2} \ldots b_{(k_{n-1}-n+1)}^{p^{n-1}} b_{(k_n-n)}^{p^n} w_n \ , \tag{18.21}$$

which is thereby expressed in terms of other monomials and hence redundant. (The multi-index L serves only as a convenient abbreviation, unrelated to eq. (18.7). The indices k_u are different, too.)

To make this more precise, we note that all terms $b^I w_j$ in the sum (18.20) have $|I| = |L| = p + p^2 + \ldots + p^n$ if $j = n$, or $|I| > |L|$ if $j > n$. We order the terms that contain w_n by defining the *weight* of any multi-index $I = (i_0, i_1, i_2, \ldots)$ as $\mathrm{wt}(I) = \sum_t t i_t$ (which is not the weight used in [23]). This makes $b^L w_n$ the heaviest term with its length, because if we improve the ordering of the indices of any other term in (18.20) by interchanging i_r and i_s, where $r < s$ and $i_r > i_s$, we increase its weight by

$$(i_s - r)p^r + (i_r - s)p^s - (i_r - r)p^r - (i_s - s)p^s = (i_r - i_s)(p^s - p^r) > 0 \ .$$

Thus $(\mathcal{R}_{k_1,k_2,\ldots,k_n})$ provides a reduction formula

$$b^L w_n = b^p_{(k_1-1)} b^{p^2}_{(k_2-2)} \cdots b^{p^n}_{(k_n-n)} w_n \equiv \sum_{I,j} \pm b^I w_j \tag{18.22}$$

in $\overline{Q}^*_*$ mod $\mathfrak{M}\mathfrak{W}^2$, where the sum is taken over certain pairs (I, j) with $j \geqslant n$, for which $|I| > |L|$, or $|I| = |L|$ and $\mathrm{wt}(I) < \mathrm{wt}(L)$.

The first nth order relation $(\mathcal{R}_{1,2,\ldots,n})$ is particularly important, as only one term of the sum (18.20) is meaningful, namely $b^{pm}_{(0)} w_n$, where $m = f(n)/2$. We observe that this monomial lies just inside the stable range of Lemma 18.14. In this simple case, we can do better with a little more attention to detail, to obtain the direct analogue of [8, Lemma 15.8].

Lemma 18.23. *In $Q^{f(n)}_* = Q(BP)^{f(n)}_*$ we have the relation*

$$b^{pm}_{(0)} w_n \equiv v_n b^m_{(0)} \qquad \mathrm{mod}\ I_n Q(BP)^{f(n)}_*$$

for each $n > 0$, where $m = f(n)/2 = 1 + p + p^2 + \ldots + p^{n-1}$.

Proof. We proceed by induction on n, starting from eq. (18.2), and work throughout mod $I_n Q^*_*$. On the left side of eq. (14.8) we have $b([p](x)) \equiv b(v_n x^{p^n} + \ldots)$, by [8, (15.5)]. Then $R(j) = L(j) \equiv 0$ for all $j < n$, and the only surviving terms in $(\mathcal{R}_{1,2,\ldots,n})$ are $b^h_{(0)} L(n) \equiv b^h_{(0)} R(n)$, where $h = p + p^2 + \ldots + p^{n-1}$. On the left, we clearly have $L(n) \equiv v_n b_{(0)}$. On the right, $b^h_{(0)} w_j \equiv 0$ for all $j < n$, by the induction hypothesis; by eq. (18.3) and dimensional reasons, the only surviving term in $R(n)$ is $b^{p^n}_{(0)} w_n$. □

Proof of Thm. 18.16. We work entirely in the quotient algebra $\overline{Q}^*_*$ defined by eq. (18.17). We first generalize (18.22) to show that

$$\mathfrak{M}\mathfrak{W}^m \subset \mathfrak{A}_m + \mathfrak{M}\mathfrak{W}^{m+1} \tag{18.24}$$

for any $m \geqslant 1$. As an $\mathbb{F}_p$-module, $\mathfrak{M}\mathfrak{W}^m$ is generated by those monomials $e^\epsilon b^I w^J$ that have $|J| \geqslant m$. These lie in $\mathfrak{A}_m$ or $\mathfrak{M}\mathfrak{W}^{m+1}$ except for the disallowed monomials that have $|J| = m$. On comparing the monomial (18.5) with eq. (18.21), we see that each such monomial has the form $b^L w_n c$, where L is given by eq. (18.21)

and $c = e^\epsilon b^L w^N$, with $|N| = m-1$. When we multiply eq. (18.22) by c, both orderings are preserved, and we express the general disallowed monomial $b^L w_n c$ as a signed sum of monomials with greater length, or the same length and lower weight, mod $\mathfrak{M}\mathfrak{W}^{m+1}$. Because there are only finitely many monomials in each bidegree, eq. (18.24) follows by induction.

For any $i > m$, eq. (18.24) gives

$$\mathfrak{A}_m + \mathfrak{M}\mathfrak{W}^i \subset \mathfrak{A}_m + \mathfrak{A}_i + \mathfrak{M}\mathfrak{W}^{i+1} = \mathfrak{A}_m + \mathfrak{M}\mathfrak{W}^{i+1} \ .$$

Then by induction on i, starting from eq. (18.24),

$$\mathfrak{M}\mathfrak{W}^m \subset \mathfrak{A}_m + \mathfrak{M}\mathfrak{W}^i$$

for all $i > m$. In any fixed bigrading, $\mathfrak{M}\mathfrak{W}^i$ is zero for large i. Thus $\mathfrak{M}\mathfrak{W}^m \subset \mathfrak{A}_m$ and we have (a) for $m > 0$. For $m = 0$, we note that every monomial in $\mathfrak{M}$ either lies in $\mathfrak{M}\mathfrak{W} \subset \mathfrak{A}_1$ or is automatically allowable and so lies in $\mathfrak{A}_0$.

On reinstating the monomials of the form w^J, which are all allowable, we see that the allowable monomials span $\overline{Q}_*^*$. Then (b) follows by Nakayama's Lemma in the form [8, Lemma 15.2(d)]. □

The ideals $\mathfrak{J}_n$. Just as the ideal $I_\infty \subset BP^*$ led to the introduction of the ideal $\mathfrak{W} \subset Q_*^*$, the ideal J_n, needed for our splitting theorems, leads to an ideal in Q_*^*.

Definition 18.25. We define the ideal $\mathfrak{J}_n = (w_{n+1}, w_{n+2}, w_{n+3}, \ldots) \subset Q_*^*$.

We need to know how $\mathfrak{J}_n$ sits inside Q_*^*. The answer is remarkably clean, in a certain range.

Lemma 18.26. *Assume $n \geqslant 0$. Then:*

(a) *If $k < f(n+1)$, $Q_*^k \cap \mathfrak{J}_n$ is the left BP^*-submodule of Q_*^k spanned by all the allowable monomials $e^\epsilon b^I w^J \in Q_*^k$ that contain an explicit factor w_t for some $t > n$;*

(b) *If $k = f(n+1)$, $Q_*^k \cap \mathfrak{J}_n$ is the left BP^*-submodule of Q_*^k spanned by all the allowable monomials as in (a), together with all disallowed monomials of the form $b^p_{(i_1)} b^{p^2}_{(i_2)} \ldots b^{p^{n+1}}_{(i_{n+1})} w_{n+1}$, where $0 \leqslant i_1 \leqslant i_2 \leqslant \ldots \leqslant i_{n+1}$.*

Remark. The first disallowed monomial in (b) is $b^{pm}_{(0)} w_{n+1}$, where $m = f(n+1)/2$. Lemma 18.23 shows it definitely does not lie in the submodule described in (a).

Proof. The stated elements obviously lie in $\mathfrak{J}_n$. To show the converse, we fix k and a large integer m, and prove by *downward* induction on h that all elements in Q_i^k of the form cw_h lie in the indicated submodule whenever $i < m$. This statement is vacuous for sufficiently large h (depending on m and k). We therefore fix $t > n$, assume the statement for all $h > t$, and prove it for $h = t$. We ignore e^ϵ throughout and assume k is even.

Case 1: $c = b^I$. The number $|I|$ of b-factors in c is $k/2 + p^t - 1$. In (a), as $k < f(n+1)$ and $t > n$, this is always less than $p + p^2 + \ldots + p^t$, which makes $cw_t = b^I w_t$ automatically allowable. The same holds in (b), except in the extreme case $b^I w_{n+1}$, which may be allowable or disallowed; either way, it is in.

Case 2: $c = b^I w_h w^J$ allowable, with $h \leqslant t$. Then $cw_t = b^I w_h w_t w^J$ remains allowable, by the form of Defn. 18.4.

Case 3: $c = aw_h$, *with* $h > t$, *any* a. Then $cw_t = (aw_t)w_h$ is in by induction, provided $i < m$.

By Thm. 18.15, these c generate $Q_*^{k+2(p^t-1)}$ as a BP^*-module. □

19. Relations in the Hopf ring for *BP*

In this section, we develop the unstable analogues of the results of section 18, working in the Hopf ring $BP_*(\underline{BP}_*)$ for BP. By taking account of $*$-decomposable elements, we can improve many of these results by one. The structure of the Hopf ring was described briefly in section 17. Before we can even state some of our results precisely, it is necessary to clarify the concept of *ideal* in a Hopf ring.

Hopf ring ideals. As it is obviously impractical to retain everything in typical Hopf ring calculations (the preceding sections should convince), we need to control carefully what is thrown away. There is an obvious relevant concept, valid in any Hopf ring H. We concentrate on the structure of H as a $*$-algebra, treating $\circ$-multiplication chiefly as a means of creating new $*$-generators from old.

Definition 19.1. We call a bigraded R-submodule $\mathfrak{I}$ of any Hopf ring H over R a *Hopf ring ideal* if the quotient $H/\mathfrak{I}$ inherits a well-defined Hopf ring structure from H (over the possibly smaller ground ring $R/\epsilon\mathfrak{I}$).

If we ignore the $\circ$-multiplication and coalgebra structure, $\mathfrak{I}$ must obviously be a $*$-ideal in the ordinary sense, i.e. an R-submodule for which $b * c \in \mathfrak{I}$ whenever $b \in H$ and $c \in \mathfrak{I}$.

Lemma 19.2. *Let* H *be a Hopf ring over* R *and* $I \subset R$ *an ideal. Let* $\mathfrak{I}$ *be the* $*$*-ideal in* H *generated by the elements* c_α. *Then* $\mathfrak{I}$ *is a Hopf ring ideal, with quotient a Hopf ring over* R/I, *if and only if:*

(i) $\psi c_\alpha \in \mathfrak{I}\otimes H + H\otimes\mathfrak{I}$ *for all* α;
(ii) $\epsilon c_\alpha \in I$ *for all* α;
(iii) $a \circ c_\alpha \in \mathfrak{I}$ *for all* $a \in H$ *and all* α;
(iv) $IH \subset \mathfrak{I}$.

Proof. The conditions are evidently necessary. Conditions (i) and (ii) ensure that $H/\mathfrak{I}$ inherits a comultiplication ψ and counit ϵ. Condition (iv) shows that $H/\mathfrak{I}$ is defined over R/I. For any $a, b \in H$, eq. (10.11) and (iii) show that $a \circ (b * c_\alpha) \in \mathfrak{I}$; this is enough to furnish $H/\mathfrak{I}$ with a $\circ$-multiplication. All the necessary identities in $H/\mathfrak{I}$ (see section 10) are inherited from H. □

Remark. It is clear from the Lemma that the sum $\mathfrak{I} + \mathfrak{J}$ of two Hopf ring ideals is another Hopf ring ideal. However, their $*$-product ideal $\mathfrak{I} * \mathfrak{J}$ (defined as the usual product of ideals) *need not* be a Hopf ring ideal, as (i) can fail. We note that (ii) and (iii) nevertheless continue to hold for $\mathfrak{I} * \mathfrak{J}$, with the help of eq. (10.11).

When $R = \mathbb{F}_p$, we can define a rather more useful ideal.

Definition 19.3. Given an ideal $\mathfrak{I}$ in a Hopf ring over $\mathbb{F}_p$, we define $F\mathfrak{I}$ as the $*$-ideal generated by $\{Fx : x \in \mathfrak{I}\}$.

The ideal $F\mathfrak{I}$ is far smaller than $\mathfrak{I}^{*p}$, and clearly *is* a Hopf ring ideal by Lemma 19.2 whenever $\mathfrak{I}$ is. (We use eq. (10.13) to verify (iii).)

The redundant generators. We proved in Lemma 15.9 that the generator b_i is redundant unless i is a power of p. As in (17.3), this implies that $BP_*(\underline{BP}_*)$ is $*$-generated as a BP^*-algebra by $\circ$-monomials of the forms (cf. eq. (18.1))

$$\begin{aligned}
&\text{(i)} \quad b^{\circ I} \circ [v^J] = b_{(0)}^{\circ i_0} \circ b_{(1)}^{\circ i_1} \circ b_{(2)}^{\circ i_2} \circ \ldots \circ [v_1^{j_1} v_2^{j_2} \ldots] \\
&\qquad\qquad = b_{(0)}^{\circ i_0} \circ b_{(1)}^{\circ i_1} \circ b_{(2)}^{\circ i_2} \circ \ldots \circ [v_1]^{\circ j_1} \circ [v_2]^{\circ j_2} \circ \ldots, \\
&\text{(ii)} \quad e \circ b^{\circ I} \circ [v^J], \\
&\text{(iii)} \quad [\lambda v^J] = [\lambda] \circ [v_1]^{\circ j_1} \circ [v_2]^{\circ j_2} \circ \ldots,
\end{aligned} \tag{19.4}$$

in the notation of eq. (15.11). To carry out computations, we need to express the redundant b_i in terms of these $*$-generators.

In order to make the finiteness of our computations apparent, we write $b(x) = 1_2 + \overline{b}(x)$ as in eq. (15.4) and use $1 \circ \overline{b}(x) = 0$. Then eq. (15.8) expands to

$$\begin{aligned}
&1_2 + \overline{b}\Big(x + y + \sum_{i,j} a_{i,j} x^i y^j\Big) \\
&\quad = \big(1_2 + \overline{b}(x)\big) * \big(1_2 + \overline{b}(y)\big) * \mathop{\Large *}_{i,j} \big\{1_2 + \overline{b}(x)^{\circ i} \circ \overline{b}(y)^{\circ j} \circ [a_{i,j}]\big\}
\end{aligned} \tag{19.5}$$

As in Lemma 15.9, if n is not a power of p, we take s as the largest power of p less than n, and the coefficient of $x^s y^{n-s}$ then yields a reduction formula for b_n. For the low b_n's we can be explicit; they are no longer trivially zero, as in section 18.

Lemma 19.6. *For $1 \leqslant i < p$ we have $b_i = b_{(0)}^{*i}/i!$.*

Proof. All that is left of eq. (19.5) in this range is $b(x+y) = b(x) * b(y)$. Hence $b(x)$ must be the exponential series $\exp(b_1 x)$, expanded using $*$-multiplication. □

Beyond this range, we must settle for inductive formulae in terms of $\circ$-monomials of the form

$$b_{i_1} \circ b_{i_2} \circ \ldots \circ b_{i_r} \circ [v^J]\,. \tag{19.7}$$

We expand the formal group law $F(x,y)$ *fully*, in the form

$$F(x,y) = x + y + \sum_{\lambda,I,i,j} \lambda v^I x^i y^j,$$

summing over appropriate quadruples (λ, I, i, j) consisting of a coefficient $\lambda \in \mathbb{Z}_{(p)}$, a multi-index I, and exponents i and j. The right side of eq. (19.5) becomes

$$\big(1_2 + \overline{b}(x)\big) * \big(1_2 + \overline{b}(y)\big) * \mathop{\Large *}_{\lambda,I,i,j} \big\{1_2 + \overline{b}(x)^{\circ i} \circ \overline{b}(y)^{\circ j} \circ [v^I]\big\}^{*\lambda},$$

where $\{1+\ldots\}^{*\lambda}$ is expanded by the binomial series as in eq. (15.5). Every element of the Hopf ring that appears here is a $*$-product of elements of the form (19.7).

This is still not enough! To make the induction succeed, we really need a reduction formula for *every* $\circ$-monomial (19.7) that contains a $\circ$-factor b_n with n not a power of p, without relying on iterated appeals to the distributive law (10.11). A reduction formula for $b_n \circ b_{h_1} \circ b_{h_2} \circ \ldots \circ b_{h_q}$, whenever n is not a power of p and the h_i are any positive integers, will suffice, as $- \circ [v^I]$ is a $*$-homomorphism and $[v^I] \circ [v^J] = [v^{I+J}]$.

We therefore $\circ$-multiply eq. (15.8) by $b(z_1) \circ b(z_2) \circ \ldots \circ b(z_q)$ (and thus work in the $(q+2)$-fold product $(\mathbb{C}P^\infty)^{q+2}$). On the right, we use the distributive law (15.6) to move all the $b(-)$'s inside the $*$-factors, to obtain

$$\begin{aligned} 1_{2q+2} + \overline{b}\Big(x + y + \sum_{\lambda,I,i,j} \lambda v^I x^i y^j\Big) \circ \overline{b}(z_1) \circ \ldots \circ \overline{b}(z_q) \\ = \big\{1_{2q+2} + \overline{b}(x) \circ \overline{b}(z_1) \circ \ldots \circ \overline{b}(z_q)\big\} \\ * \big\{1_{2q+2} + \overline{b}(y) \circ \overline{b}(z_1) \circ \ldots \circ \overline{b}(z_q)\big\} \\ * \mathop{\ast}_{\lambda,I,i,j} \big\{1_{2q+2} + \overline{b}(x)^{\circ i} \circ \overline{b}(y)^{\circ j} \circ \overline{b}(z_1) \circ \ldots \circ \overline{b}(z_q) \circ [v^I]\big\}^{*\lambda} \end{aligned} \tag{19.8}$$

The coefficient of $x^s y^{n-s} z_1^{h_1} \ldots z_q^{h_q}$ yields the desired reduction formula. Inspection of the $\circ$-monomials that appear on the right shows that they are all simpler, so that the induction makes progress. (In detail, they all have lower height, or the same height but more b-factors, if we define the *height* of the monomial (19.7) as $\sum_r i_r$.)

None of this is necessary for the other generators, (19.4)(ii). For these it is far simpler to start from Lemma 14.6, work in $Q(BP)^*_*$, and suspend by applying $e \circ -$.

The main relations. As given in Defn. 15.15, the main relations are particularly opaque. We make eq. (15.14) more useful in our situation by first expanding the p-series [8, (13.9)] for *BP in full* as

$$[p](x) = px + \sum_{\lambda,I,m} \lambda v^I x^m, \tag{19.9}$$

much as we just did for $F(x,y)$, and summing over appropriate combinations of coefficient $\lambda \in \mathbb{Z}_{(p)}$, multi-index I, and exponent m. Then eq. (15.14) becomes

$$1_2 + \overline{b}\Big(px + \sum_{\lambda,I,m} \lambda v^I x^m\Big) = \big\{1_2 + \overline{b}(x)\big\}^{*p} * \mathop{\ast}_{\lambda,I,m} \big\{1_2 + \overline{b}(x)^{\circ m} \circ [v^I]\big\}^{*\lambda} \tag{19.10}$$

where we again expand $\{1_2 + \ldots\}^{*\lambda}$ by the binomial series as in eq. (15.5).

The first main relation, the coefficient of x^p, simplifies (with the help of [8, (15.4)] and Lemma 19.6) to

$$(\mathcal{R}_1): \quad v_1 b_{(0)} = p b_{(1)} + b_{(0)}^{\circ p} \circ [v_1] - \frac{b_{(0)}^{*p}}{(p-1)!} \qquad \text{in } BP_*(\underline{BP}_2) \tag{19.11}$$

(although it is far easier to extract this as the coefficient of $x^{p-1}y$ in eq. (15.8), using [8, (15.3)]). Subsequent relations rapidly become extremely complicated and can be handled only by neglecting terms wholesale. We need some ideals.

Let $\mathfrak{V}$ be the ideal $(p, v_1, v_2, \ldots)$ in $BP_*(\underline{BP}_*)$ (more accurately, generated as a graded $*$-ideal by all the elements $p1_k$ and $v_n 1_k$ for each k). We need the unstable analogue of the ideals $\mathfrak{M}\mathfrak{W}^m$ of section 18, coming from the right action of I_∞ on Q_*^*. It is obvious how to handle the generators v_i of I_∞. For the generator p, eq. (10.13) shows that in the quotient Hopf ring $BP_*(\underline{BP}_*)/\mathfrak{V}$ over $\mathbb{F}_p = BP^*/I_\infty$, we may write $c\circ[p] = c\circ(F[1]) = F(Vc\circ[1]) = FVc$. Indeed, it is even more convenient to ignore e and work in the Hopf subring

$$\overline{H} = BP_*(\underline{BP}_{\text{even}})/\mathfrak{V} \cong H_*(\underline{BP}_{\text{even}}; \mathbb{F}_p), \tag{19.12}$$

using only those elements that do not involve the $\circ$-generator e (though of course we keep $b_{(0)} = -e^{\circ 2}$).

Definition 19.13. We define $\mathfrak{M}_0$ as the $*$-ideal in $\overline{H}$ generated by all the elements $b^{\circ I}\circ[v^J]$ with $I \neq 0$, whether allowable or not. For $m > 0$, we define $\mathfrak{M}_m$ inductively as the $*$-ideal generated by $F\mathfrak{M}_{m-1}$ and all elements $b^{\circ I}\circ[v^J]$ with $I \neq 0$, whether allowable or not, that have $|J| \geqslant m$.

Equivalently, $\mathfrak{M}_m$ is the $*$-ideal generated by all elements $F^h(b^{\circ I}\circ[v^J])$ with $I \neq 0$ and $h + |J| \geqslant m$. (Thus $\mathfrak{M}_m$ is roughly, but not quite, the Hopf ring analogue of the right BP^*-action of the ideal I_∞^m.) We thus have the decreasing sequence of ideals

$$\overline{H} \supset \mathfrak{M}_0 \supset \mathfrak{M}_1 \supset \mathfrak{M}_2 \supset \ldots .$$

We note that $\mathfrak{M}_0$ is just the obvious augmentation ideal in $\overline{H}$ consisting of all the $H_i(\underline{BP}_{\text{even}}; \mathbb{F}_p)$ with $i > 0$.

Lemma 19.14. *For all $m \geqslant 0$:*

(a) *$\mathfrak{M}_m$ is a Hopf ring ideal in the Hopf ring $\overline{H} = BP_*(\underline{BP}_{\text{even}})/\mathfrak{V}$;*

(b) *$\mathfrak{M}_m \circ [v_n] \subset \mathfrak{M}_{m+1}$ for all $n > 0$;*

(c) *$\mathfrak{M}_m \circ [v^J] \subset \mathfrak{M}_{m+|J|}$;*

(d) *$\mathfrak{M}_m \circ [p] \subset \mathfrak{M}_{m+1}$.*

Proof. We first prove (b), from which (c) follows by induction. As $- \circ [v_n]$ is a $*$-homomorphism, it is enough to check that $c\circ[v_n] \in \mathfrak{M}_{m+1}$ for the generators c of $\mathfrak{M}_m$. For $c = b^{\circ I}\circ[v^J]$, we use $[v^J]\circ[v_n] = [v^J v_n]$. For $c = Fa = a^{*p}$, where $a \in \mathfrak{M}_{m-1}$, we have $c\circ[v_n] = F(a\circ[v_n])$ by eq. (10.13). This lies in $F\mathfrak{M}_m$, by induction on m.

We next apply Lemma 19.2 to prove (a). Clearly, $\epsilon\mathfrak{M}_m = 0$. For a generator of the form $c = b^{\circ I}\circ[v^J]$, with $|J| \geqslant m$, we have $a \circ c = (a \circ b^{\circ I})\circ[v^J] \in \mathfrak{M}_m$ by (c), since $a \circ b^{\circ I} \in \mathfrak{M}_0$. Similarly, if we write $\psi b^{\circ I} = \sum_i B_i' \otimes B_i''$, we find that $\psi c = \sum_i B_i'\circ[v^J] \otimes B_i''\circ[v^J]$ has the required form, because for each i, either $B_i' \in \mathfrak{M}_0$ or $B_i'' \in \mathfrak{M}_0$ for reasons of degree.

For a generator Fc with $c \in \mathfrak{M}_{m-1}$, we use induction on m. By eq. (10.13), $a\circ(Fc) = F(Va \circ c) \in F\mathfrak{M}_{m-1}$. Also, $\psi Fc = (F\otimes F)\psi c$ has the required form.

Because $\mathfrak{M}_m$ is now known to be a Hopf ring ideal, we have $Va \in \mathfrak{M}_m$ for any $a \in \mathfrak{M}_m$. Then (d) is immediate from eq. (10.13), using $[p] = F[1]$. □

We now have the tools to handle eq. (19.10). We work entirely in $\overline{H}$, so that by [8, (15.5)], the left side is trivial. By Lemma 19.14, $\overline{b}(x)^{\circ m} \circ [v^I] \in \mathfrak{M}_{|I|}$. Most $*$-factors on the right side of eq. (19.10) are trivial mod $\mathfrak{M}_2$ and we are left with only

$$\{1_2 + F\overline{b}(x)\} * \mathop{\Large *}_{j>0} \left\{1_2 + \overline{b}(x)^{\circ p^j} \circ [v_j]\right\} \qquad \text{in } \overline{H}[[x]] \bmod \mathfrak{M}_2 \ .$$

When we pick out the coefficient of x^{p^k} and neglect also certain products, we obtain

$$R(k) \equiv Fb_{(k-1)} + \sum_{j=1}^{k} b_{(k-j)}^{\circ p^j} \circ [v_j] \qquad \text{in } \overline{H} \bmod \mathfrak{M}_2 + \mathfrak{M}_1 * \mathfrak{M}_1 \ , \quad (19.15)$$

analogous to eq. (18.3). Although the ideal here is *not* a Hopf ring ideal, (ii) and (iii) of Lemma 19.2 still hold, according to the Remark following that lemma.

The Ravenel-Wilson generators. We lift the allowable monomials of section 18 via the canonical projections $q_k \colon BP_*(\underline{BP}_k) \to Q(BP)_*^k$, so that multiplication is now to be interpreted as $\circ$-multiplication.

Definition 19.16. We *disallow* all $\circ$-monomials of the form

$$b_{(i_1)}^{\circ p} \circ b_{(i_2)}^{\circ p^2} \circ \ldots \circ b_{(i_n)}^{\circ p^n} \circ [v_n] \circ c \qquad (i_1 \leqslant i_2 \leqslant \ldots \leqslant i_n,\ n > 0), \qquad (19.17)$$

where c stands for any $\circ$-monomial in the $b_{(i)}$, $[v_j]$, and e ($c = [1]$ is permitted). All $\circ$-monomials (19.4)(i) and (ii) not of this form are declared to be *allowable*.

It follows from Thm. 18.15 (and local finiteness) that the allowable $\circ$-monomials generate $BP_*(\underline{BP}_*)$, but far more is true, by [23, Thm. 5.3, Rk. 4.9].

Theorem 19.18. (Ravenel-Wilson) *In the Hopf ring for BP:*

(a) *If k is even, denote by $\underline{BP}'_k$ the zero component of the space $\underline{BP}_k$* (so that $\underline{BP}'_k = \underline{BP}_k$ if $k > 0$). *Then $BP_*(\underline{BP}'_k)$ is a polynomial algebra over BP^* on those allowable $\circ$-monomials $b^{\circ I} \circ [v^J]$ with $I \neq 0$ that lie in it. If $k \leqslant 0$, $BP_*(\underline{BP}_k) = BP_*(BP^k) \otimes BP_*(\underline{BP}'_k)$ as in eq.* (17.4).

(b) *If k is odd, $BP_*(\underline{BP}_k)$ is an exterior algebra over BP^* on those allowable $\circ$-monomials $e \circ b^{\circ I} \circ [v^J]$ that lie in it.* □

As in section 18, we need information on where the disallowed monomials lie. The difficulty with eq. (19.15) is that it is hard to tell whether a given element lies in $\mathfrak{M}_2$. We therefore define analogous ideals in terms of the polynomial generators in Thm. 19.18 for which this problem does not exist. Again, we ignore e and neglect $\mathfrak{V}$ by working in the Hopf ring $\overline{H}$ over $\mathbb{F}_p$ (see eq. (19.12)).

Definition 19.19. We define $\mathfrak{A}_0$ as the $*$-ideal in $\overline{H}$ generated by all the *allowable* $\circ$-monomials $b^{\circ I} \circ [v^J]$ that have $I \neq 0$. For $m > 0$, we define $\mathfrak{A}_m$ inductively as the $*$-ideal generated by $F\mathfrak{A}_{m-1}$ and all the *allowable* $\circ$-monomials $b^{\circ I} \circ [v^J]$ for which $I \neq 0$ and $|J| \geqslant m$.

In other words, $\mathfrak{A}_m$ is the $*$-ideal generated by all the elements $F^h(b^{\circ I} \circ [v^J])$, where $b^{\circ I} \circ [v^J]$ is allowable, $I \neq 0$, and $h + |J| \geqslant m$.

Theorem 19.20. *For all $m \geqslant 0$, $\mathfrak{A}_m = \mathfrak{M}_m$ and is therefore a Hopf ring ideal in $\overline{H} = BP_*(\underline{BP}_{\text{even}})/\mathfrak{V} \cong H_*(\underline{BP}_{\text{even}}; \mathbb{F}_p)$.*

This result we shall prove in full. For $m = 0$, it is part of Thm. 19.18.

Higher order relations. As in section 18, we derive a more useful relation by elimination from the n relations $(\mathcal{R}_{k_1})$, $(\mathcal{R}_{k_2})$, $\dots (\mathcal{R}_{k_n})$, with multiplication now interpreted as $\circ$-multiplication. We find it simpler to return to eq. (19.10) rather than try to deal directly with eq. (19.15).

Definition 19.21. Given any positive integers $i_1, i_2, \dots, i_n$, where $n \geqslant 1$, we define $L(i_1, i_2, \dots, i_n)$ and $R(i_1, i_2, \dots, i_n)$ as the coefficient of $x_1^{p^{i_1}} x_2^{p^{i_2}} \dots x_n^{p^{i_n}}$ in

$$b(x_1)^{\circ p} \circ b(x_2)^{\circ p^2} \circ \dots \circ b(x_{n-1})^{\circ p^{n-1}} \circ b([p](x_n))$$

and

$$b(x_1)^{\circ p} \circ b(x_2)^{\circ p^2} \circ \dots \circ b(x_{n-1})^{\circ p^{n-1}} \circ P(x_n) \tag{19.22}$$

respectively, where $P(x)$ denotes the right side of eq. (19.10).

Then given any integers $0 < k_1 < k_2 < \dots < k_n$, where $n > 1$, we define the *nth order derived relation*

$$(\mathcal{R}_{k_1, k_2, \dots, k_n}): \qquad \sum_\pi \epsilon_\pi \, L(i_1, i_2, \dots, i_n) = \sum_\pi \epsilon_\pi \, R(i_1, i_2, \dots, i_n)$$

by summing over all permutations $\pi \in \Sigma_n$, where $(i_1, i_2, \dots, i_n) = \pi(k_1, k_2, \dots, k_n)$. (For $n = 1$, we recover $(\mathcal{R}_{k_1})$.)

This relation lies in $BP_*(\underline{BP}_{f(n)})$, where $f(n)$ denotes the usual numerical function (1.4). To study it, we work in $\overline{H}$. The left side of $(\mathcal{R}_{k_1, k_2, \dots, k_n})$ vanishes, as before. To handle the right side, we first rewrite (19.22) just as we did eq. (19.8), by using eq. (15.6) to move all the $\circ$-factors $b(-)$ inside the $*$-factors. The term px of $[p](x)$ produces the $*$-factor

$$\left\{1 + \overline{b}(x_1)^{\circ p} \circ \overline{b}(x_2)^{\circ p^2} \circ \dots \circ \overline{b}(x_{n-1})^{\circ p^{n-1}} \circ \overline{b}(x_n)\right\}^{*p}, \tag{19.23}$$

and the general term $\lambda v^I x^m$ produces the $*$-factor

$$\left\{1 + \overline{b}(x_1)^{\circ p} \circ \overline{b}(x_2)^{\circ p^2} \circ \dots \circ \overline{b}(x_{n-1})^{\circ p^{n-1}} \circ \overline{b}(x_n)^{\circ m} \circ [v^I]\right\}^{*\lambda},$$

to be expanded as in eq. (15.5). By the form [8, (15.5)] of the p-series, the only $*$-factors of the latter kind that are not trivial mod $\mathfrak{M}_2$ are

$$1 + \overline{b}(x_1)^{\circ p} \circ \overline{b}(x_2)^{\circ p^2} \circ \dots \circ \overline{b}(x_{n-1})^{\circ p^{n-1}} \circ \overline{b}(x_n)^{\circ p^j} \circ [v_j] \tag{19.24}$$

for $j > 0$. We can now efficiently extract the coefficient $R(i_1, i_2, \dots, i_n)$ of $x_1^{p^{i_1}} x_2^{p^{i_2}} \dots x_n^{p^{i_n}}$. From the factor (19.23) we have the term

$$F\left(b_{(i_n - 1)} \circ b^{\circ p}_{(i_1 - 2)} \circ b^{\circ p^2}_{(i_2 - 3)} \circ \dots \circ b^{\circ p^{n-1}}_{(i_{n-1} - n)}\right),$$

after some shuffling, while the factor (19.24) yields

$$b^{\circ p}_{(i_1-1)} \circ b^{\circ p^2}_{(i_2-2)} \circ \ldots \circ b^{\circ p^{n-1}}_{(i_{n-1}-n+1)} \circ b^{\circ p^j}_{(i_n-j)} \circ [v_j].$$

(We continue the convention of section 18 that meaningless terms, those involving any $b_{(i)}$ with $i < 0$, are treated as zero.) We now sum over π and j, taking the opportunity to permute the i_r in the terms with F (which introduces a sign), to obtain $(\mathcal{R}_{k_1,k_2,\ldots,k_n})$ in the desired form

$$\begin{aligned}(-1)^{n-1} \sum_{\pi} \epsilon_\pi F\left(b_{(i_1-1)} \circ b^{\circ p}_{(i_2-2)} \circ \ldots \circ b^{\circ p^{n-1}}_{(i_n-n)}\right) \\ + \sum_{\pi,j} \epsilon_\pi\, b^{\circ p}_{(i_1-1)} \circ b^{\circ p^2}_{(i_2-2)} \circ \ldots \circ b^{\circ p^{n-1}}_{(i_{n-1}-n+1)} \circ b^{\circ p^j}_{(i_n-j)} \circ [v_j] \equiv 0\end{aligned} \tag{19.25}$$

in $\overline{H}$ mod $\mathfrak{M}_2 + \mathfrak{M}_1 * \mathfrak{M}_1$. As before, the terms involving $[v_j]$ for $j < n$ cancel: when we interchange i_j and i_n, we obtain two identical terms having opposite signs. We therefore sum only over $j \geqslant n$. The terms of most interest are the two *leading* terms with $\pi = \mathrm{id}$:

$$(-1)^{n-1} F\left(b^{\circ L}\right) = (-1)^{n-1} F\left(b_{(k_1-1)} \circ b^{\circ p}_{(k_2-2)} \circ \ldots \circ b^{\circ p^{n-1}}_{(k_n-n)}\right) \tag{19.26}$$

and

$$b^{\circ pL} \circ [v_n] = b^{\circ p}_{(k_1-1)} \circ b^{\circ p^2}_{(k_2-2)} \circ \ldots \circ b^{\circ p^n}_{(k_n-n)} \circ [v_n], \tag{19.27}$$

for a certain multi-index L (different from section 18).

The reduction formula. We obtain a reduction formula for the general disallowed $\circ$-monomial (19.17) in $BP_*(\underline{BP}_k)$. First, we assume k is even. For any $n > 0$, $0 < k_1 < k_2 < \ldots < k_n$, and multi-indices M and J, the desired formula is:

$$\begin{aligned}b^{\circ p}_{(k_1-1)} \circ b^{\circ p^2}_{(k_2-2)} \circ \ldots \circ b^{\circ p^n}_{(k_n-n)} \circ b^{\circ M} \circ [v_n v^J] \\ \equiv - \sum_{\pi \neq \mathrm{id}} \epsilon_\pi\, b^{\circ p}_{(i_1-1)} \circ \ldots \circ b^{\circ p^n}_{(i_n-n)} \circ b^{\circ M} \circ [v_n v^J] \\ + (-1)^n F\left(b_{(k_1-1)} \circ b^{\circ p}_{(k_2-2)} \circ \ldots \circ b^{\circ p^{n-1}}_{(k_n-n)} \circ b^{\circ s^{-1}(M)} \circ [v^J]\right) \\ + (-1)^n \sum_{\pi \neq \mathrm{id}} \epsilon_\pi F\left(b_{(i_1-1)} \circ b^{\circ p}_{(i_2-2)} \circ \ldots \circ b^{\circ p^{n-1}}_{(i_n-n)} \circ b^{\circ s^{-1}(M)} \circ [v^J]\right) \\ \text{in } \overline{H} \text{ mod } \mathfrak{M}_{h+2} + \mathfrak{M}_{h+1} * \mathfrak{M}_{h+1}\ ,\end{aligned} \tag{19.28}$$

where we sum over permutations $\pi \in \Sigma_n$, $(i_1, i_2, \ldots, i_n) = \pi(k_1, k_2, \ldots, k_n)$, and $h = |J|$. (Terms involving $s^{-1}(M)$ with $m_0 \neq 0$ are to be omitted.) To obtain this, we first apply $- \circ b^{\circ M}$ to eq. (19.25), using eq. (15.13) to rewrite the terms involving F. The suppressed terms lie in $\mathfrak{M}_2 \circ b^{\circ M} \subset \mathfrak{M}_2$ and $(\mathfrak{M}_1 * \mathfrak{M}_1) \circ b^{\circ M} \subset \mathfrak{M}_1 * \mathfrak{M}_1$,

as we know from Lemma 19.14(a) that $\mathfrak{M}_2$ and $\mathfrak{M}_1$ are Hopf ring ideals. Then we apply the $*$-homomorphism $-\circ[v^J]$ and use Lemma 19.14(c).

Remark. Strictly speaking, this is only a reduction formula mod $\mathfrak{V}$, but it meets our present needs. One can work modulo the slightly smaller ideal $(v_1, v_2, \ldots)$ instead and extract a more complicated reduction formula that is valid in $BP_*(\underline{BP}_*)$ itself, without recourse to Nakayama's Lemma.

For odd k, the reduction formula takes the far simpler form

$$\begin{aligned} e\circ b^{\circ p}_{(k_1-1)}\circ b^{\circ p^2}_{(k_2-2)}\circ\ldots\circ b^{\circ p^n}_{(k_n-n)}\circ b^{\circ M}\circ[v_nv^J] \\ \equiv -\sum_{\pi\neq\mathrm{id}}\epsilon_\pi\, e\circ b^{\circ p}_{(i_1-1)}\circ b^{\circ p^2}_{(i_2-2)}\circ\ldots\circ b^{\circ p^n}_{(i_n-n)}\circ b^{\circ M}\circ[v_nv^J] \end{aligned}$$

mod $\mathfrak{M}_{h+2}$. To see this, one can suspend eq. (19.28) by applying $e\circ -$, which kills all $*$-products, including Fc; but it is far simpler to suspend eq. (18.22) instead.

Proof of Thm. 19.20. For $m>0$, it follows from eq. (19.28) that

$$\mathfrak{M}_m \subset \mathfrak{A}_m + \mathfrak{M}_{m+1} + \mathfrak{M}_m{*}\mathfrak{M}_m + F\mathfrak{M}_{m-1} \qquad \text{in } \overline{H}, \tag{19.29}$$

by using exactly the same orderings of monomials (reinterpreted) as in the proof of Thm. 18.16. For $m=0$, we clearly have $\mathfrak{M}_0 = \mathfrak{A}_0 + \mathfrak{M}_1$ because the generators of $\mathfrak{M}_0$ that are not in $\mathfrak{M}_1$ are all allowable.

We show by induction on m that the term $F\mathfrak{M}_{m-1}$ is not needed, that

$$\mathfrak{M}_m \subset \mathfrak{A}_m + \mathfrak{M}_{m+1} + \mathfrak{M}_m{*}\mathfrak{M}_m \tag{19.30}$$

for all $m\geqslant 1$. This is clear for $m=0$. If it holds for $m-1$, applying F yields

$$F\mathfrak{M}_{m-1} \subset F\mathfrak{A}_{m-1} + F\mathfrak{M}_m + F\mathfrak{M}_{m-1}{*}F\mathfrak{M}_{m-1}\ .$$

Each term on the right is already included in the other terms of eq. (19.29) and may be omitted.

Next, we dispose of $\mathfrak{M}_m * \mathfrak{M}_m$. On $*$-multiplying eq. (19.30) by $\mathfrak{M}_1^{*i}$ we have

$$\begin{aligned} \mathfrak{M}_m{*}\mathfrak{M}_1^{*i} &\subset \mathfrak{A}_m{*}\mathfrak{M}_1^i + \mathfrak{M}_{m+1}{*}\mathfrak{M}_1^i + \mathfrak{M}_m{*}\mathfrak{M}_m{*}\mathfrak{M}_1^i \\ &\subset \mathfrak{A}_m + \mathfrak{M}_{m+1} + \mathfrak{M}_m{*}\mathfrak{M}_1^{i+1}. \end{aligned}$$

It follows by induction on i that

$$\mathfrak{M}_m \subset \mathfrak{A}_m + \mathfrak{M}_{m+1} + \mathfrak{M}_m{*}\mathfrak{M}_1^{*i}$$

for all i. Since $\mathfrak{M}_1^{*i}$ is zero in each bigrading for large enough i, we must have $\mathfrak{M}_m \subset \mathfrak{A}_m + \mathfrak{M}_{m+1}$. As in the proof of Thm. 18.16, this implies $\mathfrak{M}_m = \mathfrak{A}_m$. □

The suspension. We can use eq. (19.28) to extract detailed information about the suspension homomorphism $e\circ -\colon Q^k_* \to PBP_*(\underline{BP}_{k+1})$ when k is odd. (When k is even, there is nothing to discuss: the allowable monomial $b^Iw^J \in Q^k_*$ suspends to the allowable $\circ$-monomial $e\circ b^{\circ I}\circ[v^J] \in PBP_*(\underline{BP}_{k+1})$.)

By Lemma 18.12(c), we can write every allowable monomial in Q_*^k uniquely in the extended canonical form

$$c = e\, b^{L-\Delta(0)}\, b^{(p-1)(L+s(L)+\ldots+s^{h-1}(L))}\, b^{s^h(M)}\, w_{n+1}^h w^J,$$

where $0 = k_0 \leqslant k_1 \leqslant k_2 \leqslant \ldots \leqslant k_n$, $n \geqslant 0$, $b^L = b_{(k_0)} b_{(k_1)}^p \ldots b_{(k_n)}^{p^n}$, M and J satisfy the conditions (18.9), and $h \geqslant 0$ is maximal. What happens to $e \circ c$ is that if $h > 0$, it is disallowed, as the derived relation $(\mathcal{R}_{k_0+1,k_1+2,\ldots,k_n+n+1})$ applies, and we pick out the leading term (19.26) mod $\mathfrak{V}$. If $h > 1$, we can repeat this cycle h times (always with the same indices k_u). In all cases, $e \circ c$ has the leading term

$$F^h(b^{\circ L} \circ b^{\circ M} \circ [v^J]), \tag{19.31}$$

where $b^{\circ L} \circ b^{\circ M} \circ [v^J]$ is allowable by Lemma 18.12(d) and primitive in $\overline{H}$ because $b^{\circ L}$ contains the factor $b_{(0)}$.

In fact, one can show that every primitive allowable $\circ$-monomial in $BP_*(\underline{BP}_{k+1})$ can be written uniquely in the form $b^{\circ L} \circ b^{\circ M} \circ [v^J]$, subject to the conditions (18.9). We have a computational verification mod $\mathfrak{V}$ of the isomorphism $Q_*^k \cong PBP_*(\underline{BP}_{k+1})$ induced by suspension.

The first nth order relation. The relation $(\mathcal{R}_{1,2,\ldots,n})$ is particularly important, as only the two leading terms are meaningful. Bendersky has pointed out (during the proof of [3, Thm. 6.2]) that with a little more attention to detail, one obtains a sharper version, the unstable analogue of Lemma 18.23.

Lemma 19.32. (Bendersky) *In $BP_*(\underline{BP}_{f(n)})$ we have the relation*

$$b_{(0)}^{\circ pm} \circ [v_n] \equiv v_n b_{(0)}^{\circ m} + (-1)^n \big(b_{(0)}^{\circ m}\big)^{*p} \qquad \mod I_n BP_*(\underline{BP}_{f(n)}), \tag{19.33}$$

for each $n > 0$, where $m = f(n)/2 = 1 + p + p^2 + \ldots + p^{n-1}$.

Proof. Although this result can be extracted from $(\mathcal{R}_{1,2,\ldots,n})$ by detailed examination, it is far simpler to return to $(\mathcal{R}_n)$. We proceed by induction on n, starting from eq. (19.11) for $n = 1$. For $n > 1$, we assume the result for all smaller n, and obtain it for n by evaluating $b_{(0)}^{\circ ph} \circ (\mathcal{R}_n)$ mod I_n, where $h = f(n-1)/2 = 1+p+p^2+\ldots+p^{n-2}$.

We recall that $(\mathcal{R}_n)$ is defined as the coefficient of x^{p^n} in eq. (19.10). On the left, we have $b_{(0)}^{\circ ph} \circ b(v_n x^{p^n} + \ldots)$ by [8, (15.5)], which provides only the term $v_n b_{(0)}^{\circ ph+1}$. The right side simplifies enormously, because $h > 0$ and $b_{(0)} \circ -$ kills $*$-decomposables; we obtain

$$b_{(0)}^{\circ ph} \circ P(x) = p b_{(0)}^{\circ ph} \circ \bar{b}(x) + \sum_{\lambda,I,m} \lambda b_{(0)}^{\circ ph} \circ \bar{b}(x)^{\circ m} \circ [v^I]\ .$$

By induction, $b_{(0)}^{\circ ph} \circ [v_j] \equiv 0$ mod I_n for all $j < n-1$, since $h = f(n-1)/2 > f(j)/2$.

Thus the only terms of interest in $[p](x)$ in our range of degrees are $v_n x^{p^n}$ and $v_{n-1} x^{p^{n-1}}$, as it follows from [8, (14.26)] and the map $BP \to K(n-1)$ of ring spectra that any terms in eq. (19.9) of the form $\lambda v_{n-1}^i x^m$ with $i > 1$ have λ divisible by

p. The term $v_n x^{p^n}$ yields $b_{(0)}^{\circ ph} \circ b_{(0)}^{\circ p^n} \circ [v_n]$, which is the leading term (19.27). By induction and eq. (15.13), $v_{n-1}x^{p^{n-1}}$ yields

$$b_{(0)}^{\circ ph} \circ b_{(1)}^{\circ p^{n-1}} \circ [v_{n-1}] \equiv (-1)^{n-1} F(b_{(0)}^{\circ h}) \circ b_{(1)}^{\circ p^{n-1}} \equiv (-1)^{n-1} F(b_{(0)}^{\circ h} \circ b_{(0)}^{\circ p^{n-1}}),$$

which is the other leading term, (19.26). □

The ideals $\mathfrak{J}_n$. For the unstable version of our splitting theorems we need the unstable analogue of the ideal $\mathfrak{J}_n$ of Defn. 18.25.

Definition 19.34. For $n \geqslant 0$, we define $\mathfrak{J}_n \subset BP_*(\underline{BP}_*)$ as the $*$-ideal generated by all elements of the form $c \circ ([v_j]-1)$, where $j > n$.

Lemma 19.35. *$\mathfrak{J}_n$ is a Hopf ring ideal in $BP_*(\underline{BP}_*)$.*

Proof. We apply Lemma 19.2; only (i) requires any comment. It holds for $[v_j]-1$, by the identity

$$\psi([v]-1) = ([v]-1) \otimes [v] + 1 \otimes ([v]-1), \tag{19.36}$$

which is valid for any $v \in BP^*$ by Prop. 11.2(a). We combine this with $\psi c = \sum_i c_i' \otimes c_i''$ to obtain

$$\psi(c \circ ([v]-1)) = \sum_i c_i' \circ ([v]-1) \otimes c_i'' \circ [v] + \sum_i c_i' \circ 1 \otimes c_i'' \circ ([v]-1), \tag{19.37}$$

which shows that (i) holds for the typical $*$-generator of $\mathfrak{J}_n$. □

Lemma 19.38. *$[v] \equiv 1 \bmod \mathfrak{J}_n$ for all $v \in J_n$.*

Proof. Suppose $v = v' + \lambda v_j v^K$ with $j > n$. As $\mathfrak{J}_n$ is a Hopf ring ideal, we have

$$[v] = [v'] * [\lambda v^K] \circ [v_j] \equiv [v'] * [\lambda v^K] \circ 1 = [v'] \qquad \bmod \mathfrak{J}_n \ .$$

The result follows by induction on the number of terms in v. □

The unstable analogue of Lemma 18.26 requires more detail but no new ideas.

Lemma 19.39. *For $k \leqslant f(n+1)$, $\mathfrak{J}_n \cap BP_*(\underline{BP}_k)$ is the $*$-ideal in $BP_*(\underline{BP}_k)$ generated by all elements that lie in $BP_*(\underline{BP}_k)$ and have any of the following forms, where v^J contains a factor v_j with $j > n$:*

(i) *(if k is even) an allowable monomial $b^{\circ I} \circ [v^J]$;*

(ii) *(if k is odd) an allowable monomial $e \circ b^{\circ I} \circ [v^J]$;*

(iii) *(if $k \leqslant 0$ and is even) $[\lambda v^J] - 1_k$, with $\lambda \in \mathbb{Z}_{(p)}$;*

(iv) *(if $k = f(n+1)$) a disallowed monomial*

$$b_{(k_1-1)}^{\circ p} \circ b_{(k_2-2)}^{\circ p^2} \circ \ldots \circ b_{(k_{n+1}-n-1)}^{\circ p^{n+1}} \circ [v_{n+1}]$$

with $0 < k_1 < k_2 < \ldots < k_{n+1}$.

Remark. To make (i) correct for $I = 0$, it is necessary to define $b^{\circ 0} = e^{\circ 0} = [1]-1$ as in Prop. 13.7, so that $b^{\circ 0} \circ [v^J] = [v^J] - 1$.

Proof. Denote by $\mathfrak{I}$ the $*$-ideal in $BP_*(\underline{BP}_k)$ generated by the stated elements. It is clear from Lemma 19.38 that $\mathfrak{I} \subset \mathfrak{J}_n$.

To show the converse, we fix k and a large m, and prove by *downward* induction on h that all elements in $BP_i(\underline{BP}_k)$ of the form $c \circ ([v_h]-1)$ lie in $\mathfrak{I}$ whenever $i < m$. This statement is vacuous for sufficiently large h (depending on m and k). We therefore fix $t > n$ and assume the statement holds for all $h > t$.

Case 1: $c = [\lambda v^J]$. (This includes the degenerate cases [1] and $1_k = [0_k]$.) Then $c \circ ([v_t]-1) = [\lambda v^J v_t] - 1$ is listed in (iii).

Case 2: $c = e^\epsilon \circ b^{\circ I}$. As in Lemma 18.26, $c \circ ([v_t]-1) = e^\epsilon \circ b^{\circ I} \circ [v_t]$ has to be allowable, except in the extreme case when $k = f(n+1)$ and $j = n+1$; either way, it is a listed generator of $\mathfrak{I}$.

Case 3: $c = e^\epsilon \circ b^{\circ I} \circ [v_h v^J]$ *allowable, where* $h \leqslant t$. From the form of Defn. 19.16, $c \circ ([v_t]-1) = e^\epsilon \circ b^{\circ I} \circ [v_h v_t v^J]$ remains allowable and is thus a listed generator of $\mathfrak{I}$.

Case 4: $c = e^\epsilon \circ b^{\circ I} \circ [v_h v^J]$, *with* $h > t$. We can write $c \circ ([v_t]-1) = e^\epsilon \circ b^{\circ I} \circ [v_t v^J] \circ ([v_h]-1)$, which lies in $\mathfrak{I}$ by induction, provided $i < m$.

By Thm. 19.18, we have enough $*$-generators c. If $c = a*d$, eqs. (10.11) and (19.36) give

$$c * ([v_t]-1) = a \circ ([v_t]-1) * d \circ [v_t] + a \circ 1 * d \circ ([v_t]-1),$$

which shows that the statement holds for $c = a * d$ whenever it holds for a and d. □

20. Additively unstable *BP*-objects

In this section, we discuss the additively unstable structures developed in sections 5 and 7 in the case $E = BP$, with particular attention to what becomes of the stable results of [8, §15]. We easily recover Quillen's theorem, that for any space X, the generators of $BP^*(X)$ all lie in non-negative degrees. Our main result Thm. 20.11 says in effect that there are no relations there either; more precisely, all relations follow from relations in non-negative degrees. We apply the theory to Landweber filtrations of an additively unstable module or algebra M, and find that the presence of additive unstable operations implies severe constraints on the degrees of the generators of M; this may be viewed as a better version of Quillen's theorem.

By Thms. 6.35 and 7.11, module and comodule structures are equivalent, with or without multiplication. The most convenient context remains the Second Answer of section 5, that an additively unstable BP-cohomology module (algebra) consists of a BP^*-module (BP^*-algebra) M equipped with coactions

$$\rho_M \colon M^k \longrightarrow M \mathbin{\widehat{\otimes}} Q(BP)^k_* \tag{20.1}$$

that (as k varies) form a homomorphism of BP^*-modules (BP^*-algebras) and satisfy the usual coaction axioms (6.33). We continue to abbreviate $Q(BP)^*_*$ to Q^*_*. The bigraded algebra Q^*_* was discussed in detail in section 18.

Connectedness. The principle is that nothing interesting ever happens in negative degrees. The first result in this direction is due to Quillen [22, Thm. 5.1].

Theorem 20.2. (Quillen) *For any space X, $BP^*(X)\hat{}$ is generated, as a BP^*-module, (topologically if X is infinite) by elements of positive degree and exactly one element of degree 0 for each component of X.*

This will be an immediate consequence of Lemmas 4.10 of [8] and 20.5 (below). Quillen's proof is geometric; in contrast, section 6 provides a global algebraic proof of the weak form of Quillen's theorem.

Theorem 20.3. *Given any integer $k < 0$, there exist for $n \geqslant 1$:*

(i) *additive unstable BP-operations r_n defined on $BP^k(-)$, with $\deg(r_n) \to \infty$ and $\deg(r_n) \geqslant |k|$ for all n;*

(ii) *elements $v(n) \in BP^*$;*

such that in any additively unstable BP-cohomology module M (e.g. $BP^*(X)\hat{}$ for any space X), any $x \in M^k$ decomposes as the (topological infinite) sum $x = \sum_n v(n) r_n x$, with $\deg(r_n x) \geqslant 0$ for all n.

In particular, M is generated (topologically) by elements of degree $\geqslant 0$.

Proof. Let $\{c_1, c_2, c_3, \ldots\}$ be the Ravenel-Wilson (or any other) basis of the free BP^*-module Q^k_*. By eq. (6.39) and the following Remark, we can write

$$x = \iota_k x = \sum_n \langle \iota_k, c_n \rangle \, x_n \tag{20.4}$$

with $x_n = r_n x$, where r_n denotes the operation dual to c_n. If $c_n \in Q^k_j$, we must have $j \geqslant 0$; then $\deg(r_n) = -\deg(c_n) = j - k \geqslant -k$ gives (i). We put $v(n) = \langle \iota_k, c_n \rangle$ and note that $\deg(x_n) = \deg(r_n) + \deg(x) = j \geqslant 0$. □

Remark. The coefficients in eq. (20.4) are readily computed from eq. (6.41) as $v(n) = Q(\epsilon) c_n$. Thus $v(n) = v^J$ if $c_n = e^\epsilon b^m_{(0)} w^J$, and vanishes for monomials c_n not of this form, so that many terms in eq. (20.4) are zero.

If M is bounded above or X is finite-dimensional, the sum is finite and no topology on M is needed.

To handle the generators in degree 0, we need a stronger hypothesis.

Lemma 20.5. *Let M be a connected* (see Defn. 7.14) *additively unstable algebra (e.g. $BP^*(X)\hat{}$ for any connected space X). Then as a topological BP^*-module, M is generated by $1_M \in M^0$ and elements of strictly positive degree. The generator 1_M is never redundant.*

Remark. Again, we may ignore the topology on M if M is bounded above or X is finite-dimensional.

Proof. We choose a basis $\{c_1, c_2, c_3, \ldots\}$ of Q^0_* with $c_1 = 1$; then given $x \in M^0$, we have eq. (20.4) with $\deg(x_n) = -\deg(c_n) > 0$ for all $n > 1$. Thus $x \equiv \langle \iota_0, 1 \rangle \, x_1$

mod L, where L denotes the BP^*-submodule of M generated (topologically) by the elements of positive degree.

For the collapse operation κ_0 introduced in Defn. 7.13, we similarly have $\kappa_0 x \equiv \langle \kappa_0, 1\rangle$ mod L. But $\langle \iota_0, 1\rangle = \langle \kappa_0, 1\rangle = 1$. As M is connected, $\kappa_0 x = \lambda 1_M$ for some $\lambda \in \mathbb{Z}_{(p)}$, by Defn. 7.14. We deduce from Thm. 20.3 that $M = L + (BP^*)1_M$. Since $\kappa L = 0$ and $\kappa(v1_M) = v1_M$ for any $v \in BP^*$, this is a direct sum decomposition. □

Primitive elements. We generalize the theory of Landweber filtrations to the additive unstable context by following the same strategy as stably. We explore a general unstable comodule M by looking for morphisms $f\colon BP^*(S^k, o) \to M$, for any $k \geqslant 0$. As a BP^*-module, $BP^*(S^k, o)$ is free on the canonical generator u_k. Thus f is determined, as a homomorphism of BP^*-modules, by the element $x = fu_k \in M$. Since $\rho_S u_k = u_k \otimes e^k$ by Prop. 12.3(a), the condition we need is clear.

Definition 20.6. Let M be any unstable comodule. If $k \geqslant 0$, we call $x \in M^k$ *additively unstably primitive* if $\rho_M x = x \otimes e^k$ in $M \mathbin{\widehat{\otimes}} Q^k_*$.

This obviously stabilizes to [8, Defn. 15.9], so that the additively unstable primitives of M form a subgroup of the stable primitives of M. *We do not define primitives in negative degrees,* for lack of a space S^k, and because e^k is meaningless. In fact, for $k < 0$, $x \otimes 1$ does not in general lie in the image of the stabilization

$$M \otimes Q(\sigma)\colon M \mathbin{\widehat{\otimes}} Q^k_* \longrightarrow M \mathbin{\widehat{\otimes}} BP_*(BP, o)\ .$$

(Perhaps it *never* does?)

Remark. One might object that we have abolished primitives in negative degrees by simply defining them away, while some alternate definition might work. However, no such definition can be satisfactory.

It is obvious from Defn. 12.6 that if $x \in M$ is primitive, so is $\Sigma x \in \Sigma M$. On the other hand, we shall find (nontrivially) in Cor. 20.12 that the only primitive in ΣM of degree zero is 0 (at least, for the kind of comodule we discuss). It follows, by suspending enough, that no definition of primitive can have both these properties and produce anything interesting in negative degrees.

It is immediate from the definition that if $x \in M^k$ is primitive,

$$\rho_M(vx) = x \otimes e^k \eta_R v \qquad \text{in } M \mathbin{\widehat{\otimes}} Q^*_* \qquad (\text{for } v \in BP^*). \tag{20.7}$$

We again recall from eq. (1.4) the numerical function

$$f(n) = \frac{2(p^n - 1)}{p - 1} = 2(p^{n-1} + p^{n-2} + \ldots + p + 1)$$

and remind that $\deg(v_n) = -(p-1)f(n)$ for $n > 0$.

Lemma 20.8. *Let $x \in M^k$ be a nonzero primitive element of the unstable BP-cohomology comodule M, and take $n > 0$.*

(a) *If $k < pf(n)$, then $v_n^i x \neq 0$ for all $i > 0$ and is not additively unstably primitive;*

(b) *If $k \geqslant pf(n)$ and $I_n x = 0$, then $v_n x$ is additively unstably primitive.*

Corollary 20.9. *If the additively unstably primitive element $x \in M$ satisfies $I_n x = 0$ and is a v_n-torsion element, then:*

(a) $\deg(v_n^i x) \geqslant pf(n)$ *whenever* $v_n^i x \neq 0$;

(b) $v_n^i x$ *is additively unstably primitive or zero for all* i.

Proof. We apply the Lemma to $v_n^i x$ by induction on i. Part (a) never applies (unless $v_n^i x = 0$); hence (b) must apply, to show that $v_n^{i+1} x$ is primitive. □

All this follows easily from Lemma 18.23.

Proof of Lemma 20.8. From eq. (20.7) we have

$$\rho_M(v_n^i x) = x \otimes e^k w_n^i \ .$$

In case (a), we note that by Defn. 18.4, $e^k w_n^i$ is a basis element of Q_*^*, so that $\rho_M(v_n^i x)$ is clearly nonzero. Even if $k \geqslant 2(p^n-1)i$, $v_n^i x$ is not primitive because $\rho_M(v_n^i x)$ is different from

$$v_n^i x \otimes e^{k-2(p^n-1)i} = x \otimes v_n^i e^{k-2(p^n-1)i}.$$

In case (b), we use the same formulae, with $i = 1$. The difference is that by Lemma 18.23, they now coincide, since $e^2 = b_{(0)}$ and $I_n x = 0$. □

Remark. For any $x \in M^k$, where $k \geqslant 0$, the coaction axiom (ii) of [8, (8.7)] forces $\rho_M x$ to have the form

$$\rho_M x = x \otimes e^k + \sum_\alpha x_\alpha \otimes c_\alpha,$$

where the c_α are other Ravenel-Wilson basis elements and $\deg(x_\alpha) > k$. Assuming that $k < pf(n)$, so that $e^k w_n$ is a basis element, let r be the operation (or functional) dual to it. Proceeding as in the proof of the Lemma, we obtain

$$r(v_n x) = x + \sum_\alpha \langle r, c_\alpha w_n \rangle \, x_\alpha,$$

which shows that $v_n x \neq 0$ if (for example) x is a module generator of M.

Landweber filtrations. The preceding results allow us to sharpen Thms. 15.10 and 15.11 of [8].

Theorem 20.10. *Let M be the BP^*-module with the single generator $x \in M^k$ and* $\mathrm{Ann}(x) = I_n$, *so that $M \cong \Sigma^k(BP^*/I_n)$.*

(a) *If $n > 0$, M admits an unstable comodule structure if and only if $k \geqslant f(n)-2$, and it is unique. The additively unstably primitive elements are those of the form $\lambda v_n^i x$, where $\lambda \in \mathbb{F}_p$, and $k + \deg(v_n^i) \geqslant f(n)$ if $i > 0$.*

(b) *If $n = 0$, $M \cong \Sigma^k BP^*$ admits an unstable comodule structure if and only if $k \geqslant 0$, and it is unique. The additively unstably primitive elements are those of the form λx, with $\lambda \in \mathbb{Z}_{(p)}$.*

Remark. Unlike the stable case, there are only finitely many primitives for $n > 0$. Of course, our definition forces this by requiring the degree of a primitive element to be non-negative. However, the theorem gives a much stronger condition.

Proof. By Thm. 20.3, we must have $k \geqslant 0$, the canonical generator x is necessarily primitive, and ρ must be given by eq. (20.7). Thus in (a), ρ will be well defined if and only if $e^k(\eta_R v) \in I_n Q_*^*$ whenever $v \in I_n$. Lemma 18.23 shows that this holds for $v = v_i$ for all $i < n$, since $k \geqslant f(n) - 2 \geqslant pf(i)$; this is sufficient. On the other hand, if $k < f(n) - 2 = pf(n-1)$, Lemma 20.8(a) (with n replaced by $n-1$) would contradict $v_{n-1}x = 0$.

Because ρ is a BP^*-module homomorphism (when it exists), the coaction axioms [8, (8.7)] need only be checked on x, where they are obvious. (Alternatively, $\Sigma^k(BP^*/I_n)$ is a quotient of the geometric comodule $BP^*(S^k, o)$.)

Since any additively unstably primitive element is also by design stably primitive, [8, Thm. 15.10] restricts the candidates for primitives to $\lambda v_n^i x$. Lemma 20.8 shows, by induction on $i \geqslant 0$, that $v_n^{i+1}x$ is additively unstably primitive if and only if $\deg(v_n^i x) \geqslant pf(n)$. This is what we want, since $|\deg(v_n)| = (p-1)f(n)$.

The proof of (b) is similar, but far simpler. □

With this restriction on the basic building blocks for an unstable module, we obtain the expected improvement in [8, Thm. 15.11].

Theorem 20.11. *Let M be an unstable BP-cohomology comodule that is finitely presented as a BP^*-module and has the discrete topology. Then there exists a filtration by subcomodules*

$$0 = M_0 \subset M_1 \subset \ldots \subset M_m = M,$$

where each M_i/M_{i-1} is generated, as a BP^*-module, by a single element x_i, whose annihilator ideal $\operatorname{Ann}(x_i) = I_{n_i}$ for some n_i, and $\deg(x_i) \geqslant f(n_i) - 2$ (if $n_i > 0$), or $\deg(x_i) \geqslant 0$ (if $n_i = 0$).

If, further, M is a spacelike BP^*-algebra (see Defn. 7.14)*, for example $BP^*(X)$ for any finite complex X, we can take each M_i to be an invariant ideal in M. At the last stage, we may take $x_m = 1$ and $n_m = 0$ or 1.*

Unfortunately, although the statement of the Theorem is exactly as expected, Landweber's method *fails*; Lemma 2.3 of [16] does not appear to be available here. (The BP^*-submodule $0\colon I_n = \{y \in M\colon I_n y = 0\}$ of M is defined but does not appear to be unstably invariant, owing to the dimensional restriction in Lemma 18.23.) Instead, we are forced to construct a suitable primitive $x_1 \in M$ directly. We would have preferred Landweber's construction because it guarantees that $\operatorname{Ann}(x_1)$ is maximal, which is useful in applications.

Proof. We start with a nonzero element $x \in M^k$ of top degree; by Thm. 20.2, $k \geqslant 0$ and x is automatically primitive. We construct a sequence of nonzero primitive elements $y_s \in M$ such that $I_s y_s = 0$, starting with $y_0 = x$. (Here, it is convenient to write $v_0 = p$.) We stop when we reach an element y_n that is v_n-torsion-free ($v_n^i y_n \neq 0$ for all $i > 0$) and put $x_1 = y_n$ and $n_1 = n$; this must occur eventually, by Lemma 20.8(a) (e. g. when $2p^n > k$). Assume we have y_s, where $s \geqslant 0$. If it is

v_s-torsion-free, we stop; this is y_n. Otherwise, take the smallest exponent q such that $v_s^q y_s = 0$ and put $y_{s+1} = v_s^{q-1} y_s$, to get $I_{s+1} y_{s+1} = 0$. By Cor. 20.9 (with s in place of n), y_{s+1} is primitive and $\deg(y_{s+1}) \geqslant pf(s)$.

We have found a primitive x_1 such that $I_n x_1 = 0$, x_1 is v_n-torsion-free, and $\deg(x_1) \geqslant pf(n-1) = f(n) - 2$. (If $n = 0$, there was no induction, and $\deg(y_0) = k \geqslant 0$.) As $\mathrm{Ann}(x_1)$ is an invariant ideal (in the stable sense), its radical ideal must be a finite intersection of invariant prime ideals in BP^*, therefore be I_m for some m. That is,

$$I_n \subset \mathrm{Ann}(x_1) \subset \sqrt{\mathrm{Ann}(x_1)} = I_m.$$

Since $v_n \notin \sqrt{\mathrm{Ann}(x_1)}$, we conclude that $m = n$ and $\mathrm{Ann}(x_1) = I_n$.

We finish as in the stable case, by setting $M_1 = (BP^*)x_1$, observing that this submodule is invariant by eq. (20.7), and replacing M by M/M_1. The induction continues until $M = 0$, and must terminate (easily, unlike the stable case), because each M^k is a finitely generated module over the Noetherian ring $\mathbb{Z}_{(p)}$ and we need consider only $k \geqslant 0$.

Now assume that M is a spacelike algebra, i.e. a product of connected algebras. This product is evidently finite, otherwise M would be uncountable. We easily reduce to the case when M is connected, which includes the case when $M = BP^*(X)$ for a connected finite complex X. By Lemma 20.5, the module $(BP^*)x$ is automatically an ideal in M; by induction, so is $(BP^*)y_s$ for each s, in particular M_1. At the last step, the module M/M_{m-1} is also an algebra; we therefore have $1 = vx_m$ and $x_m^2 = v'x_m$ for some $v, v' \in BP^*$. Then $x_m = 1x_m = vx_m^2 = vv'x_m = v'1$, which shows that $\mathrm{Ann}(1) = \mathrm{Ann}(x_m) = I_{n_m}$, and we may replace the generator x_m by 1. This implies $n_m \leqslant 1$, since $f(n) > 2$ for $n \geqslant 2$. □

Corollary 20.12. *For M as in Thm. 20.11, the suspension ΣM contains no nonzero additively unstably primitive elements in degree zero.*

Proof. We observe that

$$0 = \Sigma M_0 \subset \Sigma M_1 \subset \ldots \subset \Sigma M_m = \Sigma M$$

is a Landweber filtration of ΣM. By Thm. 20.10, the only unstable comodule of the form $\Sigma^k(BP^*/I_n)$ that has a nonzero primitive in degree zero is BP^*, which does not occur as a Landweber factor $\Sigma M_i/\Sigma M_{i-1}$ of ΣM. □

21. Unstable *BP*-algebras

In this section, we apply the theory of sections 10 and 19 to an unstable BP-cohomology algebra M. Our main application is Thm. 21.12 on Landweber filtrations of M, which contains Thm. 1.5 and improves on Thm. 20.11 by one degree.

Of course, we can always recover an additively unstable algebra from an unstable algebra simply by discarding the nonadditive operations. As a general rule, we can improve our results by one degree (but never more than one, in view of Thm. 13.6) by retaining all operations, at the cost of working in a far more complicated and unfamiliar environment. We developed the necessary machinery in section 10.

Primitive elements. It is clear from section 20 that the way to study a general unstable algebra M is to look for unstable morphisms $f\colon BP^*(S^k) \to M$ from the (relatively) well understood object $BP^*(S^k)$. Since $BP^*(S^k)$ is a free BP^*-module with basis $\{1_S, u_k\}$, f is uniquely determined, as a homomorphism of BP^*-modules, by $f1_S = 1_M$ and the element $x = fu_k \in M^k$. We extend the concept of primitive element to the unstable context, using Prop. 13.7 as a guide.

Definition 21.1. We call $x \in M^k$ (where $k \geqslant 0$) *unstably primitive* if

$$r(x) = \langle r, 1_k\rangle\, 1_M + \langle r, e_k\rangle\, x \qquad \text{for all } r, \tag{21.2}$$

where we interpret $e_0 = [1] - 1_0$ (as in Prop. 13.7).

This is a necessary and sufficient condition for f to be a morphism of unstable algebras, by eqs. (10.41), (10.16), and the Cartan formula (10.23). Among the unstable operations is the squaring operation, defined by $r(y) = y^2$ for all y, which implies that f is a homomorphism of BP^*-algebras (even if $k = 0$). When we restrict to additive operations, x is automatically additively primitive, and we have available all the results of section 20.

Many elementary properties of primitives follow directly from the definition.

Proposition 21.3. *Let M be an unstable algebra. Then:*

(a) *Unstable primitives are natural: if $x \in M$ is unstably primitive and $f\colon M \to N$ is a morphism of unstable algebras, then $fx \in N$ is also unstably primitive;*

(b) *The elements $0 \in M^k$ (for any $k \geqslant 0$) and 1_M are unstably primitive;*

(c) *If $x \in M^k$ is unstably primitive, where $k > 0$, then $x^2 = 0$;*

(d) *If $x \in M^k$ is unstably primitive, where $k > 0$, then λx is unstably primitive for any $\lambda \in \mathbb{Z}_{(p)}$;*

(e) *If $k > 0$ is odd, the unstable primitives in M^k form a $\mathbb{Z}_{(p)}$-submodule;*

(f) *If $k > 0$ is even and $x, y \in M^k$ are unstably primitive, then $x + y$ is unstably primitive if and only if $xy = 0$;*

(g) *The only nonzero unstable primitive in $BP^* = BP^*(T)$ is 1;*

(h) *Any unstable primitive $x \in M^0$ is idempotent, $x^2 = x$;*

(i) *If $x \in M^0$ is unstably primitive (and therefore idempotent), then the conjugate idempotent $1_M - x$ is also unstably primitive, but $-x$ is never unstably primitive (unless $-x = x$).*

Proof. Part (a) is trivial. Part (b) is clear from eqs. (10.41) and (10.28). As noted above, f is an algebra homomorphism, which gives (c) and (h). Then (g) follows from (b) and (h).

In (d), eq. (10.16) gives

$$r(\lambda x) = \langle r, 1_k\rangle\, 1_M + \langle r, [\lambda] \circ e_k\rangle\, x\ .$$

Since $k > 0$, Prop. 13.7(g) gives $[\lambda] \circ e_k = \lambda e_k$, which shows that λx is primitive.

We prove (e) and (f) together. If $x, y \in M^k$ are primitive, the Cartan formula (10.23) yields

$$r(x+y) = \langle r, 1_k\rangle\, 1_M + \langle r, e_k\rangle\, x + \langle r, e_k\rangle\, y + (-1)^k \langle r, e_k * e_k\rangle\, xy,$$

which is to be compared with eq. (21.2). The unwanted last term vanishes if k is odd, because e_k is then an exterior generator; but if k is even, $e_k * e_k$ is a basis element of $BP_*(\underline{BP}_k)$. For (e), we combine this with (d).

For (i), we first use eq. (10.29) to compute $r(-x) = \langle r, 1_0\rangle\, 1_M + \langle r, [-1] - 1_0\rangle\, x$, which shows that $-x$ is not primitive. We then use eqs. (10.23) and (10.41) to compute $r(1_M - x) = \langle r, [1]\rangle\, 1_M + \langle r, 1_0 - [1]\rangle\, x$, which shows that $1_M - x$ is primitive. □

We deduce that the Remark following Defn. 20.6 extends to show that unstable primitives cannot usefully be defined in negative degrees, even though the unstable suspension (see Defn. 13.4) had to be defined somewhat differently.

Corollary 21.4. *Let $M = BP^* \oplus \overline{M}$ be a based unstable BP-algebra.*

(a) *If $x \in \overline{M}$ is unstably primitive, so is $\Sigma x \in BP^* \oplus \Sigma\overline{M}$;*

(b) *If M is the kind of algebra considered in Thm. 20.11, there are no unstable primitives of degree zero in $BP^* \oplus \Sigma\overline{M}$ other than 0 and $1 \in BP^*$.*

Proof. Part (a) is clear from eq. (13.3). For (b), take any primitive $y \in BP^* \oplus \Sigma\overline{M}$ in degree 0. By Prop. 21.3(g), its augmentation in BP^* must be 0 or 1; if 1, we use Prop. 21.3(i) to replace y by $1 - y$. Then $y = \Sigma x$ for some $x \in \overline{M}$. As $y \in \Sigma\overline{M}$ is also additively primitive, Cor. 20.12 shows that $y = 0$. □

If X is the disjoint union $X_1 \coprod X_2$ of two spaces, we have $BP^*(X) = BP^*(X_1) \oplus BP^*(X_2)$, a product of unstable algebras. By Prop. 21.3, the elements $(1, 0)$ and $(0, 1)$ are primitive idempotents in $BP^*(X)$. The converse is also true, algebraically.

Theorem 21.5. *If $x \in M^0$ is an unstably primitive element in the unstable algebra M, other than 0 and 1_M, so that x and $1_M - x$ are idempotents, we have the splitting $M \cong xM \oplus (1_M - x)M$ of M as a product of unstable algebras.*

Proof. By Prop. 21.3(i), both x and $1_M - x$ are primitive and idempotent. We define the first projection $p_K\colon M \to K = xM$ by $p_K y = xy$; since x is idempotent, p_K is a homomorphism of BP^*-algebras. We define $p_L\colon M \to L = (1_M - x)M$ similarly, by $p_L y = (1_M - x)y$. These will give the desired splitting of M.

Given $y \in M$, we assume that $r_M(y)$ is in the standard form (10.22), where r_M denotes the operation of r on M. By the Cartan formula (10.36),

$$\begin{aligned} r_M(xy) &= \sum_\beta \langle r, 1_0 \circ d_\beta\rangle\, y_\beta + \sum_\beta \langle r, d_\beta - 1_0 \circ d_\beta\rangle\, xy_\beta \\ &= x r_M(y) + \sum_\beta \langle r, 1_0 \circ d_\beta\rangle\, (1_M - x) y_\beta . \end{aligned}$$

Hence $x r_M(xy) = x r_M(y)$, which shows that p_K is an unstable morphism, provided we define the action $r_K\colon K \to K$ of r on K by $r_K(z) = x r_M(z)$ for $z \in K \subset M$. All the necessary laws are inherited from M. We treat p_L similarly. □

Landweber filtrations. We repeat the theory of section 20, with an improvement of one in degree. If $x \in M^k$ is primitive in the unstable algebra M, where $k > 0$,

we compute from eq. (10.16) that

$$r(vx) = \langle r, 1_{k-h}\rangle\, 1_M + \langle r, e_k \circ [v]\rangle\, x \tag{21.6}$$

for any $v \in BP^{-h}$.

Lemma 21.7. *Let M be an unstable algebra, and $x \in M^k$ an unstably primitive element, where $k > 0$. Then the BP^*-submodule $(BP^*)x$ generated by x is an unstably invariant ideal in M, provided it is an ideal.*

Proof. We apply Lemma 8.10, with the help of eq. (21.6). □

It is still true that an element of positive top degree in M is automatically primitive, for lack of any other possible terms in $r(x)$.

We now use the additional structure of the unstable operations to sharpen Lemma 20.8. We recall once more from eq. (1.4) the numerical function

$$f(n) = \frac{2(p^n - 1)}{p - 1} = 2(p^{n-1} + p^{n-2} + \ldots + 1)\ .$$

Lemma 21.8. *Let $x \in M^k$ be a nonzero unstably primitive element of the unstable algebra M, and $n > 0$.*

(a) *If $k \leqslant pf(n)$, then $v_n^i x \neq 0$ for all $i > 0$ and is not unstably primitive;*

(b) *If $k > pf(n)$ and $I_n x = 0$, then $v_n x$ is unstably primitive.*

Corollary 21.9. *If the unstably primitive element $x \in M$ satisfies $I_n x = 0$ and is a v_n-torsion element, where $n > 0$, then:*

(a) $\deg(v_n^i x) > pf(n)$ *whenever* $v_n^i x \neq 0$.

(b) $v_n^i x$ *is unstably primitive or zero for all* i.

Proof. This is formally the same as for Cor. 20.9. □

Proof of Lemma. Part (a) adds nothing to Lemma 20.8(a) unless $k = pf(n)$, in which case we must take $i = 1$ if we are to have $\deg(v_n^i x) \geqslant 0$.

To test whether or not $v_n x$ is primitive, we have to compare

$$r(v_n x) = \langle r, 1_{k-d}\rangle\, 1_M + \langle r, e_k \circ [v_n]\rangle\, x$$

from eq. (21.6) with

$$\langle r, 1_{k-d}\rangle\, 1_M + \langle r, e_{k-d}\rangle\, v_n x = \langle r, 1_{k-d}\rangle\, 1_M + \langle r, v_n e_{k-d}\rangle\, x,$$

where we write $\deg(v_n) = -d$. For (a), we take $k = 2pm$, where $m = f(n)/2$. Lemma 19.32 expands $e_{2pm} \circ [v_n]$, to show that $r(v_n x)$ has the term $\pm\langle r, (b_{(0)}^{\circ m})^{*p}\rangle\, x$. As $b_{(0)}^{\circ m}$ is a $*$-polynomial generator of $BP_*(\underline{BP}_{2m})$, we deduce that $v_n x$ cannot be primitive or zero, whatever $\mathrm{Ann}(x)$ is. Similarly, for $i > 1$, $r(v_n^i x)$ has the term

$$\pm\langle r, (b_{(0)}^{\circ m})^{*p} \circ [v_n^{i-1}]\rangle\, x = \pm\langle r, (b_{(0)}^{\circ m} \circ [v_n^{i-1}])^{*p}\rangle\, x,$$

which shows that $v_n^i x \neq 0$.

For (b), we apply a further suspension $e_{k-2pm} \circ -$ to eq. (19.33), which kills decomposables, to yield

$$e_{k-2pm} \circ e_{2pm} \circ [v_n] \equiv v_n e_{k-2pm} \circ e_{2m} = v_n e_k \qquad \mod I_n \ ,$$

This shows that $v_n x$ is unstably primitive, a stronger statement than Lemma 20.8 provides. □

As promised, these two results improve on Lemma 20.8 and Cor. 20.9 by one degree. We use them to deduce the main theorems, which likewise improve on Thms. 20.10 and 20.11 by one.

Theorem 21.10. *Let M be the BP^*-module $BP^* \oplus (BP^*)x$, where the annihilator ideal $\mathrm{Ann}(x) = I_n$ and $\deg(x) = k > 0$. If M is made an algebra by taking $1 \in BP^*$ as the unit element and setting $x^2 = 0$, then:*

(a) *If $n > 0$, M admits an unstable algebra structure if and only if $k \geqslant f(n) - 1$, and it is unique. The nonzero unstably primitive elements in M are 1_M and the elements $\lambda v_n^i x$, where $\lambda \in \mathbb{F}_p$ $(\lambda \neq 0)$ and i satisfies $i = 0$ or $\deg(v_n^i x) > f(n)$.*

(b) *If $n = 0$, M admits a unique unstable structure. The nonzero unstably primitive elements in M are 1_M and the elements λx with $\lambda \in \mathbb{Z}_{(p)}$ $(\lambda \neq 0)$.*

Proof. In (a), we regard M as the quotient of the geometric unstable algebra $BP^*(S^k)$ with BP^*-basis $\{1_S, u_k\}$ by the ideal $I_n u_k$. The proof is formally the same as Thm. 20.10, except that we use Lemma 21.8 instead of Lemma 20.8, Cor. 21.9 instead of Cor. 20.9, and eq. (21.6) instead of eq. (20.7).

To determine the primitives in positive degrees, we first note that λx is primitive by Prop. 21.3(d) and apply Lemma 21.8 to $\lambda v_n^i x$, by induction on i. The primitives in degree zero are given already by Prop. 21.3. □

For completeness, we mention the analogous results for $k = 0$.

Proposition 21.11. *For the unstable algebra $BP^*(T) = BP^*$:*

(a) *BP^* has no proper nonzero invariant ideals;*

(b) *The unstable algebra $BP^*(S^0) \cong BP^* \oplus BP^*$ has the two copies of BP^* as its only proper nonzero invariant ideals.*

Proof. In (a), assume J is a nonzero ideal, and take $v \neq 0$ in J. As the elements $[v]$ are linearly independent in the Hopf ring, we see from eq. (11.1) that there is an operation r such that $r(v) = 1$ and $r(0) = 0$. Thus if J is invariant, we must have $1 \in J$, and therefore $J = BP^*$.

In (b), the operations are given similarly by

$$r((v, v')) = (\langle r, [v] \rangle, \langle r, [v'] \rangle) \in BP^* \oplus BP^*,$$

from which it is easy to see that any invariant ideal J that contains an element (v, v') with both v and v' nonzero must contain $1_S = (1, 1)$ and therefore everything. For other ideals J, we can apply (a). □

Theorem 21.12. *Given any spacelike* (see Defn. 7.14) *discrete unstable BP-cohomology algebra M that is finitely presented as a BP^*-module (e. g. $BP^*(X)$ for any finite complex X), there is a filtration by unstably invariant ideals*

$$0 = M_0 \subset M_1 \subset \ldots \subset M_m = M$$

in which each quotient M_i/M_{i-1} is generated, as a BP^*-module, by a single element x_i, whose annihilator ideal $\mathrm{Ann}(x_i) = I_{n_i}$ *for some* $0 \leqslant n_i < \infty$, *and* $\deg(x_i) \geqslant \max(f(n_i)-1, 0)$. *At the last step, $n_m = 0$ and we may take $x_m = 1_M$.*

Proof. This is formally identical to the algebra case of the proof of Thm. 20.11, except that we use the corresponding results from this section instead of section 20. Lemma 21.7 shows that $M_1 = (BP^*)x_1$ is indeed an invariant ideal. □

22. Additive splittings of *BP*-cohomology

Lemma 22.1 will construct idempotent operations θ_n in BP-cohomology, from which Parts (a) of our splitting theorems 1.12 and 1.16 will follow. In fact, we find a large class of θ_n, among which none seems to be preferred. At the end of the section, we give an example where *no* choice of θ_n has the obvious image $\mathbb{Z}_{(p)}[v_1, \ldots, v_n]$ on homotopy groups.

Lemma 22.1. *Assume that $k < f(n+1)$, where $n \geqslant 0$. Then there exists an additive idempotent operation $\theta_n : k \to k$ having the following properties:*

(i) *The image of $\theta_n : \underline{BP}_k \to \underline{BP}_k$ can be canonically identified with $\underline{BP\langle n\rangle}_k$;*

(ii) *The map θ_n factors to yield an H-space splitting $\overline{\theta}_n : \underline{BP\langle n\rangle}_k \to \underline{BP}_k$ of the canonical H-map $\pi\langle n\rangle : \underline{BP}_k \to \underline{BP\langle n\rangle}_k$;*

(iii) *For all spaces X, $\overline{\theta}_n$ naturally embeds $BP\langle n\rangle^k(X) \subset BP^k(X)$ as a summand, in the sense of abelian groups* (but not as BP^*-modules)*;*

(iv) *If also $k \geqslant f(n)$, the H-space $\underline{BP\langle n\rangle}_k$ does not decompose further.*

Remark. This result is best possible, in the sense that no additive θ_n exists when $k \geqslant f(n+1)$. (In more detail, choose m so that $f(m) \leqslant k < f(m+1)$; then $m > n$ and θ_m exists. Lemma 22.2 will show that if θ_n exists, we automatically have $\theta_n \circ \theta_m = \theta_n$. The modified idempotent $\theta'_n = \theta_m \circ \theta_n$ satisfies $\theta'_n \circ \theta_m = \theta'_n = \theta_m \circ \theta'_n$ and therefore decomposes $\underline{BP\langle m\rangle}_k$ further, contrary to (iv).) For $k > f(n+1)$ this is obvious, because $H_*(\underline{BP\langle n\rangle}_k)$ then has torsion [26]. The borderline case $k = f(n+1)$ will be discussed in section 23, where we find that a nonadditive θ_n *does* exist.

Proof of Thm. 1.12(a) and Thm. 1.16(a) (*assuming Lemma 22.1*)**.** The two Theorems are equivalent by [8, Thm. 3.6(a)]. As indicated, we use the splittings provided by Lemma 22.1, namely $\overline{\theta}_n : \underline{BP\langle n\rangle}_k \to \underline{BP}_k$ and, for each $j > n$, the map

$$f_j : \underline{BP\langle j\rangle}_{k+2(p^j-1)} \xrightarrow{\overline{\theta}_{j*}} \underline{BP}_{k+2(p^j-1)} \xrightarrow{v_j} \underline{BP}_k \,.$$

This $\overline{\theta}_j$ exists because

$$k + 2(p^j-1) < f(n+1) + (p-1)f(j) \leqslant pf(j) < f(j+1).$$

On homotopy groups, $\overline{\theta}_n$ induces a splitting of $BP^* \to BP^*/J_n$, while f_j induces a splitting of $J_{j-1} \to J_{j-1}/J_j$, in view of the commutative diagram

$$\begin{array}{ccccc} BP^*/J_j & \xrightarrow{\overline{\theta}_j} & BP^* & \xrightarrow{v_j} & J_{j-1} \\ & \searrow^{=} & \downarrow & & \downarrow \\ & & BP^*/J_j & \xrightarrow{\cong} & J_{j-1}/J_j \end{array}$$

in which multiplication by v_j induces the isomorphism.

We use the H-space structure of $\underline{BP}_k$ to multiply the maps θ_n and the f_j together to form a map $f\colon W \to \underline{BP}_k$ from the *restricted* product W (the union of the finite subproducts) of $\underline{BP\langle n\rangle}_k$ and the spaces $\underline{BP\langle j\rangle}_{k+2(p^j-1)}$. The homotopy groups of W are the direct *sums*

$$\pi_s(W) = \pi_s\left(\underline{BP\langle n\rangle}_k\right) \oplus \bigoplus_{j>n} \pi_s\left(\underline{BP\langle j\rangle}_{k+2(p^j-1)}\right).$$

We have enough information to conclude that f induces an isomorphism of filtered groups $f_*\colon \pi_*(W) \cong \pi_*(\underline{BP}_k)$. For connectedness reasons, the above sum is in fact a product of graded groups, which makes W homotopy equivalent to the desired *product* of spaces. Finally, Lemma 22.1 shows that all factors of W after the first are indecomposable, since

$$k + 2(p^j-1) \geqslant 2(p^j-1) = (p-1)f(j) \geqslant f(j).$$

If $k \geqslant f(n)$, so is the first. □

Construction of idempotent operations. To complete the proof, we need an idempotent operation θ_n. We actually construct the BP^*-linear functional $\langle\theta_n, -\rangle\colon Q^k_* = Q(BP)^k_* \to BP^*$ that corresponds to it in the list (6.9). We recall the coalgebra structure $(Q(\psi), Q(\epsilon))$ on Q^*_* and the ideal $\mathfrak{J}_n$ introduced in Defn. 18.25.

Lemma 22.2. *Assume the linear functional $\langle\theta_n, -\rangle\colon Q^k_* \to BP^*$ defined by the additive operation $\theta_n\colon k \to k$ satisfies the conditions:*

$$\begin{aligned} &\text{(i)} \quad \langle\theta_n, Q^k_* \cap \mathfrak{J}_n\rangle = 0; \\ &\text{(ii)} \quad \langle\theta_n, c\rangle \equiv Q(\epsilon)c \bmod J_n \qquad \text{for all } c \in Q^k_* . \end{aligned} \tag{22.3}$$

Then:

(a) *The homology homomorphism $Q(\theta_n)\colon Q^k_* \to Q^k_*$ satisfies*

(i) $Q(\theta_n)\mathfrak{J}_n = 0$;

(ii) $Q(\theta_n) \equiv \mathrm{id}\colon Q^k_* \to Q^k_* \bmod \mathfrak{J}_n$;

(b) *$Q(\theta_n)$ induces a splitting of the short exact sequence*

$$0 \longrightarrow Q^k_* \cap \mathfrak{J}_n \longrightarrow Q^k_* \longrightarrow Q^k_*/(Q^k_* \cap \mathfrak{J}_n) \longrightarrow 0$$

of left BP^*-modules;

(c) $\pi\langle n\rangle \circ \theta_n = \pi\langle n\rangle : \underline{BP}_k \to \underline{BP\langle n\rangle}_k$;

(d) *The operation θ_n is idempotent and has the properties listed in Lemma 22.1.*

We shall write $Q^k_*/\mathfrak{J}_n$ for the tedious but more accurate expression $Q^k_*/(Q^k_* \cap \mathfrak{J}_n)$.

Remark. From a more invariant point of view, $Q(\epsilon)$ induces the quotient augmentation $\overline{Q(\epsilon)}: Q^k_*/\mathfrak{J}_n \to BP^*/J_n$. The conditions (22.3) on $\langle \theta_n, -\rangle$ are conveniently expressed by the commutative diagram

$$\begin{array}{ccc} Q^k_* & \overset{\langle\theta_n,-\rangle}{\cdots\cdots\cdots\to} & BP^* \\ \Big\downarrow{\scriptstyle \pi} & \swarrow & \Big\downarrow \\ Q^k_*/\mathfrak{J}_n & \xrightarrow{\overline{Q(\epsilon)}} & BP^*/J_n \end{array} \tag{22.4}$$

in which the vertical arrows are the obvious projections. In words, we plan to lift $\overline{Q(\epsilon)}$ to a homomorphism of BP^*-modules $Q^k_*/\mathfrak{J}_n \to BP^*$ and define $\langle \theta_n, -\rangle$ as the composite. This is easy if $Q^k_*/\mathfrak{J}_n$ is a free BP^*-module (and in view of (b), impossible otherwise).

Proof. We enlarge diag. (22.4) to the commutative diagram

$$\begin{array}{ccccccc} Q^k_* & \xrightarrow{Q(\psi)} & Q^*_* \otimes Q^k_* & \overset{1\otimes\langle\theta_n,-\rangle}{\cdots\cdots\to} & Q^*_* \otimes BP^* & \xrightarrow{\lambda_R} & Q^k_* \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle 1\otimes\pi} & \swarrow & \Big\downarrow & & \Big\downarrow{\scriptstyle \pi} \\ Q^k_*/\mathfrak{J}_n & \xrightarrow{\overline{Q(\psi)}} & Q^*_* \otimes Q^k_*/\mathfrak{J}_n & \xrightarrow{1\otimes\overline{Q(\epsilon)}} & Q^*_* \otimes BP^*/J_n & \xrightarrow{\overline{\lambda}_R} & Q^*_*/\mathfrak{J}_n \end{array}$$

of BP^*-module homomorphisms, where $\overline{Q(\psi)}$ and $\overline{\lambda}_R$ are quotients of $Q(\psi)$ and λ_R. By Lemma 6.51(c), we recover $Q(\theta_n)$ as the top row, while the bottom row reduces by diag. (6.31) to the identity homomorphism of $Q^k_*/\mathfrak{J}_n$. Thus the diagonal provides a splitting $j: Q^k_*/\mathfrak{J}_n \to Q^k_*$ such that $j \circ \pi = Q(\theta_n)$ and $\pi \circ j = 1$.

This is enough to establish (a), that $Q(\theta_n)$ is idempotent with kernel exactly $Q^k_* \cap \mathfrak{J}_n$. Part (b) is merely a restatement of (a). It follows that θ_n also is idempotent.

By [8, Lemma 3.9], the idempotent operation θ_n is represented in Ho by the idempotent map $\theta_n = i_2 \circ p_2$ on the product $W = W_1 \times W_2$ of H-spaces, where $i_2: W_2 \to W$ and $p_2: W \to W_2$. Corollary 12.4 gives the effect of θ_n on homotopy groups: eq. (22.3)(i) shows that $\theta_{n*}v = 0$ if $v \in J_n$, while (ii) shows that

$$\theta_{n*}v \equiv Q(\epsilon)(e^{k+h}\eta_R v) = v \bmod J_n \qquad \text{in } \pi_*(\underline{BP}_k) \cong BP^*$$

for all $v \in BP^{-h}$. These two statements identify $\pi_*(W_2)$ with BP^*/J_n; more precisely, the composite $f = \pi\langle n\rangle \circ i_2: W_2 \to \underline{BP}_k \to \underline{BP\langle n\rangle}_k$ induces the desired

isomorphism on homotopy groups and is thus an isomorphism of abelian group objects in *Ho*.

We need (c) to be sure our identifications are correct. Now that we know $\underline{BP\langle n\rangle}_k$ is a summand of $\underline{BP}_k$, it is enough to work in $QBP_*(-)$. By construction, $Q\pi\langle n\rangle_*$ kills $\mathfrak{J}_n$; this, with (a)(ii), gives $Q\pi\langle n\rangle_* \circ Q\theta_{n*} = Q\pi\langle n\rangle_*$.

We can now define the splitting $\overline{\theta}_n = i_2 \circ f^{-1}\colon \underline{BP\langle n\rangle}_k \to \underline{BP}_k$ of $\pi\langle n\rangle$, so that $\pi\langle n\rangle \circ \overline{\theta}_n = 1$. From (c), we have $\pi\langle n\rangle = \pi\langle n\rangle \circ \theta_n = \pi\langle n\rangle \circ i_2 \circ p_2 = f \circ p_2$, which shows that the idempotent $\overline{\theta}_n \circ \pi\langle n\rangle = \overline{\theta}_n \circ f \circ p_2 = i_2 \circ p_2 = \theta_n$ is as expected. Now we can read off properties (i), (ii), and (iii) of Lemma 22.1.

Property (iv) was proved in [26], but also follows from Cor. 12.4. Suppose there is a splitting

$$\underline{BP}_k \simeq W_1 \times \underline{BP\langle n\rangle}_k \simeq W_1 \times W \times W'$$

of H-spaces that induces the decomposition $BP^* = J_n \oplus G \oplus G'$ on homotopy groups, where $1 \in G$, and let r be the idempotent that splits off W', so that $\langle r, 1\rangle = 0$ and $\langle r, Q^k_* \cap \mathfrak{J}_n\rangle = 0$. Suppose that W' is $(k+h-1)$-connected, where we must have $h > 0$. Then $\langle r, c\rangle = 0$ for all $c \in Q^k_i$ whenever $i < k + h$.

Choose a nonzero element $v \in BP^{-h}$ that lies in G' and is not divisible by p. Then $r_* v = v$ in homotopy and $v \notin I_1 + J_n$ (recall that $I_1 = (p)$). Obviously, $v \in I_\infty = I_{n+1} + J_n$. There must be some integer m, satisfying $1 \leqslant m \leqslant n$, such that $v \in I_{m+1} + J_n$ but $v \notin I_m + J_n$. We write $v = py_0 + \sum_{j=1}^m v_j y_j + z$, with $z \in J_n$. Since

$$k + h \geqslant f(n) + 2(p^j - 1) = f(n) + (p-1)f(j) \geqslant pf(j),$$

we have enough factors e to apply Lemma 18.23 for each $j \leqslant m$, in the form

$$\begin{aligned} \langle r, e^{k+h} w_j \eta_R y_j\rangle &\equiv \langle r, v_j e^{k+h-2(p^j-1)} \eta_R y_j\rangle \qquad \bmod I_j \\ &= v_j \langle r, e^{k+h-2(p^j-1)} \eta_R y_j\rangle = 0\,. \end{aligned}$$

By Cor. 12.4, $r_* v \equiv 0 \bmod I_m$, which contradicts our choices of v and m. □

Proof of Lemma 22.1. Lemma 18.26(a) makes it obvious that linear functionals $\langle \theta_n, -\rangle$ exist as in diag. (22.4), so that Lemma 22.2 applies. □

Example. Even in the simplest case, namely $\theta_1\colon \underline{BP}_2 \to \underline{BP}_2$ for $p = 2$, θ_{1*} *never* induces the obvious splitting on homotopy groups. (Presumably, this failure is completely general.) We compute $\theta_{1*} v_1^3$ in terms of the Hazewinkel generators [11]. The element $b_{(0)}^4 w_1^3 \in Q^2_*$ is not allowable; instead,

$$b_{(0)}^4 w_1^3 = -\frac{12}{7} v_1 b_{(0)} b_{(1)} w_1 + \left(v_1^3 + \frac{4}{7} v_2\right) b_{(0)} - \frac{10}{7} v_1^2 b_{(1)} - \frac{4}{7} b_{(0)}^4 w_2 - \frac{8}{7} b_{(2)},$$

as can be checked by stabilizing and working in $BP_*(BP, o)$. By construction, $\langle \theta_1, -\rangle$ takes the values 1 on $b_{(0)}$, λv_2 on $b_{(2)}$ for some $\lambda \in \mathbb{Z}_{(2)}$, and zero on the other

allowable monomials that appear. Thus by Cor. 12.4,

$$\theta_{1*}v_1^3 = v_1^3 + \frac{4-8\lambda}{7}v_2 \ ,$$

which always contains a term in v_2.

Remark. It is often useful to arrange the operations $\theta_n\colon k \to k$ compatibly as n and k vary. However, we emphasize that Thm. 1.12 as stated requires no compatibility conditions whatever.

For fixed n, compatibility in k is easily arranged. Given $\theta_n\colon k \to k$ that satisfies conditions (22.3), the looped operation $\Omega\theta_n\colon k-1 \to k-1$ has the functional

$$Q_*^{k-1} \xrightarrow{e} Q_*^k \xrightarrow{\langle \theta_n, - \rangle} BP^*$$

and clearly again satisfies (22.3). We may choose $\theta_n\colon k \to k$ arbitrarily for $k = f(n+1)-1$ and use this approach for all lower k.

For fixed k, we have θ_n for all sufficiently large n. The compatibility condition $\theta_n \circ \theta_{n+1} = \theta_n$ (equivalently, $\operatorname{Ker}\theta_{n+1} \subset \operatorname{Ker}\theta_n$) is automatic, from Lemma 22.2. The other condition, $\theta_{n+1} \circ \theta_n = \theta_n$ (equivalently, $\operatorname{Im}\theta_n \subset \operatorname{Im}\theta_{n+1}$), does *not* hold in general, but can be arranged for all n simultaneously by replacing each θ_n by $\theta'_n = \ldots \circ \theta_{n+2} \circ \theta_{n+1} \circ \theta_n$. (The infinite composite presents no difficulty, as $Q(\theta_n) = \mathrm{id}\colon Q_i^k \to Q_i^k$ for $i < k + 2(p^{n+1}-1)$.) This results in a sequence of commuting idempotents θ_n that satisfy $\theta_n \circ \theta_m = \theta_m \circ \theta_n = \theta_n$ whenever $n < m$.

23. Unstable splittings of *BP*-cohomology

In this section, we improve the splitting in Lemma 22.1 by one by allowing the idempotent operation θ_n to be nonadditive. We defer the proof until after stating Lemma 23.5. For this, we need the more detailed relations in the Hopf ring developed in section 19.

Lemma 23.1. *Assume that $k = f(n+1)$, where $n \geqslant 0$. Then there is a nonadditive operation $\theta_n\colon k \to k$ having the following properties:*

(a) *It satisfies the axioms* [8, (3.11)] *and so is idempotent;*

(b) *It has a coimage* $\operatorname{Coim}\theta_n$ *which is represented by the H-space* $\underline{BP\langle n\rangle}_k$*;*

(c) *Its representing map $\theta_n\colon \underline{BP}_k \to \underline{BP}_k$ factors to yield a section $\overline{\theta}_n\colon \underline{BP\langle n\rangle}_k \to \underline{BP}_k$ (not an H-map) of the canonical H-map $\pi\langle n\rangle\colon \underline{BP}_k \to \underline{BP\langle n\rangle}_k$.*

Proof of Thms. 1.12 and 1.16, for $k = f(n+1)$ (*assuming Lemma 23.1*). This is almost identical to the proof given in section 22 for $k < f(n+1)$, except that we apply [8, Lemma 3.10] instead of [8, Lemma 3.9]. The maps f_j appearing there are still H-maps; only $\overline{\theta}_n$ is not. We can still represent $\operatorname{Ker}\theta_n$ by $\prod_{j>n} \underline{BP\langle j\rangle}_{k+2(p^j-1)}$.

If any of the spaces decomposed as a product, we could apply the loop space functor Ω to obtain an H-space decomposition of $\underline{BP}_{k-1}$, using additive operations, which would contradict the part of Thm. 1.12 already proved. □

Of course, we know from Lemma 22.1 that for $k = f(n+1)$, $\theta_n: k \to k$ can never be additive and that $\overline{\theta}_n$ is never an H-map. However, looping gives an *additive* idempotent operation $\Omega\theta_n: k-1 \to k-1$, which will be one of those provided by Lemma 22.1. We have the converse, which we prove after stating Lemma 23.5.

Theorem 23.2. *Let $\theta_n: k-1 \to k-1$ be any of the additive idempotent operations provided by Lemma 22.1. Then:*

(a) *If $k-1$ is even, θ_n can be delooped uniquely to an additive idempotent operation $k \to k$ as in Lemma 22.1;*

(b) *If $k-1$ is odd, θ_n can be delooped (not uniquely) to a* nonadditive *idempotent operation $k \to k$ as in Lemma 23.1.*

The next two lemmas constitute the unstable analogue of Lemma 22.2. They are far more complicated, because instead of $Q(\psi)$, we have only the natural transformation $\psi: U \to UU$. This requires knowledge of the homology homomorphisms r_* induced by each operation r, which is provided by Thms. 10.19 and 10.33 and the properties of each $\circ$-generator of $BP_*(\underline{BP}_*)$. *We warn that as a consequence, the form of the proofs runs totally counter to traditional proofs involving cohomology operations.* We abbreviate $\langle r, \mathfrak{J}_n \cap BP_*(\underline{BP}_k)\rangle$ to $\langle r, \mathfrak{J}_n\rangle$, etc.

Lemma 23.3. *If the unstable operation $r: k \to m$ satisfies $\langle r, \mathfrak{J}_n\rangle = 0$, then the homology homomorphism $r_*: BP_*(\underline{BP}_k) \to BP_*(\underline{BP}_m)$ satisfies $r_*\mathfrak{J}_n = 0$.*

Proof. Our plan is to show that $r_*c = 0$ in three steps, depending on the form of $c \in \mathfrak{J}_n$, simultaneously for *all* operations $r: k \to m$ that satisfy $\langle r, \mathfrak{J}_n\rangle = 0$, where $c \in BP_*(\underline{BP}_k)$ determines k and m is arbitrary.

Case 1: $c = [v_j] - 1$, where $j > n$. By hypothesis, $\langle r, [v_j]\rangle = \langle r, 1\rangle$. Then by Prop. 11.2(g),

$$r_*([v_j] - 1) = [\langle r, [v_j]\rangle] - [\langle r, 1\rangle] = 0.$$

Case 2: $c = a \circ ([v_j]-1)$, where $j > n$. Thus c is a $*$-generator of $\mathfrak{J}_n$. We apply Thm. 10.33(c); the operations r''_α defined by eq. (10.35) satisfy our hypothesis

$$\langle r''_\alpha, d\rangle = \pm\langle r, c_\alpha \circ d\rangle = 0 \qquad \text{for all } d \in \mathfrak{J}_n$$

because $c_\alpha \circ d \in \mathfrak{J}_n$, $\mathfrak{J}_n$ being a Hopf ring ideal by Lemma 19.35. Using eq. (19.36) to compute the iterated coproduct $\Psi([v_j]-1)$, we see that every term of r_*c in eq. (10.37) contains a factor $r''_{\alpha *}([v_j]-1)$, which vanishes by Case 1.

Case 3: $c = a * b$, with b as in Case 2. Since such elements span $\mathfrak{J}_n$ as a BP^*-module, this will complete the proof. We apply Thm. 10.19(c); the operations r''_α defined by eq. (10.21) satisfy our hypothesis

$$\langle r''_\alpha, d\rangle = \pm\langle r, c_\alpha * d\rangle = 0 \qquad \text{for all } d \in \mathfrak{J}_n$$

because $\mathfrak{J}_n$ is a $*$-ideal. Using eq. (19.37) to compute the iterated coproduct Ψb, we see that every term of r_*c in eq. (10.25) contains a factor of the form $r''_{\alpha *}(b' \circ ([v_j]-1))$, which vanishes by Case 2. □

Lemma 23.4. *Let $r\colon k \to m$ be an unstable operation.*

(a) *If r satisfies $\langle r, c\rangle \in J_n$ for all $c \in BP_*(\underline{BP}_k)$, then $r_*c \equiv (\epsilon c)1_m$ mod $\mathfrak{J}_n$ for all $c \in BP_*(\underline{BP}_k)$;*

(b) *If r satisfies $\langle r, c\rangle \equiv Q(\epsilon)q_k c \bmod J_n$ for all $c \in BP_*(\underline{BP}_k)$, then $r_*c \equiv c$ mod $\mathfrak{J}_n$ for all $c \in BP_*(\underline{BP}_k)$.*

Proof. We prove (a) in five steps, depending on the form of c, simultaneously for all $r\colon k \to m$ that satisfy the hypothesis, where $c \in BP_*(\underline{BP}_k)$ determines k and m is arbitrary. We work throughout mod $\mathfrak{J}_n$, which is a Hopf ring ideal by Lemma 19.35.

Case 1: $c = [v]$, for any $v \in BP^*$. By Prop. 11.2(g) and Lemma 19.38, $r_*[v] = [\langle r, [v]\rangle] \equiv 1$. This includes the special case $c = 1 = [0]$.

Case 2: $c = e$. By Prop. 13.7(h) and Lemma 19.38, $r_*e \equiv 1 * 1\circ e = 1 * 0 = 0$.

Case 3: $c = b_i$, where $i > 0$. By Prop. 15.3, working formally in $BP_*(\underline{BP}_m)[[x]]$,

$$r_*b(x) = [\langle r, 1_2\rangle] * \mathop{*}_{j>0} b(x)^{\circ j}\circ[\langle r, b_j\rangle] \equiv \mathop{*}_{j>0} b(x)^{\circ j}\circ 1 = \mathop{*}_{j>0} \epsilon b(x)^{\circ j} = 1\ .$$

The coefficient of x^i gives $r_*b_i \equiv 0$.

Case 4: $c = a\circ b$, where $b = e$ or $b = b_i$ for some $i > 0$. We apply Thm. 10.33(c); the operations r''_α defined by eq. (10.35) satisfy the hypothesis $\langle r''_\alpha, d\rangle = \pm\langle r, c_\alpha \circ d\rangle \in J_n$ for all d. Then using Prop. 13.7(d) or Prop. 15.3(c) to compute the iterated coproduct Ψb, we see that every term of r_*c in eq. (10.37) contains a factor $r''_{\alpha *}e$ or $r''_{\alpha *}b_j$ with $j > 0$, which lies in $\mathfrak{J}_n$ by Case 2 or Case 3. This, with Case 1, takes care of all the $*$-generators (19.4) of $BP_*(\underline{BP}_*)$.

Case 5: $c = a * d$, with d as in Case 4. We apply Thm. 10.19(c) and again find that each r''_α satisfies our hypothesis $\langle r''_\alpha, g\rangle = \pm\langle r, c_\alpha * g\rangle \in J_n$ for all g. In the iterated coproduct $\Psi d = \sum_j \otimes_\alpha d_{j,\alpha}$, every term contains a factor $d_{j,\alpha}$ to which Case 4 applies. Thus every term of r_*c in eq. (10.25) has a factor $r''_{\alpha *}d_{j,\alpha} \equiv 0$.

As every $*$-monomial in the $\circ$-generators of $BP_*(\underline{BP}_*)$ is included in Cases 1 and 5 (by writing $[v] * [v'] = [v+v']$), this completes the proof of (a).

For (b), we recall from eq. (10.42) that $\langle \iota_k, c\rangle = Q(\epsilon)q_k c$, so that (a) applies to $r - \iota_k$. We apply eq. (10.17) to $r = (r-\iota_k) + \iota_k$ to deduce that for any $c \in E_*(\underline{E}_k)$, $r_*c \equiv \sum_i (\epsilon c'_i)c''_i = c$, where as usual we write $\psi c = \sum_i c'_i \otimes c''_i$. □

We need one more result before we prove Lemma 23.1 and Thm. 23.2. The structure of $BP_*(\underline{BP}_k)/\mathfrak{J}_n$ is much more opaque when $k = f(n+1)$. We defer the proof until after Lemma 23.12.

Lemma 23.5. *For $k \leqslant f(n+1)$, where $n \geqslant 0$:*

(a) *$BP_*(\underline{BP}_k)/\mathfrak{J}_n$ is a free BP^*-module;*

(b) *The homomorphism $Q(BP)_*^{k-1}/\mathfrak{J}_n \to BP_*(\underline{BP}_k)/\mathfrak{J}_n$ induced by suspension is a split monomorphism of BP^*-modules.*

Note that we have two different ideals $\mathfrak{J}_n$ here. One is an ideal in the algebra Q_*^* in the ordinary sense, while the other is a Hopf ring ideal in $BP_*(\underline{BP}_*)$.

Proof of Lemma 23.1 (*assuming Lemma 23.5*). To apply the method of Lemma 22.1, we need an operation $\theta_n\colon k \to k$ that satisfies $\theta_*\mathfrak{J}_n = 0$ and $\theta_* \equiv \mathrm{id} \bmod \mathfrak{J}_n$. In view of Lemma 23.3 and Lemma 23.4(b), these conditions are ensured by (and in fact equivalent to) the following conditions on the linear functional $\langle\theta_n, -\rangle$:

$$\begin{aligned} &\text{(i) } \langle\theta_n, \mathfrak{J}_n\rangle = 0; \\ &\text{(ii) } \langle\theta_n, c\rangle \equiv Q(\epsilon)q_k c \bmod J_n \text{ for all } c \in BP_*(\underline{BP}_k). \end{aligned} \tag{23.6}$$

Therefore we need to fill in the diagram

$$\begin{array}{ccccc} Q_*^{k-1} & \longrightarrow & BP_*(\underline{BP}_k) & \overset{\langle\theta_n,-\rangle}{\cdots\cdots\blacktriangleright} & BP^* \\ \downarrow & & \downarrow & \nearrow & \downarrow \\ Q_*^{k-1}/\mathfrak{J}_n & \longrightarrow & BP_*(\underline{BP}_k)/\mathfrak{J}_n & \longrightarrow & BP^*/J_n \end{array} \tag{23.7}$$

analogous to diag. (22.4) with a lifting $BP_*(\underline{BP}_k)/\mathfrak{J}_n \to BP^*$ of the homomorphism $BP_*(\underline{BP}_k)/\mathfrak{J}_n \to BP^*/J_n$ induced by $Q(\epsilon)\circ q_k$, which then defines $\langle\theta_n, -\rangle$. Lemma 23.5(a) makes this easy to do.

For (a), we must verify the axioms [8, (3.11)] on θ_n. The first holds trivially, for dimensional reasons. The second is the identity $\theta_n(x+z) = \theta_n(x)$, for $z = y - \theta_n(y)$. We assume that the standard form $r(x) = \sum_\alpha \langle r, c_\alpha\rangle\, x_\alpha$ holds for all r, as in eq. (10.3). Then by eq. (10.20), $\theta_n(x+z) = \sum_\alpha x_\alpha \theta''_\alpha(z)$, where the operation θ''_α is defined as having the functional $\langle\theta''_\alpha, c\rangle = \langle\theta_n, c_\alpha * c\rangle$. Because $z = (\iota_k - \theta_n)(y)$, we have only to prove that $(\iota_k - \theta_n)^*\theta''_\alpha = \langle\theta_n, c_\alpha\rangle\, 1$ in $BP^*(\underline{BP}_k)$ for each α. We compute the associated linear functional as

$$\langle(\iota_k - \theta_n)^*\theta''_\alpha, c\rangle = \langle\theta''_\alpha, (\iota_k - \theta_n)_* c\rangle = \langle\theta_n, c_\alpha * (\iota_k - \theta_n)_* c\rangle\ .$$

By Lemma 23.4(a), $(\iota_k - \theta_n)_* c \equiv (\epsilon c)1 \bmod \mathfrak{J}_n$. As $\langle\theta_n, -\rangle$ kills $\mathfrak{J}_n$ by Lemma 23.3 and $\mathfrak{J}_n$ is an ideal, this agrees with $\langle\theta_n, (\epsilon c)c_\alpha\rangle = \langle\theta_n, c_\alpha\rangle \epsilon c$. Now we can apply [8, Lemma 3.10] to construct the coimage of θ_n.

For (b) and (c), we have to check that θ_n acts as desired on homotopy groups. By Lemma 13.9, θ_{n*} is given on $v \in BP^{-h} \cong \pi_{k+h}(\underline{BP}_k)$ by $\theta_{n*}v = \langle\theta_n, e^{\circ\, k+h}\circ[v]\rangle$. For $v \in J_n$, we have $[v] \equiv 1 \bmod \mathfrak{J}_n$ by Lemma 19.38, so that $\theta_{n*}v = 0$ by (i). For any v, (ii) gives $\theta_{n*}v \equiv Q(\epsilon)q_k(e^{\circ\, k+h}\circ[v]) = v \bmod J_n$. □

Proof of Thm. 23.2 (*assuming Lemma 23.5*). Part (a) is trivial and belongs in section 22, as suspension induces an isomorphism $Q_*^{k-1} \cong Q_*^k$ and preserves the conditions (22.3).

In (b), we must have $k \leqslant f(n+1)$ for θ_n to exist. In effect, the lifting $BP_*(\underline{BP}_k)/\mathfrak{J}_n \to BP^*$ in diag. (23.7) is prescribed on $Q_*^{k-1}/\mathfrak{J}_n$. As we have by Lemma 23.5(b) a split monomorphism with free cokernel, it is easy to extend the given lifting over $BP_*(\underline{BP}_k)/\mathfrak{J}_n$. □

Resolutions. Lemma 23.5 is easy to prove when $k < f(n+1)$. In the borderline case $k = f(n+1)$, the presence of the extra disallowed monomials in Lemma 19.39 makes it necessary to do some homological algebra.

Lemma 23.8. *In the sequence of homomorphisms of BP*-modules*

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\epsilon} M \longrightarrow 0, \tag{23.9}$$

assume that:

(i) *Each C_i is free of finite type;*

(ii) *We have exactness at C_0 and M;*

(iii) $\partial_1 \circ \partial_2 = 0$ (we do not assume exactness at C_1);

(iv) *The sequence*

$$C_2 \otimes \mathbb{F}_p \xrightarrow{\partial_2 \otimes \mathbb{F}_p} C_1 \otimes \mathbb{F}_p \xrightarrow{\partial_1 \otimes \mathbb{F}_p} C_0 \otimes \mathbb{F}_p \xrightarrow{\epsilon \otimes \mathbb{F}_p} M \otimes \mathbb{F}_p \tag{23.10}$$

is exact at $C_1 \otimes \mathbb{F}_p$ (as well as at $C_0 \otimes \mathbb{F}_p$).

Then:

(a) *The sequence* (23.9) *is split exact in the sense that:*

(i) *C_0 splits as $C_0 \cong M \oplus \partial_1 C_1$;*

(ii) *C_1 splits as $C_1 \cong \partial_1 C_1 \oplus \partial_2 C_2$;*

(b) *M is a free BP*-module; explicitly, if L_0 is a free module and the module homomorphism $g_0 \colon L_0 \to C_0$ induces an isomorphism*

$$L_0 \otimes \mathbb{F}_p \xrightarrow{g_0 \otimes \mathbb{F}_p} C_0 \otimes \mathbb{F}_p \longrightarrow \operatorname{Coker}(\partial_1 \otimes \mathbb{F}_p) \cong M \otimes \mathbb{F}_p,$$

the composite $\epsilon \circ g_0 \colon L_0 \to M$ is an isomorphism.

Proof. We build the following commutative diagram, which includes the projections from diag. (23.9) to diag. (23.10),

$$\begin{array}{ccccccccc}
L_2 & & L_1 & & L_0 & & & & \\
\downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_0} & & & & \\
C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{\epsilon} & M & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
C_2 \otimes \mathbb{F}_p & \xrightarrow{\partial_2 \otimes \mathbb{F}_p} & C_1 \otimes \mathbb{F}_p & \xrightarrow{\partial_1 \otimes \mathbb{F}_p} & C_0 \otimes \mathbb{F}_p & \xrightarrow{\epsilon \otimes \mathbb{F}_p} & M \otimes \mathbb{F}_p & &
\end{array}$$

It is easy to construct g_0 as in (b), by lifting a basis of $\operatorname{Coker}(\partial_1 \otimes \mathbb{F}_p)$ to C_0. Similarly, we construct $g_1 \colon L_1 \to C_1$, with L_1 free, that induces an isomorphism $L_1 \otimes \mathbb{F}_p \cong \operatorname{Coker}(\partial_2 \otimes \mathbb{F}_p) \cong \operatorname{Im}(\partial_1 \otimes \mathbb{F}_p)$, and again $g_2 \colon L_2 \to C_2$, with L_2 free, that induces $L_2 \otimes \mathbb{F}_p \cong \operatorname{Im}(\partial_2 \otimes \mathbb{F}_p)$.

Then by Nakayama's Lemma in the form [8, Lemma 15.2(a)], the homomorphism $L_0 \oplus L_1 \to C_0$ with components g_0 and $\partial_1 \circ g_1$ is an isomorphism, and similarly $L_1 \oplus$

$L_2 \cong C_1$. These allow us to write $g_i\colon L_i \subset C_i$ for $i = 0, 1, 2$, and the isomorphisms simplify to $C_0 = L_0 \oplus \partial_1 L_1$ and $C_1 = L_1 \oplus \partial_2 L_2$. The latter gives $\partial_1 C_1 = \partial_1 L_1$, which shows that $M \cong \mathrm{Coker}[\partial_1 \otimes \mathbb{F}_p] \cong L_0$ is free. Moreover, because $\partial_1 | L_1$ is monic, $\partial_2 C_2 = \partial_2 L_2$ and we have split exactness at C_1. □

For our application, we take a polynomial algebra

$$R = BP^*[x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots]$$

on generators of negative degree, with $\deg(x_i) \to -\infty$ and $\deg(y_i) \to -\infty$ as $i \to \infty$, to make R a BP^*-module of finite type. We consider the quotient ring $M = R/\mathfrak{J}$ as a BP^*-module, where the ideal $\mathfrak{J} = (x_1^p - c_1, x_2^p - c_2, \ldots)$. The elements c_i are to be in some sense negligible. We construct what we hope is the beginning (or end) of an R-free resolution of M,

$$C_2 = \bigoplus_{i<j} R u_i u_j \xrightarrow{\partial_2} C_1 = \bigoplus_i R u_i \xrightarrow{\partial_1} C_0 = R \longrightarrow M \longrightarrow 0, \quad (23.11)$$

with R-linear differentials given by $\partial_1 u_i = x_i^p - c_i$ and $\partial_2 u_i u_j = (x_i^p - c_i)u_j - (x_j^p - c_j)u_i$. This is part of a familiar Koszul-type resolution if the c_i are in fact zero, and the structure of M is then clear. Lemma 23.8 supplies conditions under which M has the expected size, even when $c_i \neq 0$.

Lemma 23.12. *Assume that the sequence* (23.11) *induces an exact sequence* (23.10)*, and that the set B of monomials in R of the form*

$$x^I y^J = x_1^{i_1} x_2^{i_2} \ldots y_1^{j_1} y_2^{j_2} \ldots,$$

with $i_t < p$ for all t, yields an $\mathbb{F}_p$-basis of $\mathrm{Coker}(\partial_1 \otimes \mathbb{F}_p)$*. Then the sequence* (23.11) *is exact, $M = R/\mathfrak{J}$ is a free BP^*-module, and B yields a BP^*-basis of it.* □

Proof of Lemma 23.5. Nakayama's Lemma [8, Lemma 15.2] and Lemma 23.12 allow us to work mod $\mathfrak{V}$ everywhere.

For (a), we apply Lemma 23.12 to $BP_*(\underline{BP}_k)/\mathfrak{J}_n$, using the detailed information on $\mathfrak{J}_n$ provided by Lemma 19.39. The $*$-ideal $\mathfrak{J}_n \cap BP_*(\underline{BP}_k) \subset BP_*(\underline{BP}_k)$ has two kinds of generator: the first kind are standard polynomial generators, but the second kind (which occur only if $k = f(n+1)$) are disallowed; we express them in terms of allowable monomials by means of eq. (19.28), of which the leading term (19.26) is of most interest.

We therefore classify the Ravenel-Wilson polynomial generators of $BP_*(\underline{BP}_k)$ into three types:

(i) The allowable $b^{\circ I} \circ [v^J]$ in which v^J contains some factor v_j with $j > n$;

(ii) Monomials of the form $b_{(k_0)} \circ b_{(k_1)}^{\circ p} \circ \ldots \circ b_{(k_n)}^{\circ p^n}$, where $0 \leqslant k_0 \leqslant k_1 \leqslant \ldots \leqslant k_n$; (23.13)

(iii) All other allowable monomials $b^{\circ I} \circ [v^J]$.

The first type visibly lie in $\mathfrak{J}_n$, and we ignore them, by taking R in Lemma 23.12 as the quotient polynomial ring (using $*$-multiplication, of course) on the second and

third types, which serve as the x_i and y_i respectively. The interesting generators of $\mathfrak{J}_n$ then have the form $x_i^p - c_i$.

There are five types of term in the reduction formula (19.28) for the monomial $b^{\circ p}_{(k_1-1)} \circ b^{\circ p^2}_{(k_2-2)} \circ \ldots \circ b^{\circ p^{n+1}}_{(k_{n+1}-n-1)} \circ [v_{n+1}]$:

(i) $b^{\circ p}_{(i_1-1)} \circ b^{\circ p^2}_{(i_2-2)} \circ \ldots \circ b^{\circ p^{n+1}}_{(i_{n+1}-n-1)} \circ [v_{n+1}]$;

(ii) $F(b_{(k_1-1)} \circ b^{\circ p}_{(k_2-2)} \circ \ldots \circ b^{\circ p^n}_{(k_{n+1}-n-1)})$;

(iii) $F(b_{(i_1-1)} \circ b^{\circ p}_{(i_2-2)} \circ \ldots \circ b^{\circ p^n}_{(i_{n+1}-n-1)})$;

(iv) Terms in $\mathfrak{A}_2$;

(v) Terms in $\mathfrak{A}_1 * \mathfrak{A}_1$;

where $(i_1, i_2, \ldots, i_{n+1})$ denotes any nontrivial permutation of $(k_1, k_2, \ldots, k_{n+1})$.

Because the suffixes in (i) are out of order, (i) is an example of a type (i) generator in (23.13), which has been discarded. The term we want is (ii), which is x_i^p. We can take care of (iii) and (iv) by filtering R by powers of the ideal $(y_1, y_2, \ldots)$ and working with the associated graded groups; if we have exactness in diag. (23.10) after filtering, we had exactness before. In effect, we may ignore the y_i's. We take care of (v) by filtering again, this time by powers of the ideal $\mathfrak{A}_1 + (u_1, u_2, \ldots)$ in (23.10). This done, we have effectively reduced c_i to zero, when we have exactness. Thus $BP_*(\underline{BP}_k)/\mathfrak{J}_n$ is a free BP^*-module, and we have constructed a basis.

For (b), we have only to show that we have a monomorphism mod $\mathfrak{V}$. By Lemma 18.26(a) and Lemma 18.12(c), $Q_*^{k-1}/\mathfrak{J}_n$ is a free BP^*-module with a basis consisting of the monomials of the extended canonical form (18.13)

$$e\, b^{L-\Delta_0}\, b^{L+s(L)+\ldots+s^{h-1}(L)}\, b^{s^h(M)}\, w_m^h w^J,$$

that lie in Q_*^{k-1} and have no factor w_j with $j > n$, where $b^L = b_{(k_0)} b^p_{(k_1)} \ldots b^{p^m}_{(k_m)}$, $0 = k_0 \leqslant k_1 \leqslant \ldots \leqslant k_m$, $m \geqslant 0$, $h \geqslant 0$, and the conditions (18.9) on M and J hold. After suspension, we find the leading term (19.31), namely $F^h(b^{\circ L} \circ b^{\circ M} \circ [v^J])$, which by Lemma 18.12(d) is the p^hth power of an allowable monomial.

There are two cases:

Case $m < n$. The element $b^{\circ L} \circ b^{\circ M} \circ [v^J]$ is a generator y_i of type (iii) in (23.13), and therefore harmless.

Case $m \geqslant n$. Since $j_t = 0$ for all $t \leqslant m$ and $t > n$, we must have $J = 0$. Also, $h = 0$. We must have $m = n$, otherwise we would have $k > f(n+1)$. We have a generator x_i of type (ii), but it is not raised to a power.

By Lemma 23.12, the elements $F^h y_i$ and x_i (for certain i) map to part of a basis of $\overline{H}_*^k/\mathfrak{J}_n$, which is sufficient. (Because $k_0 = 0$, it is clear that these elements lie in $P\overline{H}_*^k$. In view of the suspension isomorphism $Q_*^{k-1}/\mathfrak{V} \cong P\overline{H}_*^k$ in [23, Thm. 5.3], all we really need to know is that enough basis elements of $P\overline{H}_*^k$ in each degree remain linearly independent in $\overline{H}_*^k$ mod $\mathfrak{J}_n$.) □

Index of symbols

This index lists most symbols in roughly alphabetical order (English, then Greek), with brief descriptions and references. Several symbols have multiple roles.

A additive comonad, Thm. 5.8.
A' additive comonad, (6.23).
$-_A$ (subscript) additively unstable context.
$\overline{A}$ additive comonad, on modules, §9 (only).
$\overline{A}$ augmentation ideal in algebra A.
$\mathcal{A}$ etc. generic category.
$\mathcal{A}^{\mathrm{op}}$ dual category of $\mathcal{A}$, [8, §6].
$\mathcal{A}$ $= E^*(E, o)$, Steenrod algebra for E, §2.
$\mathcal{A}_k$ $= E^*(\underline{E}_k)$, the operations on degree k, §2.
$\mathfrak{A}_m$ ideal in $Q(BP)^*_*$, §18.
$\mathfrak{A}_m$ Hopf ring ideal in $\overline{H}$, Defn. 19.19.
Ab, *Ab** category of (graded) abelian groups, [8, §6].
Alg category of E^*-algebras, [8, §6].
a_i, $a_{(i)}$ Hopf ring element for $H(\mathbb{F}_p)$, Prop. 17.9.
a_i, $a_{(i)}$ Hopf ring element for $K(n)$, Prop. 17.16.
$a_{(i)}$ additive element for $K(n)$, (16.21).
$a_{i,j}$ coefficient in formal group law, [8, (5.14)].
BG classifying space of group G.
$B(i,k)$ coefficient in $b(x)^i$, Prop. 14.4.
BP Brown-Peterson spectrum, [8, §2].
$BP\langle n\rangle$ modified BP, §1.
b^I etc. monomial.
$b^{\circ I}$ etc. $\circ$-monomial, (15.11).
b_i additive element, Prop. 14.4.
b_i Hopf ring element, Prop. 15.3.
$b_{(i)}$ accelerated b_i, Defns. 14.7, 15.10.
$b(x)$ formal power series, (14.2), Defn. 15.1.
$\overline{b}(x)$ series $b(x)$ without the 1 term, (15.4).
$\mathbb{C}$ the field of complex numbers.
$\mathbb{C}P^n$, $\mathbb{C}P^\infty$ complex projective space.
Coalg category of E^*-coalgebras, [8, §6].
c etc. generic Hopf ring element.
c_i, $c_{(i)}$ Hopf ring element for $H(\mathbb{F}_2)$, Prop. 17.7.
c_i Hopf ring element for $H(\mathbb{F}_p)$, Prop. 17.9.
c_i Hopf ring element for $K(n)$, Prop. 17.16.
DM dual of E^*-module M, [8, Defn. 4.8].
d duality homomorphism, [8, (4.5)].
E generic ring spectrum.
E^* coefficient ring of E-(co)homology, [8, §§3, 4].
$E^*(-)$ E-cohomology, [8, §3].
$E^*(-)\hat{}$ completed E-cohomology, [8, Defn. 4.11].
$E_*(-)$ E-homology, [8, §4].
$\underline{E}_n$ nth space of Ω-spectrum E, [8, Thm. 3.17].
e suspension element, Props. 12.3, 13.7.
e_k unstable k-fold suspension element, Prop. 13.7.
Fc $= c^{*p}$, Frobenius operator, §10.
$F\mathfrak{I}$ Hopf ring ideal, Defn. 19.3.
$F(x,y)$ formal group law, [8, (5.14)].
F^aM generic filtration submodule, [8, Defn. 3.36].
FAlg category of filtered E^*-algebras, [8, §6].
F^LDM generic filtration submodule of DM, [8, Defn. 4.8].
F_M etc. corepresented functor, [8, §8].
FMod, *FMod** (graded) category of filtered E^*-modules, [8, §6].
$\mathbb{F}_p$ field with p elements.

$F_R(X,Y)$ right formal group law, (14.5), (15.8).
$F^sE^*(X)$ skeleton filtration, [8, (3.33)].
f generic map or module homomorphism.
f^*, f_* homomorphism induced by map f, [8, (6.3)].
$f(n)$ numeric function, (1.4).
G generic group.
$\mathcal{Gp}(\mathcal{C})$ category of group objects in $\mathcal{C}$, [8, §7].
g_i coefficient in p-series, [8, (13.9)].
H, $H(R)$ Eilenberg-MacLane spectrum, [8, §2].
$\overline{H}$ quotient Hopf ring, (19.12).
$\mathcal{Ho}$, $\mathcal{Ho}'$ homotopy category of (based) spaces, [8, §6].
I identity functor.
I etc. generic multi-index.
$|I|$ length of multi-index I, §18.
I_n, I_∞ ideal in BP^*, (1.1).
i_1, i_2 injection in coproduct, [8, §2].
id identity morphism or permutation.
J_n ideal in BP^*, (1.6).
$\mathfrak{J}_n$ ideal in $Q(BP)^*_*$, Defn. 18.25.
$\mathfrak{J}_n$ Hopf ring ideal, Defn. 19.34.
$K_{\mathcal{C}}$ unit object in (symmetric) monoidal category $\mathcal{C}$, [8, §7].
$K(n)$ Morava K-theory, [8, §2].
KU complex K-theory Bott spectrum, [8, §2, Defn. 3.30].
L infinite lens space.
$L(k)$ left side of main relation $(\mathcal{R}_k)$.
$L(i_1,\ldots,i_n)$ coefficient, Defns. 18.18, 19.21.
M etc. generic (filtered) module or algebra.
$M\hat{}$, $\widehat{M}$ completion of filtered M, [8, Defn. 3.37].
$\mathfrak{M}$ ideal in $Q(BP)^*_*$, §18.
$\mathfrak{M}_n$ Hopf ring ideal, Defn. 19.13.
$\mathcal{Mod}$, $\mathcal{Mod}^*$ (graded) category of E^*-modules, [8, §6].
MU unitary Thom spectrum, [8, §2].
o generic basepoint, point spectrum.
PA the primitives in coalgebra A, [8, (6.13)].
$PE^*(\underline{E}_k)$ the additive operations, Prop. 2.7.
$PE_*(X)$ the primitives in homology of space X, Defn. 4.13.
$PE^*(X)$ the primitives in cohomology of H-space X, Defn. 3.1.
$P(n)$ modified BP spectrum, §1.
p fixed prime number.
p_1, p_2 projection from product, [8, §2].
$[p](x)$ p-series, [8, (13.9)].
$[p]_R(x)$ right p-series, (14.8), (15.14).
$-_Q$ (subscript) additive unstable context, shifted degree.
QA the indecomposables of algebra A, [8, (6.10)].
$QE^*(X)$ the indecomposables of cohomology of space X, (3.5).
$QE_*(X)$ the indecomposables of homology of H-space X, Defn. 4.3.
$Q(E)^*_*$ bigraded algebra, Defn. 6.1.
$Q(r)$ homology homomorphism induced by operation r, (6.48).
Q^*_* $= Q(BP)^*_*$, abbreviation.
$\overline{Q}^*_*$ quotient algebra of Q^*_*, (18.17).
$Q(\epsilon)$ counit of $Q(E)^*_*$, (6.28).
$Q(\eta)$ unit morphism of $Q(E)^*_*$, (6.17).
$Q(\sigma)$ stabilization on $Q(E)^*_*$, (6.3).
$Q(\phi)$ multiplication in $Q(E)^*_*$, (6.16).
$Q(\psi)$ comultiplication on $Q(E)^*_*$, (6.27).
$\mathbb{Q}$ field of rational numbers.
q map to one-point space T.
q_k projection to $Q(E)^k_*$, (6.2).
$\mathbb{R}P^\infty$ real projective space.
$R(k)$ right side of main relation $(\mathcal{R}_k)$.
$R(i_1,\ldots,i_n)$ coefficient, Defns. 18.18, 19.21.
$(\mathcal{R}_k)$ kth main relation, (14.10), (15.16).
$(\mathcal{R}_{k_1,\ldots,k_n})$ nth order relation, Defns. 18.18, 19.21.

r generic cohomology operation.
$\langle r, - \rangle$ E^*-linear functional defined by operation r, (6.9), (10.1).
S stable comonad, [8, Thm. 10.12].
$-_S$ (subscript) stable context.
S^1 unit circle, as space or group.
S^n unit n-sphere.
$\overline{S}$ comonad S on modules, §9 (only).
Stab, Stab^* (graded) stable homotopy category, [8, §6].
Set category of sets, [8, §6].
$\mathit{Set}^{\mathbb{Z}}$ category of graded sets, [8, §7].
$s(I)$, $s^h(I)$ shifted multi-index I, Defn. 15.12.
T the one-point space.
T^+ 0-sphere, T with basepoint added.
$T(n)$ torus group.
t $\in H^1(\mathbb{R}P^\infty)$, generator of $H^*(\mathbb{R}P^\infty)$, (16.1).
U unstable comonad, Thm. 8.8.
$-_U$ (subscript) unstable context.
U, $U(n)$ unitary group.
u $\in KU^{-2}$, generator.
u $\in E^1(L)$, exterior generator of $E^*(L)$, §§16, 17.
u_1 canonical generator of $E^*(S^1)$, [8, Defn. 3.23].
u_n canonical generator of $E^*(S^n)$, [8, §3].
V generic (often forgetful) functor.
V Verschiebung operator, §10.
$\mathfrak{V}$ ideal in $Q(BP)^*_*$, §18.
$\mathfrak{V}$ ideal in $BP_*(\underline{BP}_*)$, §19.
v generic element of E^*.
v $= \eta_R u \in KU_2(KU, o)$, Thm. 16.15.
$[v]$ $\in E_0(\underline{E}_*)$, Defn. 10.8.
v_n Hazewinkel generator of BP^*, $K(n)^*$, [11].
$\mathfrak{W}$ ideal in $Q(BP)^*_*$, §18.
w generic element of $\eta_R E^*$, Prop. 12.3.
w_n $= \eta_R v_n$, §16.
$\mathrm{wt}(I)$ weight of multi-index I, §18.
X etc. generic space.
X^+ space X with basepoint adjoined.
x generic cohomology class or module element.
x $\in E^*(\mathbb{C}P^\infty)$, Chern class of Hopf line bundle, [8, Lemma 5.4].
$x(\theta)$ Chern class of line bundle θ, [8, Defn. 5.1].
Y skeleton of lens space L, [8, §14].
$\mathbb{Z}$ the ring of integers.
$\mathbb{Z}/p$ the group of integers mod p.
$\mathbb{Z}_{(p)}$ $\mathbb{Z}$ localized at p.
z_F morphism for a (symmetric) monoidal functor F, [8, §7].
α etc. generic index.
β_i $\in E_{2i}(\mathbb{C}P^n)$, [8, Lemma 5.3].
γ_i $\in E_{2i+1}(U(n))$, [8, Lemma 5.11].
$\Delta\colon X \to X \times X$ diagonal map.
Δ_0 $= (1, 0, 0, \ldots)$, multi-index, §18.
ϵ generic counit morphism.
ζ pth power map on $\mathbb{C}P^\infty$, [8, (13.9)].
ζ_F pairing for (symmetric) monoidal functor F, [8, §7].
η generic unit morphism.
η_R right unit, Defns. 6.19, 10.8.
θ generic anything.
θ_n idempotent cohomology operation on BP, Lemmas 22.1, 23.1.
$\overline{\theta}_n$ splitting of $\pi\langle n \rangle$, Lemmas 22.1, 23.1.
ι $\in E^0(E, o)$, universal class, [8, §9].
ι_n $\in E^n(\underline{E}_n)$, universal class, [8, Thm. 3.17].
κ_n collapse operation, Defn. 7.13.
$\Lambda(-)$ exterior algebra.
λ generic action.
λ numerical coefficient.
λ_L left E^*-action on $Q(E)^*_*$, §6.
λ_R right E^*-action on $Q(E)^*_*$, (6.21).
μ addition or multiplication in generic group object, [8, §7].
ν inversion morphism in generic group object, [8, §7].
ξ Hopf line bundle over $\mathbb{C}P^n$.
ξ generic line or vector bundle.
ξ_i element for $H(\mathbb{F}_2)$, (16.1).

ξ_i element for $H(\mathbb{F}_p)$, Thm. 16.5.
ξv action of v on E^*-module, [8, (7.4)].
π generic permutation in Σ_n.
$\pi_*(X)$ homotopy groups of space X.
$\pi\langle n\rangle\colon BP \to BP\langle n\rangle$ projection, (1.8).
ρ generic coaction.
ρ_M coaction on module M.
ρ_X coaction on $E^*(X)$ or $E^*(X)\hat{}$.
Σ, Σ^k suspension isomorphism, [8, (3.13), Defn. 6.6].
ΣX, $\Sigma^k X$ suspension of space X.
ΣM, $\Sigma^k M$ suspension of module M, [8, Defn. 6.6].
Σ_n permutation group on $\{1, 2, \ldots, n\}$.
$\sigma\colon A \to S$ natural transformation of comonads, Thm. 5.8.
$\overline{\sigma}\colon \overline{A} \to \overline{S}$ natural transformation of comonads, on modules, §9 (only).
$\sigma_k\colon \underline{E}_k \to E$ stabilization map, [8, Defn. 9.3].
$\tau\colon U \to A$ natural transformation of comonads, Thm. 8.8.
τ_i element for $H(\mathbb{F}_p)$, (16.4).
ϕ generic multiplication.
χ canonical antiautomorphism of Hopf algebra.
Ψ iterated coproduct, (10.18).
ψ generic comultiplication.
ΩX loop space on based space X.
Ωr looped operation, Prop. 2.12.
ω zero morphism of generic group object, [8, §7].

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