ELSEVIER Topology and its Applications 59 (1994) 117–136

TOPOLOGY AND ITS APPLICATIONS

Brown–Peterson homology of elementary p-groups II

David Copeland Johnson *,[a], W. Stephen Wilson **,[b], Dung Yung Yan ***,[c]

[a] Department of Mathematics, University of Kentucky, Lexington, KY 40506, USA
[b] Department of Mathematics, The Johns Hopkins University, Baltimore, MD 21218, USA
[c] Department of Mathematics, National Tsing Hua University, Hsinchu, Taiwan 30043, Taiwan

Received 12 January 1992; revised 31 August 1993

Abstract

We compute the complete Abelian group structure of the Brown–Peterson homology of BV, the classifying space for $V=(\mathbb{Z}/p)^n$, the elementary Abelian p-group of rank n.

Keywords: Complex bordism; Brown–Peterson homology; Classifying space; Elementary Abelian p-group

AMS (MOS) Subj. Class.: 55N22

1. Introduction

Let us fix a prime p and a positive integer n. Let $V=(\mathbb{Z}/p)^n$ be the elementary Abelian p-group of rank n. Its classifying space is BV, the n-fold product of $B\mathbb{Z}/p$'s. In this paper, we establish the complete Abelian group structure of the Brown–Peterson homology of BV, $BP_*(BV^+)$. This builds on and strengthens the results of this paper's predecessor [8].

Recall there is a BP-module homology theory $BP\langle k\rangle_*()$ with coefficients $BP\langle k\rangle_* \cong \mathbb{Z}_{(p)}[v_1,\ldots,v_k]$. ($BP\langle 1\rangle_*()$ is a summand of p-localized connective K-theory.) A corollary to our work is:

* Corresponding author. E-mail: johnson@ms.uky.edu.
** E-mail: wsw@chow.mat.jhu.edu.
*** E-mail: dyyan@math.nthu.edu.tw.

SSDI 0166-8641(93)E0109-2

Theorem 1.1. *Let $V = (\mathbb{Z}/p)^n$ as above. There is an additive isomorphism*

$$BP_*(BV^+) \cong BP\langle n\rangle_*(BV^+) \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{(p)}[v_{n+1}, v_{n+2}, \dots].$$

By additive isomorphism, we mean an isomorphism as Abelian groups ($\mathbb{Z}_{(p)}$-modules) not as BP_*-modules. This theorem is in the spirit of and motivated by the work of Bahri, Bendersky, Davis and Gilkey [1] who show that

$$BP_*(BG^+) \cong BP\langle 1\rangle_*(BG^+) \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{(p)}[v_2, v_3, \dots]$$

when G is a p-group with cyclic cohomology.

We now turn to the main result of [8] which requires a review of notation. If X is a pointed space or a spectrum, let $BP_*(X)$ denote its reduced Brown–Peterson homology. The unreduced Brown–Peterson homology of a space X is $BP_*(X^+)$. Let $L_0 = BP_*$ and for $0 < k$, let L_k be the free BP_*-module on generators one in each degree $2i$, $0 < i < p^k$. For BP_*-modules M and N, let MN be $M \otimes_{BP_*} N$ and M^k (or $\otimes^k M$) be the k-fold iterated BP_*-tensor product $M \otimes_{BP_*} \cdots \otimes_{BP_*} M$. M^0 is taken to be BP_*.

Theorem 1.2 [8]. *Let $V = (\mathbb{Z}/p)^n$ as above. There is a BP_*-module filtration of $BP_*(BV^+)$ whose associated graded module is*

$$\bigoplus_{\substack{0 \leqslant k \leqslant n \\ i_0+i_1+\cdots+i_k=n-k \\ 0 \leqslant i_j}} \binom{n}{i_0} L_0^{i_0} L_1^{i_1} \cdots L_k^{i_k} \otimes^k BP_*(B\mathbb{Z}/p).$$

To get this theorem to give us the additive structure of $BP_*(BV^+)$, we have three tasks.

(I) We must show that the filtration splits additively.

(II) In analogy with Theorem 1.1, we must show

$$\otimes^k BP_*(B\mathbb{Z}/p) \cong T_{k,*} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{(p)}[v_{k+1}, v_{k+1}, \dots] \tag{1}$$

additively, where $T_{k,*}$ is defined to be

$$\otimes^k BP_*(B\mathbb{Z}/p) \otimes_{BP_*} BP\langle k\rangle_*.$$

(III) We must compute the Abelian group structure of $T_{k,*}$. Our additive version of Theorem 1.2 is then:

Theorem 1.3. *Let $V = (\mathbb{Z}/p)^n$ as above. There are the following additive isomorphisms:*

(i)

$$BP_*(BV^+) \cong \bigoplus_{\substack{0 \leqslant k \leqslant n \\ i_0+i_1+\cdots+i_k=n-k \\ 0 \leqslant i_j}} \binom{n}{i_0} L_0^{i_0} L_1^{i_1} \cdots L_k^{i_k} T_{k,*} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{(p)}[v_{k+1}, \dots],$$

(ii)

$$T_{k,j} \cong (\mathbb{Z}/p)^{d_{j,1}} \oplus (\mathbb{Z}/p^2)^{d_{j,2}} \oplus \cdots$$

where

$$d_{j,h} = d_{j-2(p^k-1),h-1}, \quad h > 1,$$

$$d_{j,1} + d_{j,2} + \cdots = \binom{i-1}{k-1} \quad \text{if } j = 2i - k,$$

$$T_{k,j} = 0 \quad \text{if } j - k \text{ is odd.}$$

To paraphrase (ii), *the* $\mathbb{Z}$*-extensions presenting* $T_{k,*}$ *are all nontrivial*; $T_{k,*}$ *has the largest (and therefore fewest) possible summands.*

Theorem 1.3 complements, rather than replaces, Theorem 1.2. Of course, we really compute the additive structure of $BP_*(\wedge B\mathbb{Z}_{(p)})$ and use the fact that $BV = B\mathbb{Z}_{(p)} \times \cdots \times B\mathbb{Z}_{(p)}$ is stably a wedge sum of $\wedge^k B\mathbb{Z}_{(p)}$'s. An important, but understated theme of [8] is that the bottom class of $BP_*(\wedge B\mathbb{Z}_{(p)})$—related to the "toral class" in the bordism of BV—determines the major part of the structure of $BP_*(\wedge B\mathbb{Z}/p)$. Our job is to show that this theme remains valid modulo p^s, $s > 0$. Our proof comes from a close examination of the Brown–Peterson homology of the spectrum $D(n)$ which Mitchell uses in his proof of the Conner–Floyd Conjecture [11].

Our paper is organized as follows. Section 2 studies the effect of the Brown–Peterson $[p]$-series on the iterated tensor product $\otimes^n BP_*(B\mathbb{Z}/p)$. Sections 3 and 4 introduce the spectrum $D(n)$ and important relations modulo p^s in its Brown–Peterson homology.

Section 5 is a retelling of [8] modulo p^s. A classic fact is that a short exact sequence of finite Abelian p-groups splits if the sequence is exact mod p^s for all s. The final Section 6 uses this fact to consolidate proofs for Theorems 1.1 and 1.3 (see Theorem 6.10, Corollary 5.5 and Theorem 6.8, respectively.)

An error in [8]. The two first named authors wish to acknowledge an error in Section 6 of the predecessor paper [8]. The statements of the numbered lemmas, propositions, and theorem of the section are indeed correct, but the claim of the introductory paragraph of the section is false. The techniques of [8] do *not* lead to a new proof of the Conner–Floyd Conjecture. The problem arises in the misuse of the "Ravenel Trick" applied to diagram (6.8) to claim that "there are no differentials in the two spectral sequences in the indirect route...". The present paper makes no use of Section 6 and thus is independent of this error. However, Section 6 of [8] is still of use. We do know that the Conner–Floyd Conjecture has a positive solution. That known fact coupled with the computations of Lemmas 6.4, 6.5 and 6.6 of [8] can be used to show the collapse of the Adams spectral sequence converging to $BP_*(B(Z/p)^n)$.

2. Preliminaries on the $[p]$-series

For the fixed prime p, we have the Brown–Peterson coefficient ring $BP_* \cong \mathbb{Z}_{(p)}[v_1, v_2, \dots]$ where $v_k \in BP_{2p^k-2}$. We let $v_0 = p$. We have the prime ideals $I_0 = (0)$, $I_n = (p, v_1, \dots, v_{n-1})$, and $I_\infty = \bigcup_n I_n = (p, v_1, v_2, \dots)$; these are invariant under BP-operations. The BP-formal group law gives the $[p]$-series:

$$[p](x) = \sum_{0 \leqslant j} a_j x^{j+1}, \quad a_j \in BP_{2j}. \tag{2}$$

Here $a_0 = p$, $a_{p^n-1} \equiv v_n$ modulo I_n, and $a_j \in I_n$ for $j < p^n - 1$. This series gives the relations in $BP_*(B\mathbb{Z}/p)$. Let us denote the standard generator in $BP_{2i-1}(B\mathbb{Z}/p)$ by $[i]$, $0 < i$. (In [8], we call this generator z_i.)

Convention 2.1. $[i] = 0$ for $i \leqslant 0$.

$BP_*(B\mathbb{Z}/p)$ is generated by the $[i]$ subject only to the relations:

$$\sum_{0 \leqslant j} a_j [i - j] = 0 \tag{3}$$

[2,9].

Definition 2.2. For the fixed prime p and positive integer n, $T = \otimes^n_{BP_*} BP_*(B\mathbb{Z}/p)$. For integers $i_1, i_2, \dots, i_n$, let $[i_1, i_2, \dots, i_n]$ denote the n-fold tensor product $[i_1] \otimes [i_2] \otimes \cdots \otimes [i_n]$ in T. With our Convention 2.1, this is zero if any $i_j \leqslant 0$. The dimension of $[i_1, \dots, i_n]$ is $2i_1 + \cdots + 2i_n - n$. This section's main result is that, modulo certain terms,

$$p^s[i_1, \dots, i_n] \equiv \pm v_n^s\left[i_1 - s(p-1), \dots, i_n - s(p^n - p^{n-1})\right].$$

The precise statement is given in Corollary 2.9.

Order the (bracketed) n-tuples of positive integers lexicographically. So $[I] = [i_1, \dots, i_n] < [J] = [j_1, \dots, j_n]$ provided that $i_s = j_s$, $s < t$, and $i_t < j_t$. For the (bracketed) n-tuple $[J]$, define F_J to be the BP_*-submodule generated by all $[I]$ with $[I] \leqslant [J]$. This lexicographic filtration is used in the proof of the following:

2.3 [8,3.4]. Every element w of T has a *unique* expression

$$w = \sum c_{I,L} v^L [I]$$

where $v^L = v_n^{l_n} v_{n+1}^{l_{n+1}} \cdots$ for $L = (l_n, l_{n+1}, \dots)$, $l_j \geqslant 0$ (all but finitely many are 0). The $[I]$ range over all n-tuples of positive integers. The coefficients $c_{I,L}$ are limited to the set $\{0, 1, \dots, p-1\}$.

Lemma 2.4. *Suppose the generator* $[J] = [j_1, \dots, j_n]$ *in* T *has dimension* $2(j_1 + \cdots + j_n) - n < n + 2t(p^n - p^j)$ *for* $0 \leqslant j < n$. *Then* $v_j^t[J] = 0$.

Proof. Consider the representation of $v_j^t[J]$ under 2.3:

$$v_j^t[J]=\sum c_{I,L}v^L[I]. \tag{4}$$

Since $v_j^t \in I_\infty^t$, the only possible nonzero $c_{I,L}$ are those with L having $l_n+l_{n+1}+\cdots \geqslant t$. For the right side of (4) to be nonzero, it must have dimension at least that of $v_n^t[1,\ldots,1]$: $2t(p^n-1)+n$. By hypothesis, the dimension of the left side of (4) is less than $n+2t(p^n-p^j)+2t(p^j-1)=n+2t(p^n-1)$. □

Definition 2.5. Let $[I]=[i_1,\ldots,i_n]$. For $s \geqslant 0$, $0 \leqslant k \leqslant n$, $[I-s\Delta_k]=[i_1,\ldots,i_k-s, i_{k+1},\ldots,i_n]$. Let $\Gamma=(1, p, p^2,\ldots,p^{n-1})$. So

$$\begin{aligned}[I \pm s(p-1)\Gamma] &= [I \pm s(p-1)(\Delta_1+p\Delta_2+\cdots+p^{n-1}\Delta_n)] \\ &= [i_1 \pm s(p-1), i_2 \pm s(p^2-p),\ldots,i_n \pm s(p^n-p^{n-1})].\end{aligned}$$

Lemma 2.6. *Let $a_j \in BP_{2j}$, be the $[p]$-series coefficients. Let $[I]=[i_1,\ldots,i_n]$ and $0<k\leqslant n$. Modulo terms of lexicographic filtration lower that of $[i_1,\ldots,i_k-(p^k-p^{k-1}), 1,\ldots,1]$,*

$$a_{p^{k-1}-1}[I] \equiv -a_{p^k-1}[I-(p^k-p^{k-1})\Delta_k].$$

Proof. We induct on k. The initial $k=1$ case follows the pattern of the inductive case. By (3) in the kth factor, $a_{p^{k-1}-1}[I]=-a_{p^k-1}[I-(p^k-p^{k-1})\Delta_k]+S$,

$$S=-\sum_{j\neq p^k-1,p^{k-1}-1} a_j[I-(j+1-p^{k-1})\Delta_k]. \tag{5}$$

We show that all the terms of the remaining sum (5) have the desired low filtration.

Case 1: $j>p^k-1$ (*when* $k=1$, *this is the only case*; $a_j=0$ *for* $0<j<p-1$). The term $[I-(j+1-p^{k-1})\Delta_k]$ is of lower filtration than $[i_1,\ldots,i_{k-1}, i_k-p^k+p^{k-1}, 1,\ldots,1]$.

Case 2: $j<p^k-1$ *and* $j\neq m(p^{k-1}-1)$ *any* m. For dimensional reasons. each summand of a_j must have a factor a_{p^l-1} for $l<k-1$ (i.e., $a_j \in I_{k-1}$.) By induction, $a_{p^l-1}[I-(j+1-p^{k-1})\Delta_k]$ is of filtration at most that of $[I-(p^l-p^{l-1})\Delta_l+(p^{k-1}-1-j)\Delta_k]$ and less than that of $[i_1,\ldots,i_{k-1}, i_k-p^k+p^{k-1}, 1,\ldots,1]$. Thus $a_j[I-(j+1-p^{k-1})\Delta_k]$ is of appropriately low filtration.

Case 3: $j=m(p^{k-1}-1)<p^k-1$, $j\neq p^{k-1}-1$. Here $1<m\leqslant p$; so $j+1=mp^{k-1}-m+1$ must have a p^l, $l<k-1$, in its p-adic expansion. By [5], $a_j \in I_{l+1} \subseteq I_{k-1}$. The proof continues exactly as in Case 2. □

Corollary 2.7. *Let $[I]=[i_1,\ldots,i_n]$ and recall Definition 2.5. Modulo terms of filtration less than that of $[I-s(p-1)\Gamma]$,*

$$p^s[I] \equiv (-1)^{ns}(a_{p^n-1})^s[I-s(p-1)\Gamma].$$

Proof. A first induction on k, $1 \leqslant k \leqslant n$, proves that

$$p^s[I] \equiv (-1)^{ks}(a_{p^k-1})^s[I - s(p-1)\Delta_1 - \cdots - s(p^k - p^{k-1})\Delta_k]$$

modulo terms of filtration less than that of $[I - s(p-1)\Gamma]$. This is proved by a second induction on l, $1 \leqslant l \leqslant s$, that:

$$\begin{aligned} p^s[I] \equiv (-1)^{l+(k-1)s}(a_{p^k-1})^l(a_{p^{k-1}-1})^{s-l}[&i_1 - s(p-1), \dots, \\ &i_{k-1} - s(p^{k-1} - p^{k-2}), i_k - l(p^k - p^{k-1}), i_{k+1}, \dots] \end{aligned} \tag{6}$$

modulo terms of filtration less than that of $[I - s(p-1)\Gamma]$. We apply Lemma 2.6 to the right side of (6) to get

$$\begin{aligned} p^s[I] \equiv\ & (-1)^{l+1+(k-1)s}(a_{p^k-1})^{l+1}(a_{p^{k-1}-1})^{s-l-1} \\ & \cdot[i_1 - s(p-1), \dots, i_{k-1} - s(p^{k-1} - p^{k-2}), \\ & \quad i_k - (l+1)(p^k - p^{k-1}), i_{k+1}, \dots] \\ & + (-1)^{l+(k-1)s}(a_{p^k-1})^l(a_{p^{k-1}-1})^{s-l-1}x \end{aligned} \tag{7}$$

where x is of filtration less than that of

$$[i_1 - s(p-1), \dots, i_{k-1} - s(p^{k-1} - p^{k-2}), i_k - (l+1)(p^k - p^{k-1}), 1, \dots, 1].$$

If $l < s-1$, $(a_{p^{k-1}-1})^{s-l-1}x$ has lexicographic filtration less than that of

$$[i_1 - s(p-1), \dots, i_{k-1} - s(p^{k-1} - p^{k-2}), 1, \dots, 1]$$

thus less than that of $[I - s(p-1)\Gamma]$. This is by Lemma 2.6. If $l = s-1$, the filtration of x is less than that of

$$[i_1, -s(p-1), \dots, i_k - s(p^k - p^{k-1}), 1, \dots, 1],$$

less than that of $[I - s(p-1)\Gamma]$. In either case, the remainder in (7) has the desired low filtration. □

In the iterated tensor product T, $[1, \dots, 1]$ is the class of lowest filtration, i.e., the bottom class. An immediate consequence of Lemma 2.6 is the classical result of Conner and Floyd:

Corollary 2.8 [3]. *For* $0 \leqslant j < n$, $a_{p^j-1}[1, \dots, 1] = v_j[1, \dots, 1] = 0$.

The Conner–Floyd Conjecture proved by Ravenel and Wilson [13] and by Mitchell [11] states that the BP_*-annihilator ideal of $[1, \dots, 1]$ (or of its image in $BP_*(BV^+)$, $V = (\mathbb{Z}/p)^n$) is exactly $I_n = (p, v_1, \dots, v_{n-1})$.

Since $[1, \dots, 1]$ is of lowest filtration. Corollary 2.7 has the following useful form.

Corollary 2.9. *In* T,

$$\begin{aligned}(-1)^{ns}v_n^s[1,\ldots,1] &= (-1)^{ns}(a_{p^n-1})^s[1,\ldots,1]\\ &= p^s\left[1+s(p-1),\ldots,1+s(p^n-p^{n-1})\right]\\ &= p^s[J+s(p-1)\Gamma]\end{aligned}$$

where $[J]=[1,\ldots,1]$.

3. The spectrum $D(n)$

We follow the lead of Mitchell in deducing qualitative information about $BP_*(BV^+)$ from the Brown–Peterson homology of a certain spectrum $D(n)$. Some important properties of $D(n)$ are: the Adams spectral sequence converging to $BP_*(D(n))$ collapses; the BP_*-projective dimension of $BP_*(D(n))$ is n; and $BP_*(D(n))$ is the BP_*-homomorphic image of the n-fold tensor product T.

Let $Sp^q(S^0)$ be the q-fold symmetric product of the sphere spectrum S^0. The iterated diagonal gives a map

$$Sp^{p^{n-1}}(S^0)\to Sp^{p^n}(S^0)$$

whose cofibre is our spectrum $D(n)$.

Warning. Here we are following the notation of [12]. In [11], this spectrum has the name "$M(n)$" which [12] uses for a related but different spectrum.

The canonical complex line bundle λ has a K-theoretic negative $-\lambda$, a virtual bundle. Let L_{-2}^∞ be the Thom spectrum of $-\lambda$ over $B\mathbb{Z}/p$. Let homology be with $\mathbb{Z}/p$-coefficients. Let $x\in H^1B(\mathbb{Z}/p)$ and $y\in H^2(B\mathbb{Z}/p)$ be the usual exterior and polynomial generators. (If $p=2$, $x^2=y$.) Let $y^{-1}\in H^{-2}(L_{-2}^\infty)$ denote the Thom class of $-\lambda$. Let $x^ey^s\in H^{e+2s}(L_{-2}^\infty)$ be the image of $x^ey^{s+1}\in H^{e+2s+2}(B\mathbb{Z}/p)$ under the Thom isomorphism, $e=0$ or 1, $s\geqslant -1$. Then a basis for $H^*(L_{-2}^\infty)$ is given by $\{x^ey^s\colon e=0,1,\ s\geqslant -1\}$.

The action of the Steenrod reduced powers is given by:

$$P^j(y^s)=\binom{s}{j}y^{s+j(p-1)},$$

$$P^j(xy^s)=\binom{s}{j}xy^{s+j(p-1)} \tag{8}$$

[12, p. 289]. Note that $\binom{-1}{j}=(-1)^j$ and that $\binom{-1+k(p-1)}{j}=0$ for $j\geqslant k(p-1)$.

3.1. We say that a sequence of nonnegative integers $J=(j_1,j_2,\ldots,j_n)$ is admissible if $j_{s+1}\geqslant pj_s$. (Note that the order of the indices in the n-tuple J is opposite the order usually used when writing an admissible monomial.) We say that J is of length k if $j_1=0,\ldots,j_{n-k}=0$, $j_{n-k+1}\neq 0$. Let $P^J=P^{j_n}\cdots P^{j_2}P^{j_1}$. If $J=(j_1, j_2,\ldots,j_n)$ is of length $k<n$, $P^J=P^{j_n}\cdots P^{j_{n-k+1}}=P^{J'}$ where $J'=(j_{n-k+1},\ldots,j_n)$. If the length J is greater than 1, $P^J(xy^{-1})=0$. If $J=(j)$ (of length 1 or 0),

$$P^J(xy^{-1})=P^j(xy^{-1})=(-1)^j xy^{-1+j(p-1)}.$$

Let $E=E[Q_0, Q_1,\ldots]$ be the exterior subalgebra of the mod p Steenrod algebra A; E is generated by the dimension $(2p^i-1)$ Milnor elements Q_i. $H^*(D(n))$ has a fundamental class $u_n\in H^0(D(n))$.

3.2. Over $E[Q_0,\ldots,Q_{n-1}]$, $H^*(D(n))$ is free with basis $\{P^J u_n$: J admissible of length $\leqslant n\}$ [11, (1.2)]. For example $H^*(D(1))$ has a $\mathbb{Z}/p$-basis $\{Q_0^e P^i u_1$: $e=0$ or 1, $i\geqslant 0\}$. Mitchell and Priddy [12] construct a map of spectra $f:\Sigma L_{-2}^{\infty}\to D(1)$ such that $f^*(u_1)=\Sigma xy^{-1}$. In nonnegative dimensions, f^* is an isomorphism (only $\Sigma y^{-1}\in H^{-1}(\Sigma L_{-2}^{\infty})$ fails to be in the image of f^*). There is a multiplication m: $\wedge^n D(1)\to D(n)$ which induces in cohomology $m^*(u_n)=\otimes^n u_1\in\otimes^n H^*(D(1))\cong H^*(\wedge^n D(1))$. Here and throughout this section, unadorned tensor products are over $\mathbb{Z}/p$. Mitchell's proof of the Conner–Floyd Conjecture [11] uses the composition (9) whose initial map is the Thom isomorphism associated to the n-fold product of $-\lambda$ over $BV=B(\mathbb{Z}/p)^n$.

$$\begin{aligned} BP_*(BV^{-1})\cong\Sigma^{2n}BP_*(\wedge^n L_{-2}^{\infty}) &\xrightarrow{(\wedge^n f)_*}\Sigma^n BP_*(\wedge^n D(1))\\ &\xrightarrow{m_*}\Sigma^n BP_*(D(n)). \end{aligned}\tag{9}$$

This has a mod p cohomology analog:

$$\begin{aligned} F'\colon \Sigma^n H^*(D(n)) &\xrightarrow{m^*}\Sigma^n H^*(\wedge^n D(1))\\ &\xrightarrow{(\wedge^n f)^*}\Sigma^{2n}H^*(\wedge^n L_{-2}^{\infty})\cong H^*(BV^+). \end{aligned}\tag{10}$$

In $\otimes^n\Sigma^2 H^*(L_{-2}^{\infty})\cong\Sigma^{2n}H^*(\wedge^n L_{-2}^{\infty})$ (respectively in $\otimes^n H^*(B\mathbb{Z}/p)\cong H^*(\wedge^n B\mathbb{Z}/p)$) call a monomial $b_1\otimes b_2\otimes\cdots\otimes b_n$ of length k if exactly k many b_i are of dimension greater than 1. We shall be interested in $b_i=\Sigma^2 xy^s$, $s\geqslant -1$ (respectively, $b_i=xy^s$, $s\geqslant 0$). In these cases, $b_1\otimes b_2\otimes\cdots\otimes b_n$ will be of length k if exactly $n-k$ many b_i are $\Sigma^2 xy^{-1}$ (respectively x). The monomial $b_1'\otimes b_2'\otimes\cdots\otimes b_n'$ is of higher lexicographic order than the monomial $b_1\otimes b_2\otimes\cdots\otimes b_n$ provided that the dimensions of b_i' and b_i agree for $i<k$ and that the dimension of b_k' exceeds that of b_k.

Lemma 3.3. *Let $J=(j_1,\ldots,j_k)$ be an admissible sequence of length k.*

$$P^J(\otimes^n\Sigma^2 xy^{-1})=\underbrace{\Sigma^2 xy^{-1}\otimes\cdots\otimes\Sigma^2 xy^{-1}}_{n-k}\otimes\Sigma^2 xy^{-1+j_1(p-1)}\otimes\cdots\otimes\Sigma^2 xy^{-1+j_k(p-1)}+w$$

where w is a linear combination of monomials of higher order.

Proof. 3.1 and the Cartan Formula. □

Corollary 3.4. *Let $J=(j_1,\dots,j_k)$ be an admissible sequence of length k. Let F' be as in* (10). *Then*

$$F'\left(P^J\Sigma^n u_n\right)=\underbrace{x\otimes\cdots\otimes x}_{n-k}\otimes xy^{j_1(p-1)}\otimes\cdots\otimes xy^{j_k(p-1)}+w$$

where w is a linear combination of monomials of higher lexicographic order.

Consider diagram (11).

$$\begin{array}{ccccc} T=\otimes^n_{BP_*}BP_*(B\mathbb{Z}/p) & \xrightarrow{\kappa} & BP_*(BV^+) & \xrightarrow{F} & \Sigma^n BP_*(D(n)) \\ & & \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho} \\ & & H_*(BV^+) & \xrightarrow{(F')^*} & \Sigma^n H_*(D(n)) \end{array} \tag{11}$$

Recall the notation for generators in T (of Section 2). $\rho\kappa([1,\dots,1,1+j_1(p-1),\dots,1+j_k(p-1)])$ is dual to

$$x\otimes\cdots x\otimes xy^{j_1(p-1)}\otimes\cdots\otimes xy^{j_k(p-1)}.$$

Definition 3.5. Let $J=(j_1,\dots,j_n)$ be an admissible sequence of length $\leqslant n$. Define $\langle J\rangle\in BP_*(D(n))$ to be

$$\langle J\rangle=\Sigma^{-n}F\kappa\left([1+j_1(p-1),\dots,1+j_n(p-1)]\right).$$

Note $\langle J\rangle$ has dimension $2(j_1+\cdots+j_n)(p-1)$.

Proposition 3.6. *Let*

$$K=\left(\underbrace{0,\dots,0}_{n-l},k_1,\dots,k_l\right)$$

be an admissible sequence of length l, $k_1\neq 0$. Let

$$[\tilde{K}]=\left[\underbrace{1,\dots,1}_{n-l},1+k_1(p-1),\dots,1+k_l(p-1)\right]$$

be the generator of T such that $\langle K\rangle=\Sigma^{-n}F\kappa([\tilde{K}])$. Let $[M]=[m_1,\dots,m_n]$ be a generator of T of lexicographic order less than or equal to that of $[\tilde{K}]$. Under the duality

$$H^*(D(n))\otimes H_*(D(n))\xrightarrow{\langle,\rangle}\mathbb{Z}/p,$$

$$\langle P^K u_n,\rho\Sigma^{-n}F\kappa([M])\rangle=\begin{cases}1 & \text{if } [M]=[\tilde{K}],\\ 0 & \text{if } [M]\neq[\tilde{K}].\end{cases}$$

Proof. Unless $[M]$ is 0, $[M]$ must be of the form

$$[\underbrace{1,\dots,1}_{n-l}, 1+i_1,\dots,1+i_l], \quad i_a \leqslant k_a(p-1).$$

Then $\rho\kappa([M])$ is dual to

$$\underbrace{x\otimes\cdots\otimes x}_{n-l}\otimes xy^{i_1}\otimes\cdots\otimes xy^{i_l}.$$

By Corollary 3.4 and (11),

$$\begin{aligned}&\langle P^K u_n, \rho\Sigma^{-n}F\kappa([M])\rangle\\ &\quad=\langle P^K\Sigma^n u_n, \rho F\kappa([M])\rangle=\langle F'(P^K\Sigma^n u_n), \rho\kappa([M])\rangle\\ &\quad=\langle x\otimes\cdots\otimes x\otimes xy^{k_1(p-1)}\otimes\cdots\otimes xy^{k_l(p-1)}, \rho\kappa([M])\rangle\\ &\qquad+\langle w, \rho\kappa([M])\rangle\end{aligned}$$

where w is a linear combination of monomials of higher lexicographic order. The monomials in w give zero contribution. The only term which can give a nonzero contribution is

$$\underbrace{x\otimes\cdots\otimes x}_{n-l}\otimes xy^{k_1(p-1)}\otimes\cdots\otimes xy^{k_l(p-1)}$$

itself. So $\langle P^K\Sigma^n u_n, \rho F\kappa([M])\rangle$ is nonzero if and only if $i_a = k_a(p-1)$ for $a = 1,\dots,l$. □

4. On $BP_*(D(n))$

We continue our study of the spectrum $D(n)$, but now our attention turns to its Brown–Peterson homology. We start with Adams spectral sequence calculations following Mitchell [11]. Then we show how Corollary 2.7 begets an important relation in $BP\langle n\rangle_*(D(n))$. For this we need a modulo p^s version of the key theorem of our paper [8]: modulo p^s, v_n^{s-1} times the bottom class of $BP\langle n+1\rangle_*(D(n))$ is v_{n+1}-torsion free.

We assume p a prime and n a positive integer remain fixed. Let $m \geqslant n-1$. Recall the spectrum $BP\langle m\rangle$ has homotopy $BP\langle m\rangle_* \cong \mathbb{Z}_{(p)}[v_1,\dots,v_m]$ and has mod p cohomology $H^*(BP\langle m\rangle) \cong A/A(Q_0,\dots,Q_m)$. Let us abuse the notation of Definition 3.5 slightly and let $\langle J\rangle \in BP\langle m\rangle_*(D(n))$ be the image of its homonym in $BP_*(D(n))$. Here $J=(j_1,\dots,j_n)$ is an admissible sequence of length $\leqslant n$. We also abuse notation and let ρ: $BP\langle m\rangle_*(D(n)) \to H_*(D(n))$ be the Thom homomorphism for whichever m is under study. Let $E=E[Q_0, Q_1,\dots]$. Think of $\rho(\langle J\rangle)$ as an element of $\mathrm{Ext}_E^{0,*}(H^*(D(n)), \mathbb{Z}/p)$ dual to $P^J u_n$. Recall

$\mathrm{Ext}_E^{*,*}(\mathbb{Z}/p, \mathbb{Z}/p) \cong \mathbb{Z}/p[v_0, v_1, \dots]$, $v_i \in \mathrm{Ext}_E^{1,2p^i-1}(\mathbb{Z}/p, \mathbb{Z}/p)$. For $m \geqslant n-1$, we have Adams spectral sequences:

$$\mathrm{Ext}_A(H^*(BP\langle m\rangle \wedge D(n)); \mathbb{Z}/p) \cong \mathrm{Ext}_{E[Q_1,\dots,Q_m]}(H^*(D(n)), \mathbb{Z}/p)$$
$$\Rightarrow BP\langle m\rangle_*(D(n)). \tag{12}$$

Recall (3.2) that $H^*(D(n))$ is $E[Q_0, \dots, Q_{n-1}]$-free on the $P^J u_n$, J admissible of length $\leqslant n$. So we have 4.1 and 4.2.

4.1. $\mathrm{Ext}_{E[Q_0,\dots,Q_{n-1}]}(H^*(D(n)), \mathbb{Z}/p)$ is a free $\mathbb{Z}/p$-vector space on the $\rho(\langle J\rangle)$, J admissible of length $\leqslant n$.

4.2. For $m \geqslant n$, $\mathrm{Ext}_{E[Q_0,\dots,Q_m]}(H^*(D(n)), \mathbb{Z}/p)$ is a free $\mathbb{Z}/p[v_n, \dots, v_m]$-module on the $\rho(\langle J\rangle)$, J admissible of length $\leqslant n$. Since the E_2-terms of the Adams spectral sequences (12) for $m = n-1$ and $m \geqslant n$ are concentrated in even degrees, they collapse yielding Propositions 4.3 and 4.5.

Proposition 4.3. *$BP\langle n-1\rangle_*(D(n))$ can be identified with the $\mathbb{Z}/p$-vector subspace of $H_*(D(n))$ with basis $\rho(\langle J\rangle)$, J admissible of length $\leqslant n$. The $\mathbb{Z}/p$-dimension of $BP\langle n-1\rangle_j(D(n))$ is e_j where*

$$\Sigma_j e_j x^j = \left(1 - x^{2(p-1)}\right)^{-1} \cdots \left(1 - x^{2(p^n-1)}\right)^{-1}. \tag{13}$$

Proof. All follows from the collapsed Adams spectral sequence and the computation 4.1 except the computation of the Poincaré series (13). For that, we note the monomial $(x^{2p-2})^{i_1} \cdots (x^{2(p^n-1)})^{i_n}$ corresponds with the admissible sequence

$$\left(i_n, i_{n-1} + pi_n, \dots, i_1 + pi_2 + \cdots + p^{n-1}i_n\right)$$

of length $\leqslant n$. □

Corollary 4.4. *The BP_*-projective dimension of $BP_*(D(n))$ is n. For $m \geqslant n$, $v_n, \dots, v_m$ act injectively on $BP\langle m\rangle_*(D(n))$ [11].*

Proof. We apply [6]. Since v_{n-1} annihilates the bottom class of $BP_*(D(n))$, the BP_*-projective dimension of $BP_*(D(n))$ is at least n. It is exactly n, for the collapsed Adams spectral sequences imply $BP\langle n\rangle_*(D(n))$ maps onto $BP\langle n-1\rangle_*(D(n))$. □

Proposition 4.5. *For $m \geqslant n$, any element of $BP\langle m\rangle_*(D(n))$ has a unique representation as a sum*

$$w = \sum_{L,J} c_{L,J} v^L \langle J\rangle \tag{14}$$

where $v^L = v_n^{l_n} \cdots v_m^{l_m}$, $L = (l_n, \dots, l_m)$, $J = (j_1, \dots, j_n)$ is an admissible sequence of length $\leqslant n$, and $0 \leqslant c_{L,J} < p$.

Proof. The unique representations (14) correspond exactly with the elements of the Adams E_2-term computed in 4.2. □

Recall from Definition 2.5 that $s\Gamma = (s, sp, \ldots, sp^{n-1})$. For $s > 0$, $s\Gamma$ is an admissible sequence of length n. If $J = (j_1, \ldots, j_n)$ is an admissible sequence with $j_1 \geqslant s$, note that $J - s\Gamma = (j_1 - s, \ldots, j_n - sp^{n-1})$ is also an admissible sequence (of length $\leqslant n$). By Definition 3.5, we have:

4.6. Let $J = (j_1, \ldots, j_n)$ be an admissible sequence with $j_1 \geqslant s$. Let $[\tilde{J}] = [1 + j_1(p-1), \ldots, 1 + j_n(p-1)]$. Then

$$\langle J - s\Gamma \rangle = \langle j_1 - s, \ldots, j_n - sp^{n-1} \rangle = \Sigma^{-n} F\kappa\left(\left[\tilde{J} - s(p-1)\Gamma\right]\right).$$

Proposition 4.7. *Let $J = (j_1, \ldots, j_n)$ be an admissible sequence of length $\leqslant n$. Let $s > j_1$. Then $p^s\langle J \rangle = 0$ in $BP_*(D(n))$ (or in $BP\langle m \rangle_*(D(n))$).*

Proof. Let $[\tilde{J}] = [1 + j_1(p-1), \ldots, 1 + j_n(p-1)]$ so that $\langle J \rangle = \Sigma^{-n} F\kappa([\tilde{J}])$ by Definition 3.5. Since $s > j_1$, $[\tilde{J} - s(p-1)\Gamma]$ has nonpositive first term and so is zero. The same is true for terms of lower filtration than $[\tilde{J} - s(p-1)\Gamma]$. Corollary 2.7 tells us $p^s[\tilde{J}] = 0$. □

Proposition 4.8. *Let $J = (j_1, \ldots, j_n)$ be an admissible sequence with $j_1 \geqslant s$. Then in $BP\langle n \rangle_*(D(n))$,*

$$p^s\langle J \rangle = (-1)^{ns} v_n^s \langle J - s\Gamma \rangle + v_n^s a \tag{15}$$

where $a \in BP\langle n \rangle_*(D(n))$ satisfies

$$\langle P^K u_n, \rho(a) \rangle = 0 \tag{16}$$

for any admissible sequence K of lexicographic order at least that of $J - s\Gamma$.

Proof. Let $[\tilde{J}] = [1 + j_1(p-1), \ldots, 1 + j_n(p-1)]$ so that $\langle J \rangle = \Sigma^{-n} F\kappa([\tilde{J}])$. Think of $\langle J \rangle \in BP\langle n \rangle_*(D(n))$ and $[\tilde{J}] \in T \otimes_{BP_*} BP\langle n \rangle_*$. Note that $p^s\langle J \rangle \in I_\infty^s BP\langle n \rangle_*(D(n)) = (v_n^s) BP\langle n \rangle_*(D(n))$ by Proposition 4.5. By Corollary 2.7,

$$\begin{aligned} p^s\langle J \rangle &= \Sigma^{-n} F\kappa\left(p^s[\tilde{J}]\right) \\ &= (-1)^{ns} (a_{p^n - 1})^s \langle J - s\Gamma \rangle + \sum_M c_M v_n^{s+t} \Sigma^{-n} F\kappa([M]). \end{aligned} \tag{17}$$

In (17), the value of $t \geqslant 0$ is such that $\langle J \rangle$ and $v_n^{s+t}\Sigma^{-n}[M]$ have the same dimension. The terms $[M] = [m_1, \ldots, m_n]$ of $T \otimes_{BP_*} BP\langle n \rangle_*$ have filtration less than that of $[\tilde{J} - s(p-1)\Gamma]$. If $K = (k_1, \ldots, k_n)$ is an admissible sequence of lexicographic order at least that of $J - s\Gamma$, then the lexicographic order of $[M]$ is strictly less than that of $[1 + k_1(p-1), \ldots, 1 + k_n(p-1)]$. By Proposition 3.6, $\langle P^K u_n, \rho \Sigma^{-n} F\kappa([M]) \rangle = 0$. Let $a = \Sigma_M c_M v_n^t \Sigma^{-n} F\kappa([M])$ to establish our proposition. □

(Note that $\Sigma^{-n}F\kappa([M])$ in the above may not be of the form $\langle L\rangle$ for some admissible sequence. That is $((m_1-1)/(p-1),\dots,(m_n-1)/(p-1))$ may not be a sequence of integers, much less an admissible sequence.)

Theorem 4.9. *Consider the formal sum* (18) *in* $BP\langle n+1\rangle_*(D(n))$,

$$w=\sum_{a,b,J} c_{a,b,J}v_n^a v_{n+1}^b\langle J\rangle, \tag{18}$$

satisfying

(i) $0\leqslant a<s$,

(ii) $0\leqslant c_{a,b,J}<p$.

Then w *is zero modulo* p^s *only if each coefficient is zero. In particular,* $v_n^{s-1}v_{n+1}^b\langle 0,\dots,0\rangle\not\equiv 0 \bmod p^s$ *for all* b.

Proof. Suppose that $w=p^sz$, $z\in BP\langle n+1\rangle_*(D(n))$. Then

$$w\in I_\infty^s BP\langle n+1\rangle_*(D(n))=(v_n, v_{n+1})^s BP\langle n+1\rangle_*(D(n))$$

by Corollary 4.4. Thus all b are greater than 0 in (18). Consider the exact sequence

$$BP\langle n+1\rangle_*(D(n))\xrightarrow{v_{n+1}}BP\langle n+1\rangle_*(D(n))\xrightarrow{\rho'}BP\langle n\rangle_*(D(n)).$$

We have two cases: $\rho'(z)$ is zero or not.

Case 1: $\rho'(z)\neq 0$. By Proposition 4.5 we can write

$$z=\sum_I d_I v_n^t\langle I\rangle+v_{n+1}x \tag{19}$$

where $x\in BP\langle n+1\rangle_*(D(n))$, the dimension of $v_n^t\langle I\rangle$ is that of z, $0\leqslant d_I<p$, and the sequences $I=(i_1,\dots,i_n)$ are admissible. By 4.6, $p^s\langle I\rangle=0$ if the $i_1\leqslant s$. So we may assume that all the admissible sequences I of (19) have lexicographic order at least that of $s\Gamma=(s, sp,\dots, sp^{n-1})$.

We apply ρ' to the equation $w=p^sz$. Since all the b are greater than 0 in (18), $\rho'(w)=0$. Obviously $\rho'(p^sv_{n+1}x)=0$. So (19) gives us:

$$0=\sum_I d_I v_n^t p^s\langle I\rangle \tag{20}$$

in $BP\langle n\rangle_*(D(n))$. Among those I with $d_I\neq 0$ in (20), let H be the admissible sequence having the highest lexicographic order. Then (20) becomes

$$0=d_H v_n^{t+s}\langle H-s\Gamma\rangle+v_n^{t+s}y \tag{21}$$

where y satisfies

$$\langle P^k u_n, \rho(y)\rangle=0 \tag{22}$$

for any admissible sequence K of lexicographic order $H-s\Gamma$ or higher. This contradicts the unique representation of zero given in Proposition 4.5.

Case 2: $\rho'(z)=0$. In low dimensions, ρ' is an isomorphism: so $w=p^sz=0$. In higher dimensions, if $\rho'(z)=0$, $z=v_{n+1}x$ for some $x\in BP\langle n\rangle_*(D(n))_*$. Multiplication by v_{n+1} is injective in $BP\langle n+1\rangle_*(D(n))$ (Corollary 4.4), so we have by (18),

$$\sum_{a,b,J} c_{a,b,J} v_n^a v_{n+1}^{b-1} \langle J\rangle = p^s x. \tag{23}$$

By an induction on dimensions, this implies the coefficients $c_{a,b,J}$ are all zero. □

Corollary 4.10. *Each element of* $\mathbb{Z}/p^s \otimes BP\langle n+1\rangle_*(D(n))$ *has a unique expression* (18). *Multiplication by* v_{n+1} *in* $BP\langle n+1\rangle_*(D(n))$ *is injective modulo* p^s.

Proof. We need a counting argument to show that the expressions (18) account for all the elements of $\mathbb{Z}/p^s \otimes BP\langle n+1\rangle_*(D(n))$. This is easy for $s=1$. Then one uses an induction using the exact sequence induced by

$$0 \to \mathbb{Z}/p \to \mathbb{Z}/p^{s+1} \to \mathbb{Z}/p^s \to 0.$$

Obviously, v_{n+1} multiplication will not send w in (18) to zero. □

5. $BP_*(BV^+)$ modulo p^s

Recall $BV = B(\mathbb{Z}/p)^n$ and $T = \otimes^n_{BP_*} BP_*(B\mathbb{Z}/p)$ are related by the Künneth map κ: $T \to BP_*(BV^+)$. In [8], we show κ is injective and use this fact to construct a BP_*-filtration of $BP_*(BV^+)$. This section employs our knowledge of $BP_*(D(n))$ to show that the filtration of [8] splits additively. A key step is to show κ: $T \to BP_*(BV^+)$ remains monic when reduced modulo p^s, $s>0$.

We use the accumulated notation of the previous sections. Recall that the BP_*-generators of T are given by $[J] = [j_1, \ldots, j_n]$ of dimension $2j_1 + \cdots + 2j_n - n$. The bottom class of T is $[1, \ldots, 1]$ of dimension n.

Theorem 5.1. *Let* $V = (\mathbb{Z}/p)^n$ *and let* $s>0$ *be a fixed integer. Let* $y = 1 \in \mathbb{Z}/p^s$. *Then:*

(a) *The* BP_**-annihilator ideal of* $y \otimes \kappa([1, \ldots, 1])$ *in* $\mathbb{Z}/p^s \otimes BP(BV^+)$ *is* $(p, v_1, \ldots, v_{n-1}, v_n^s)$.

(b) *In the sense of Definition* 2.17 *of* [8], $1 \otimes \kappa(T)$ *has a Landweber presentation which is free over* BP_*/I_{n+1} *on the classes represented by*

$$v_n^{s-i} y \otimes \kappa([J]), \quad 0 < i \leqslant s,\ [J] = [j_1, \ldots, j_n],\ 0 < j_a.$$

(c) *Each element* w *of* $1 \otimes \kappa(T)$ *has a unique representation* (24)

$$w = \sum_{i,J,L} c_{i,J,L} v^L v_n^{s-i} y \otimes \kappa([J]) \tag{24}$$

where:

(i) $v^L = v_{n+1}^{l_1} v_{n+2}^{l_2} \ldots,\ L = (l_1, l_2, \ldots)$;

(ii) $[J] = [j_1, \ldots, j_n]$, $0 < j_a$;

(iii) $0 \leqslant c_{i,J,L} < p$.

The obvious analogs (without κ*'s) for these statements also hold for* $\mathbb{Z}/p^s \otimes T$.

Proof. By Corollaries 2.7 and 2.8, $(p, v_1, \dots, v_{n-1}, v_n^s)$ annihilates $y \otimes [1, \dots, 1]$ in $\mathbb{Z}/p^s \otimes T$. By Theorem 4.9, $v_n^{s-1} v_{n+1}^t y \otimes \Sigma^{-n} F\kappa([1, \dots, 1])$ is nonzero in $\mathbb{Z}/p^s \otimes BP\langle n+1 \rangle_*(D(n))$; hence $v_n^{s-1} v_{n+1}^t y \otimes \kappa([1, \dots, 1])$ is nonzero in $\mathbb{Z}/p^s \otimes BP_*(BV^+)$. Since $y \otimes \kappa([1, \dots, 1])$ is a primitive element in a $BP_* BP$-comodule, the techniques of Lemma 1.9 of [7] show that the annihilator ideal of $y \otimes \kappa([1, \dots, 1])$ is exactly $(p, v_1, \dots, v_{n-1}, v_n^s)$.

Order the elements $v_n^{s-i} y \otimes \kappa([J])$, $0 \leqslant i \leqslant s$, $[J] = [j_1, \dots, j_n]$, by lexicographically ordering the $(1+n)$-tuples $(i, j_1, \dots, j_n)$ of positive integers. Define $F_{i,J} \subset \mathbb{Z}/p^s \otimes \kappa(T)$ to be the BP_*-submodule spanned by all the $v_n^{s-h} y \otimes \kappa([K])$ of lower or equal order than that of $v_n^{s-i} y \otimes \kappa([J])$. Let $E_0 = E_0(\mathbb{Z}/p^s \otimes T)$ be the graded BP_*-module associated to the filtration $\{F_{i,J}\}$. If $1 < i$, v_n-multiplication lowers the filtration of $v_n^{s-i} y \otimes \kappa([J])$. If $i = 1$, $v_n v_n^{s-1} y \otimes \kappa([J]) = y \otimes v_n^s \kappa([J]) = p^s y \otimes \kappa([J + s\Gamma]) + \Sigma_I y \otimes c_I \kappa([I])$, where the $c_I \in BP_*$, and $[I]$ has lower lexicographic order than $[J]$. The element $[J + s\Gamma]$ has higher lexicographic order, but $p^s y = 0$ (Corollary 2.7). By Lemma 2.6 multiplication by v_a, $0 \leqslant a < n$, lowers lexicographic order. Thus multiplication by elements of $I_{n+1} = (p, \dots, v_n)$ is filtration lowering: E_0 is a BP_*/I_{n+1}-module.

The proof of BP_*/I_{n+1}-freeness in (b) and the proof of (c) is exactly as in the proof of Corollary 3.3 of [8]. □

Remark. Our first approach to Theorem 5.1 used the computations of [14] for a $p = 2$ proof.

Corollary 5.2. *Let* $T = \otimes_{BP_*}^n BP_*(B\mathbb{Z}/p)$ *and let* $T\langle m \rangle = T \otimes_{BP_*} BP\langle m \rangle_* = T \otimes_{BP_*} \mathbb{Z}_{(p)}[v_1, \dots, v_m]$. *Then*:

(a) $T\langle n-1 \rangle$ *is a* $\mathbb{Z}/p$*-vector space as a* BP_**-module. The* $\mathbb{Z}/p$*-dimension of* $T\langle n-1 \rangle$ *in degree* $2i - n$ *is* $\binom{i-1}{n-1}$.

(b) *Modulo* p, $v_n \equiv 0$ *on* $T\langle n \rangle$.

(c) *Multiplication by* v_n *is injective on* $T\langle n \rangle$.

(d) *For* $m > n$, v_m*-multiplication on* $T\langle m \rangle$ *is injective modulo* p^s.

Proof. Parts (a) and (c) follow from [8] (see the last paragraph of p. 429). In degree $2i - n$, $T\langle n-1 \rangle$ has a basis of those $[J] = [j_1, \dots, j_n]$ with $j_1 + \cdots + j_n = i$. There are $\binom{i-1}{n-1}$ many.

By Theorem 5.1, a typical element of $\mathbb{Z}/p^s \otimes T\langle m \rangle$, $m > n$, is of the form (24) except the v^L is restricted to $v^L = v_{n+1}^{l_1} \cdots v_m^{l_{m-n}}$; part (d) follows. Part (b) is obvious from the $s = 1$ version of Theorem 5.1. □

Now recall the main theorem of [8]. We have a BP_*-filtration (25) of $BP_*(\wedge^n B\mathbb{Z}/p)$.

$$0 \subset F_1 = T \subset F_1 \subset \cdots \subset F_{2^n} = BP_*(\wedge^n B\mathbb{Z}/p). \tag{25}$$

The associated graded module $\oplus_i F_i/F_{i-1}$ is isomorphic to:

$$\bigoplus_{i_1+\cdots+i_k=n-k} L_1^{i_1}\cdots L_k^{i_k}\otimes_{BP_*}^k BP_*(B\mathbb{Z}/p), \tag{26}$$

where the notation is as given in the Introduction. For a fixed $s>0$, tensor the exact sequences

$$0\to F_{i-1}\to F_i\to F_i/F_{i-1}\to 0 \tag{27}$$

with $\mathbb{Z}/p^s$ to get a simple spectral sequence

$$E^1=\operatorname{Tor}_t^{\mathbb{Z}}(\mathbb{Z}/p^s,\ F_i/F_{i-1})\Rightarrow \mathbb{Z}/p^s\otimes BP_*(\wedge^n B\mathbb{Z}/p). \tag{28}$$

Here $t=0$ or 1 and there is just one possible differential: d^1: $\operatorname{Tor}_1\to\operatorname{Tor}_0$. Just as in the proof of Theorem 5.1 of [8], we may assume collapse of such a spectral sequence converging to $\mathbb{Z}/p^s\otimes BP_*(\wedge^{n-1}B\mathbb{Z}/p)$. We can use the "$X_i$" technique used there to show that no $\operatorname{Tor}_0^{\mathbb{Z}}(\mathbb{Z}/p^s,\ F_i/F_{i-1})$ receives a nonzero differential for $i>1$. Since $\operatorname{Tor}_0^{\mathbb{Z}}(\mathbb{Z}/p^s,\ F_1/F_0)=\mathbb{Z}/p^s\otimes T$ injects into $\mathbb{Z}/p^s\otimes BP_*(BV^+)$ passing through $\mathbb{Z}/p^s\otimes BP_*(\wedge^n B\mathbb{Z}/p)$, it also receives no nonzero differential. The spectral sequence (28) collapses; thus the exact sequences (27) remain when tensored with Z/p^s, any $s>0$. The sequences (27) are of locally finite graded $Z_{(p)}$-modules. By the following lemma, a standard fact about Abelian groups, we see that the inclusion of F_{i-1} into F_i in (27) is $Z_{(p)}$-split. Thus the BP_*-filtration (25) is $Z_{(p)}$-split.

Lemma 5.3. *A short exact sequence of finitely generated $\mathbb{Z}_{(p)}$ modules*

$$0\longrightarrow A\xrightarrow{\ f\ }B\xrightarrow{\ g\ }C\longrightarrow 0$$

is additively split (i.e., split over $\mathbb{Z}_{(p)}$) if and only if $\mathbb{Z}/p^s\otimes f$ is injective for every s.

Proof. The following counterexample might illustrate the lemma:

$$0\longrightarrow Z/4\xrightarrow{\ f\ }Z/2\otimes Z/8\xrightarrow{\ g\ }Z/4\longrightarrow 0.$$

Here $f(a+4Z)=(a+2Z,\ 2a+8Z)$ and $g(b+2Z,\ c+8Z)=2b+c+4Z$. Both f and $Z/2\otimes f$ are injective, but $Z/4\otimes f$ is not injective.

The "only if" statement of the lemma is obvious. By the fundamental theorem of finitely generated $Z_{(p)}$-modules, the "if" statement reduces to the case where A is a cyclic module with generator a. The module B is a finite direct sum of cyclic modules B_i with generators b_i. Let q_i: $B\to B_i$ be the projections of the direct sum. If a is a torsion class, suppose s is the largest exponent so that $p^s a$ is not zero. (If a is torsion free, let s be so large that multiplication by p^s kills all of the torsion in B.) Since f is monic, there are one or more j such that $p^s q_j(f(a))$ is not zero. Suppose that for all such j, $q_j(f(a))$ is divisable by p. Then $f(p^s a)$ is zero modulo p^{s+1}; this contradicts our hypothesis. So we have one j such that $q_j(f(a))=b_j$ (up to a unit of $\mathbb{Z}_{(p)}$) and a and b_j have the same order. Define a left inverse for f, h: $B\to A$, on the generators as follows: $h(b_j)=a$ and $h(b_i)=0$ for i not equal to j. □

Corollary 5.4. *The BP_*-module $BP_*(\wedge^n B\mathbb{Z}/p)$ and the BP_*-module given in* (26) *are additively isomorphic.*

Stably, the Cartesian product of two pointed spaces $X \times Y$ is homotopic to $X \vee (X \wedge Y) \vee Y$. Recall in the notation of the Introduction, $L_0 = BP_* = (L_0)^{i_0}$, any $i_0 > 0$. Corollary 5.4 and the standard Pascal's triangle counting prove the following.

Corollary 5.5. *Let $V = (\mathbb{Z}/p)^n$ be the rank-n elementary p-group with classifying space BV. There is an additive isomorphism*

$$BP_*(BV^+) \cong \bigoplus_{\substack{i_0+i_1+\cdots+i_k=n \\ 0 \leqslant i_a}} \binom{n}{i_0} L_0^{i_0} L_1^{i_1} \cdots L_n^{i_n} \otimes_{BP_*}^k BP_*(B\mathbb{Z}/p). \tag{29}$$

Note that for $m > n$, the analog of Theorem 5.1 would hold for $\mathbb{Z}/p^s \otimes BP\langle m\rangle_*(BV^+)$. In particular, when $m = n+1$, $\mathbb{Z}/p^s \otimes BP\langle n+1\rangle_*(\wedge^n B\mathbb{Z}/p)$ and $\mathbb{Z}/p^s \otimes BP\langle n+1\rangle_*(BV^+)$ have filtrations whose associated graded modules are sums of suspensions of tensor products of the form

$$\mathbb{Z}/p^s \otimes \left(\otimes_{BP_*}^k BP_*(B\mathbb{Z}/p) \otimes_{BP_*} BP\langle n+1\rangle_* \right) \tag{30}$$

for $1 \leqslant k \leqslant n$. By Corollary 5.2, v_{n+1}-multiplication on (30) is injective. By induction over filtrations, we see that v_{n+1}-multiplication on

$$\mathbb{Z}/p^s \otimes BP\langle n+1\rangle_*\left(\wedge^n B\mathbb{Z}_{(p)}\right)$$

and on $\mathbb{Z}/p^s \otimes BP\langle n+1\rangle_*(BV^*)$ is injective.

Proposition 5.6. *Let $V = (\mathbb{Z}/p)^n$. The v_{n+1}-multiplication on*

$$BP\langle n+1\rangle_*(\wedge^n B\mathbb{Z}/p)$$

and on $BP\langle n+1\rangle_*(BV^+)$ is injective modulo p^s for all s.

6. Additive structures

Let $V = (\mathbb{Z}/p)^n$, the rank-n elementary p-group. We have seen that the building blocks for the additive structure of $BP_*(BV^+)$ or of $BP_*(\wedge^n B\mathbb{Z}/p)$ are the iterated tensor products $\otimes_{BP_*}^k BP_*(B\mathbb{Z}/p)$, $k \leqslant n$. In this section, we compute the Abelian group structures of these basic tensor products. As a corollary to our techniques, we show $BP_*(X)$ is additively isomorphic to $BP\langle n\rangle_*(X) \otimes \mathbb{Z}_{(p)}[v_{n+1}, v_{n+2}, \ldots]$ for the important examples $X = BV^+$, $\wedge^n B\mathbb{Z}/p$, and $D(n)$.

Let A be a connective locally finitely generated BP_*-module. For $m > 0$, define $A\langle m\rangle$ to be $A \otimes_{BP_*} BP\langle m\rangle_* = A \otimes_{BP_*} \mathbb{Z}_{(p)}[v_1, \ldots, v_m]$. From [6], we know that if X is a connective spectrum with $BP_*(X)$ having BP_*-projective dimension $\leqslant m+1$, then $BP\langle m\rangle_*(X) \cong (BP_*(X))\langle m\rangle$. There are four recurring hypotheses involving the $A\langle m\rangle$. We shall see that all four conditions are satisfied by the BP_*-modules $T = \otimes^n_{BP_*} BP_*(B\mathbb{Z}/p)$ and $BP_*(D(n))$.

6.1. $A\langle n-1\rangle$ is a $\mathbb{Z}/p$-vector space as a BP_*-module: v_i acts trivially on $A\langle n-1\rangle$, all $i \geqslant 0$.

6.2. Modulo p, v_n acts trivially on $A\langle n\rangle$.

6.3. Multiplication by v_n is injective on $A\langle n\rangle$.

6.4. For $m > n$ and $s > 0$, v_m-multiplication is injective on $A\langle m\rangle$ modulo p^s.

Lemma 6.5. *Suppose A is a connective, locally finitely generated BP_*-module satisfying* 6.4, *then A is additively isomorphic to* $A\langle n\rangle \otimes \mathbb{Z}_{(p)}[v_{n+1}, v_{n+2},\dots]$.

Proof. For $m > n$, we have the short exact sequences

$$0 \longrightarrow A\langle m\rangle \xrightarrow{v_m} A\langle m\rangle \longrightarrow A\langle m-1\rangle \longrightarrow 0 \tag{31}$$

which remain exact when tensored with $\mathbb{Z}/p^s$, $s > 0$. By Lemma 5.3, sequence (31) is $\mathbb{Z}_{(p)}$-split. Thus we have the additive isomorphisms

$$A\langle m\rangle \cong A\langle m-1\rangle \otimes \mathbb{Z}_{(p)}[v_m], \quad m > n.$$

By iteration, $A \cong A\langle n\rangle \otimes \mathbb{Z}_{(p)}[v_{n+1}, v_{n+2},\dots]$ additively. □

Lemma 6.6. *Suppose A is a connective, locally finitely generated BP_*-module satisfying* 6.1, 6.2 *and* 6.3. *Let e_j be the $\mathbb{Z}/p$-dimension of $A\langle n-1\rangle$ in degree j. Then in degree j, $A\langle n\rangle$ has e_j many summands. The Abelian group structure in this degree is given by*

$$(\mathbb{Z}/p)^{d_{j,1}} \oplus (\mathbb{Z}/p^2)^{d_{j,2}} \oplus (\mathbb{Z}/p^3)^{d_{j,3}} \oplus \cdots \tag{32}$$

where

$$d_{j,1} + d_{j,2} + \cdots = e_j, \quad d_{j,k} = d_{j-2(p^n-1),k-1},\ k > 1.$$

Proof. By 6.1, v_i-multiplication is trivial on $A\langle n-1\rangle = A \otimes_{BP_*} \mathbb{Z}_{(p)}[v_1,\dots,v_{n-1}]$. Thus $A\langle n-1\rangle \cong \mathbb{Z}/p \otimes_{BP_*} A \otimes_{BP_*} \mathbb{Z}_{(p)}[v_1,\dots,v_{n-1}, v_n] = \mathbb{Z}/p \otimes_{BP_*} A\langle n\rangle$. By 6.3, we have a short exact sequence of graded Abelian groups

$$0 \longrightarrow A\langle n\rangle \xrightarrow{v_m} A\langle n\rangle \longrightarrow \mathbb{Z}/p \otimes_{BP_*} A\langle n\rangle \longrightarrow 0 \tag{33}$$

where v_n-multiplication raises the degree by $2p^n - 2$. By 6.2, all extensions in (33) are nontrivial, of the form

$$0 \to \mathbb{Z}/p^{d-1} \to \mathbb{Z}/p^d \to \mathbb{Z}/p \to 0. \quad \square$$

Lemma 6.7. *Suppose X is a connective locally finite spectrum such that:*

(i) *the BP_*-projective dimension of $BP_*(X)$ is n;*

(ii) *the v_{n+1}-multiplication in $BP\langle n+1\rangle_*(X)$ is injective modulo p^s, $s>0$; then $BP_*(X)$ satisfies 6.3 and 6.4.*

Proof. By (i) and [6], $(BP_*(X))\langle m\rangle \cong BP\langle m\rangle_*(X)$ for $m>n-1$. For the same reasons, v_m-multiplication is injective on $BP\langle m\rangle_*(X)$, $m>n$. Let Y be a $\mathbb{Z}/p^s$-Moore spectrum so that

$$\operatorname{Tor}^{BP_*}_*(BP_*(Y), A) \cong \operatorname{Tor}^{\mathbb{Z}}_*(\mathbb{Z}/p^s, A).$$

The Künneth sequence for $BP\langle n+1\rangle_*(Y\wedge X)$ has the form

$$0 \to \mathbb{Z}/p^s \otimes BP\langle n+1\rangle_*(X) \to BP\langle n+1\rangle_*(Y\wedge X)$$
$$\to \operatorname{Tor}^{\mathbb{Z}}_1(\mathbb{Z}/p^2, BP\langle n+1\rangle_*(X)) \to 0. \tag{34}$$

The Tor term is a sub-BP_*-module of $BP\langle n+1\rangle_*(X)$ which has injective v_{n+1}-multiplication. So hypotheses (ii) and the five lemma tell us that v_{n+1}-multiplication is injective on $BP\langle n+1\rangle_*(Y\wedge X)$. By [6], v_m-multiplication on $BP\langle m\rangle_*(Y\wedge X)$ is then injective for all $m>n$. By the m-analog of (34), v_m-multiplication on $\mathbb{Z}/p^s\otimes BP\langle m\rangle_*(X)$ is injective for all $m>n$. This confirms 6.4. □

Theorem 6.8. *Let $T=\otimes^n_{BP_*} BP_*(B\mathbb{Z}/p)$, the n-fold BP_*-tensor product of $BP_*(B\mathbb{Z}/p)$. Let $T_{n,j}$ be the j-degree group ($\mathbb{Z}_{(p)}$-module) of $T\langle n\rangle = T\otimes_{BP_*} BP\langle n\rangle_*$; so $T_{n,*}=T\langle n\rangle$. Then we have the following additive isomorphisms:*

(a) $T\cong T_{n,*}\otimes \mathbb{Z}_{(p)}[v_{n+1}, v_{n+2},\ldots]$;

(b) $T_{n,j}\cong (\mathbb{Z}/p)^{d_{j,1}}\oplus(\mathbb{Z}/p^2)^{d_{j,2}}\oplus\cdots$ *where*

(i) $d_{j,t}=0$ *if* $j+n\not\equiv 0$ *modulo* 2,

(ii) $d_{j,t}=d_{j-2(p^n-1),t-1}$, $t>1$,

(iii) $d_{j,1}+d_{j,2}+\cdots=\binom{i-1}{n-1}$, $j=2i-n$.

Proof. By Corollary 5.2, T satisfies 6.1–6.4. The theorem follows from Lemma 6.6 using the dimension count of $\mathbb{Z}/p\otimes T_{n,j}$ given in Corollary 5.2. □

Theorem 6.9. *For the spectrum $D(n)$, there are additive isomorphisms:*

(a) $BP_*(D(n))\cong BP\langle n\rangle_*(D(n))\otimes\mathbb{Z}_{(p)}[v_{n+1}, v_{n+2},\ldots]$;

(b) $BP\langle n\rangle_j(D(n))\cong(\mathbb{Z}/p)^{d_{j,1}}\oplus(\mathbb{Z}/p^2)^{d_{j,2}}\oplus\cdots$ *where*

(i) $d_{j,t}=d_{j-2(p^n-1),t-1}$, $t>1$,

(ii) $d_{j,1}+d_{j,2}+\cdots=e_j$ *where* $\Sigma_j e_j x^j=(1-x^{2(p-1)})^{-1}\cdots(1-x^{2(p^n-1)})^{-1}$.

Proof. By Corollaries 4.4 and 4.10, the hypotheses of Lemma 6.7 are satisfied. The theorem then follows from Lemmas 6.5 and 6.7 using the counting of Proposition 4.3. □

Theorem 6.10. *Let $V=(\mathbb{Z}/p)^n$, the rank-n elementary p-group. There are additive isomorphisms*:

(a) $BP_*(BV^+) \cong BP\langle n\rangle_*(BV^+) \otimes \mathbb{Z}_{(p)}[v_{n+1}, v_{n+2}, \ldots]$;

(b) $BP_*(\wedge^n B\mathbb{Z}/p) \cong BP\langle n\rangle_*(\wedge^n B\mathbb{Z}/p) \otimes \mathbb{Z}_{(p)}[v_{n+1}, v_{n+2}, \ldots]$.

Proof. Let $X = BV^+$ or $\wedge^n B\mathbb{Z}/p$. In [8], the BP_*-projective dimension of $BP_*(X)$ is found to be n. The other hypothesis of Lemma 6.7 is confirmed by Proposition 5.6. The theorem then follows by Lemma 6.5. □

We end with an example due to Davis [4]. In even dimensions, $BP_*(\wedge^2 B\mathbb{Z}/2)$ is $\otimes^2_{BP_*} BP_*(B\mathbb{Z}/2)$ which is additively isomorphic to $T_{2,*} \otimes \mathbb{Z}_{(2)}[v_3, v_4, \ldots]$. Here $T_{2,2j-2} \cong (\mathbb{Z}/2)^3 \oplus \cdots \oplus (\mathbb{Z}/2^u)^3 + (\mathbb{Z}/2^{u+1})^t$ where $j = 2 + 3u + t$, $0 \leqslant t < 3$. Just apply Theorem 6.8.

Acknowledgement

The first named author thanks Tony Bahri, Marty Bendersky, Don Davis, Peter Gilkey, and Peter Landweber for their questions and interest. The authors thank the referee for the referee's thorough, constructive, and prompt reading of our typescript.

References

[1] A. Bahri, M. Bendersky, D.M. Davis and P.B. Gilkey, The complex bordism of groups with periodic cohomology, Trans. Amer. Math. Soc. 316 (1989) 673–687.
[2] P.E. Conner and E.E. Floyd, Cobordism theories, in: Seattle Conference on Differential and Algebraic Topology (1963) mimeographed.
[3] P.E. Conner and E.E. Floyd, Differential Periodic Maps (Springer, Berlin, 1964).
[4] D.M. Davis, A strong nonimmersion theorem for real projective spaces, Ann. of Math. 120 (1984) 517–528.
[5] D.C. Johnson, On the $[p]$-series in Brown–Peterson homology, J. Pure Appl. Algebra 48 (1987) 263–270.
[6] D.C. Johnson and W.S. Wilson, Projective dimension and Brown–Peterson homology, Topology 12 (1973) 327–353.
[7] D.C. Johnson and W.S. Wilson, *BP* operations and Morava's extraordinary *K*-theories, Math. Z. 144 (1975) 55–75.
[8] D.C. Johnson and W.S. Wilson, The *BP* homology of elementary *p*-groups, Amer. J. Math. 102 (1982) 427–454.
[9] M. Kamata, The structure of the bordism group $U_*(B\mathbb{Z}/p)$, Osaka J. Math. 7 (1970) 409–416.
[10] P.S. Landweber, Homological properties of comodules over MU_*MU and BP_*BP, Amer. J. Math. 98 (1976) 591–610.
[11] S.A. Mitchell, A proof of the Conner–Floyd conjecture, Amer. J. Math. 106 (1984) 889–891.
[12] S.A. Mitchell and S.B. Priddy, Stable splittings derived from the Steinberg module, Topology 22 (1983) 285–298.
[13] D.C. Ravenel and W.S. Wilson, The Morava *K*-theories of Eilenberg–MacLane spaces and the Conner–Floyd conjecture, Amer. J. Math. 102 (1980) 691–748.
[14] D.Y. Yan, The Brown–Peterson homology of the classifying spaces *BO* and *BO*(*n*), Preprint (1991).