Topology Vol. 15, pp. 131–141. Pergamon Press, 1976. Printed in Great Britain

ON NOVIKOV'S EXT^1 MODULO AN INVARIANT PRIME IDEAL

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(*Received* 11 *April* 1975)

INTRODUCTION

IN HIS WORK on complex cobordism [18] Novikov introduced a spectral sequence converging to the stable homotopy of a space, depending only on the complex cobordism of the space as a module over the ring of primary complex cobordism operations. This spectral sequence can be localized at a prime p and one can work, as Novikov did, with the smaller theory, $BP^*(\ )$, known as Brown–Peterson cohomology [5]. Adams [4] translated the construction into homology, and we have:

$$E_2^{**}(X) \simeq \mathrm{Ext}^{**}_{BP_*BP}(BP_*, BP_*(X)) \Rightarrow \pi_*^S(X)_{(p)}.$$

Novikov computed the first line, $\mathrm{Ext}^{1,*}_{BP_*BP}(BP_*, BP_*)$, of the spectral sequence for X a sphere and showed that for p odd it was canonically isomorphic with the image of J. Later, Quillen's work [19], [4] gave a good grip on the structure of the BP operations and was used by Zahler [24] for low dimensional calculations. Quillen's results, together with Hazewinkel's construction [6] of algebra generators for

$$BP_* \simeq \pi_* BP \simeq \mathbb{Z}_{(p)}[v_1, v_2, \ldots],$$

$|v_n| = 2(p^n - 1)$, and the existence of the cobar construction for BP [12] (see Section 3), have now made computations with the Novikov spectral sequence really feasible.

The ideals

$$I_n = (p, v_1, \ldots, v_{n-1}) \subset BP_*, \qquad 0 \le n \le \infty,$$

are invariant under the action of BP operations. In fact, they exhaust the list of invariant prime ideals [9]. Thus BP_*/I_n is a BP_*BP comodule, and a theorem of Landweber [9] implies that for $0 < n < \infty$

$$\mathrm{Ext}^{0,*}_{BP_*BP}(BP_*, BP_*/I_n) \simeq \mathbb{F}_p[v_n].$$

The purpose of this paper is to give proofs of our calculation of

$$\mathrm{Ext}^{1,*}_{BP_*BP}(BP_*, BP_*/I_n) = \mathrm{Ext}^{1,*}_{BP^*BP}(BP^*, BP^*/I_n).$$

These results were announced in [14].

Because of the realizability of BP_*/I_n as $BP_*(X)$ for $n = 1, 2, 3, 4$, and $p > 2(n-1)$ ($n = 2$, [2]; $n = 3$, [21]; $n = 4$, [22]), these computations have immediate applications to stable homotopy (see [7]). In particular, the calculation of $\mathrm{Ext}^{1,*}_{BP_*BP}(BP_*, BP_*/(p))$ and $\mathrm{Ext}^{1,*}_{BP_*BP}(BP_*, BP_*/(p, v_1))$ are essential first steps in our work with Ravenel on computing $\mathrm{Ext}^{2,*}_{BP_*BP}(BP_*, BP_*)$ and detecting the entire gamma family of stable homotopy elements [13].

In §1 we state our main results and complete the description of $\mathrm{Ext}^{1,*}_{BP_*BP}(BP_*, BP_*/I_n)$ for an odd prime p by quoting work of Morava and Moreira. The necessary information on BP is then recalled and in Section 3 we construct the cobar complex. Next we compare BP_*BP with the Hopf algebra dual to the reduced powers and thereby handle the case $n = \infty$. The other cases are proved in Section 5 by constructing elements and using a downward induction on n. In the last section we apply this result to classify invariant ideals containing I_n and generated by $n+2$ elements.

Throughout this paper p will denote a prime and BP will be the Brown–Peterson spectrum associated with p.

We would like to thank Dave Johnson whose joint work with the second author on BP

operations [8] led us naturally into this area of research. We also thank Raph Zahler whose early partial results on $\mathrm{Ext}^{1,*}(BP_*, BP_*/(p))$ [25] stimulated our interest in the problem. We are grateful to both of the above for helping to find applications of this work to stable homotopy [7]. Special thanks go to Jack Morava whose work [15], [16] has revolutionized the field, in particular his Theorem 1.4 here showed us we had nearly complete results about Ext^1. We want to thank Doug Ravenel who pointed out to us that our proofs still worked at the prime 2 if $n > 1$, and with whom subsequent collaboration has made this work the first step in a much more ambitious program [13]. Both authors were partially supported by the NSF.

§1. STATEMENT OF RESULTS

Let I_n denote the invariant prime ideal $(p, v_1, \ldots, v_{n-1}) \subset BP_*$. We have a short exact sequence

$$0 \to BP_*/I_n \xrightarrow{v_n} BP_*/I_n \to BP_*/I_{n+1} \to 0$$

which induces a 'Bockstein' exact couple (see 4.4 below)

$$\begin{array}{ccc} \mathrm{Ext}^{**}(BP_*, BP_*/I_n) & \xrightarrow{v_n} & \mathrm{Ext}^{**}(BP_*, BP_*/I_n) \\ {\scriptstyle \delta_n} \nwarrow & & \swarrow {\scriptstyle \rho_n} \\ & \mathrm{Ext}^{**}(BP_*, BP_*/I_{n-1}) & \end{array}$$

where δ_n has bidegree $(1, 2-2p^n)$. Here and below we suppress BP_*BP from our notation.

$\mathrm{Ext}^{**}(BP_*, BP_*/I_n)$ is a bigraded algebra (see the remarks following Lemma 3.4), so $\mathrm{Ext}^{1,*}(BP_*, BP_*/I_n)$ is a module over

$$\mathrm{Ext}^{0,*}(BP_*, BP_*/I_n) \simeq \mathbf{F}_p[v_n] \qquad (\mathbf{Z}_{(p)} \quad \text{if} \quad n = 0).$$

We shall describe $\mathrm{Ext}^{1,*}(BP_*, BP_*/I_n)$ in terms of this module structure.

Recall that $BP_*BP \simeq BP_*[t_1, t_2, \ldots]$ with $|t_n| = 2(p^n - 1)$. In the cobar construction for BP_*BP with coefficients in BP_*/I_n, $n > 0$, $t_1^{p^i}$ is a cycle representing a non-zero class

$$h_i \in \mathrm{Ext}^{1,p^iq}(BP_*, BP_*/I_n)$$

where $q = 2(p-1)$. Clearly h_i is taken to h_i by reduction, ρ_n.

Let $p > 2$ and $0 \le n < \infty$, or $p = 2$ and $2 \le n < \infty$. For $r > 0$, write $r = ap^s$ with a prime to p. Write $s = kn + i + 1$ with $0 \le i < n$. If $n = 0$, let $q(r) = q_0(ap^s) = s + 1$. If $0 < n < \infty$, let

$$q(r) = q_n(ap^s) = \begin{cases} p^s & \text{if } a = 1 \\ p^s + (p-1)\sum_{l=0}^{k-1} p^{ln+i} & \text{otherwise.} \end{cases}$$

Recall that $I_0 = (0)$ and $v_0 = p$. In §5 we explicitly construct certain elements

$$c_n(r) \in \mathrm{Ext}^{1,2r(p^{n+1}-1)-2q(r)(p^n-1)}(BP_*, BP_*/I_n).$$

THEOREM 1.1. *Let $p > 2$, $0 \le n < \infty$, or $p = 2$, $2 \le n < \infty$.*

(a) *The set $\{h_i : 0 \le i < n\}$ generates a free $\mathbf{F}_p[v_n]$ submodule of $\mathrm{Ext}^{1,*}(BP_*, BP_*/I_n)$ of rank n.*

(b) *The v_n-torsion submodule of $\mathrm{Ext}^{1,*}(BP_*, BP_*/I_n)$ is a sum of cyclic $\mathbf{F}_p[v_n]$ ($\mathbf{Z}_{(p)}$ for $n = 0$) modules on generators $c_n(r)$ for $r = ap^s > 0$, a prime to p, satisfying*

(i) $v_n^{q(r)} c_n(r) = 0$

$v_n^{q(r)-1} c_n(r) = \delta_n(v_{n+1}^r) \neq 0$

(ii) $h_{s+n} = c_n(p^s) + v_n^{p^s(p-1)} h_s \qquad s \ge 0, n > 0$

(iii)
$$\begin{aligned} \rho_n(c_n(ap^s)) &= av_1^{r-1}h_0 && \textit{if } n = 0 \\ &= h_{s+n} && \textit{if } a = 1 \\ &= av_{n+1}^{a-1}h_n && \textit{if } s = 0 \\ &= 2av_2^{ap^s - p^{s-1}}h_0 && \textit{if } n = 1, s > 1, a \neq 1 \\ &= av_{n+1}^{ap^s - p^{s-1}}h_i && \textit{otherwise} \end{aligned}$$

where $s - 1 \equiv i \bmod n$, $0 \le i < n$.

COROLLARY 1.2. *Let $p > 2, 0 < n < \infty$, or $p = 2, 1 < n < \infty$. All relations in the $\mathbf{F}_p[v_n]$-submodule* *of* $\mathrm{Ext}^{1,*}(BP_*, BP_*/I_n)$ *generated by the* h_i, $i \geq 0$, *are consequences of*

$$v_n^{p^s} h_{s+n} = v_n^{p^{s+1}} h_s,\ s \geq 0.$$

COROLLARY 1.3. *Let* $p > 2$, $0 \leq n < \infty$, *or* $p = 2$, $2 \leq n < \infty$. *Suppose* $I \subset BP_*$ *is an invariant ideal containing* I_n *and generated by* $n+2$ *elements and no fewer. Then* $I = (p, v_1, \ldots, v_{n-1}, v_n^t, y)$ *where* $t > 0$ *and* y *involves only the generators* v_n, v_{n+1} *and* v_{n+2}.

Remark. In §6 we give a complete classification of all such ideals; this is a corollary of a description of the groups $\mathrm{Ext}^{0,*}(BP_*, BP_*/(I_n, v_n^t))$ which follows from the proof of Theorem 1.1.

In order to offer a complete description of $\mathrm{Ext}^{1,*}(BP_*, BP_*/I_n)$ for odd primes we summarize the work of others on the free $\mathbb{F}_p[v_n]$ summands. In particular, we need the following important result of Morava ([15] and private communications). More recent proofs of this result have been obtained by Moreira and Ravenel. All three use different techniques.

THEOREM 1.4. (Morava). *Let* p *be an odd prime and* $0 < n < \infty$. *The rank of* $\mathrm{Ext}^{1,*}(BP_*, BP_*/I_n)$ *over* $\mathbb{F}_p[v_n]$ *is* 1 *for* $n = 1$ *and* $n+1$ *for* $n > 1$.

For $n > 1$, Theorem 1.1(a) gives us all but one generator of $\mathrm{Ext}^{1,*}(BP_*, BP_*/I_n)$ mod v_n-torsion; for $n = 1$, we have everything. To complete our description we state the main property of the remaining free summand for $n > 1$. This was conjectured by us in [14]. The actual construction of the extra generator w_n is very complicated and is done explicitly by Moreira in his thesis.

THEOREM 1.5. (Moreira[17]). *For* p *an odd prime and* $1 < n < \infty$, *there is an element*

$$w_n \in \mathrm{Ext}^{1,k}(BP_*, BP_*/I_n)$$

where $k = 2(p^n - 1)^2/(p-1)$, *which generates a free* $\mathbb{F}_p[v_n]$*-module summand and reduces to*

$$0 \neq \rho_n(w_n) = v_{n+1}^{1+p+\cdots+p^{n-2}} h_{n-1}.$$

For example, a representing cycle for w_2 in the cobar construction is given by $v_2 t_1^{p(p+1)} + v_3 t_1^p - v_2^p t_2 - v_2 t_2^p$.

The main technical result we need for all of our computations concerns the right unit, $\eta_R: BP_* \to BP_*BP$.

LEMMA 5.1. *Let* $n > 0$. *Then*

$$\eta_R(v_{n+1}) \equiv v_{n+1} + v_n t_1^{p^n} - v_n^p t_1 \bmod I_n.$$

§2. RECOLLECTIONS ON BP AND BP_*BP

We begin by reviewing the necessary facts about BP (see [19] and [4]). Fix a prime p. There is an associative ring-spectrum BP with homology algebra

$$H_*(BP; \mathbb{Z}) = \mathbb{Z}_{(p)}[m_1, m_2, \ldots]$$

for canonical generators m_n, $|m_n| = 2(p^n - 1)$. Thus by the Künneth theorem,

$$H_*(BP \wedge BP; \mathbb{Z}) = H_*(BP; \mathbb{Z})[t_1, t_2, \ldots]$$

where $|t_n| = 2(p^n - 1)$; and t_n may be chosen inductively so that $t_0 = 1$ and

$$\eta_R(m_n) = \sum_{i+j=n} m_i t_j^{p^i} \qquad 2.1$$

where η_R is induced from $BP \simeq S^0 \wedge BP \to BP \wedge BP$. The map $BP \simeq BP \wedge S^0 \to BP \wedge BP$ induces $\eta_L: m_n \to m_n$.

If we allow η_L and η_R to describe left and right H_*BP-module structures on $H_*(BP \wedge BP)$ then we have for any spectrum X a diagram

$$\begin{array}{ccc} H_*(BP \wedge X) \simeq H_*(BP \wedge S^0 \wedge X) & \to & H_*(BP \wedge BP \wedge X) \\ & \searrow^{\psi_X} & \uparrow m \\ & & H_*(BP \wedge BP) \otimes_{H_*BP} H_*(BP \wedge X) \end{array} \qquad 2.2$$

in which the Künneth map m is an isomorphism. In case $X = BP$, $\psi_{BP} = \Delta$ is described inductively by

$$\sum_{i+j=n} m_i(\Delta t_j)^{p^i} = \sum_{i+j+k=n} m_i t_j^{p^i} \otimes t_k^{p^{i+j}}. \qquad 2.3$$

The switch map $BP \wedge BP \to BP \wedge BP$ induces c such that

$$m_n = \sum_{i+j+k=n} m_i t_j^{p^i} c(t_k^{p^{i+j}}) \qquad 2.4$$

and the multiplication $BP \wedge BP \to BP$ induces $\varepsilon: t_n \mapsto 0$, $m_n \mapsto m_n$.

These maps restrict to the subalgebras

$$BP_* = \pi_*(BP) \subset H_*(BP)$$

and

$$BP_*BP = \pi_*(BP \wedge BP) = BP_*[t_1, t_2, \dots] \subset H_*(BP \wedge BP).$$

Hazewinkel[6, 26] has shown that the elements $v_i \in H_*(BP)$ determined by

$$v_n = pm_n - \sum_{i=1}^{n-1} m_i v_{n-i}^{p^i} \qquad 2.5$$

generate the subalgebra $\pi_*(BP) = Z_{(p)}[v_1, v_2, \dots]$. By convention $m_0 = 1$, $v_0 = p$.

§3. THE COBAR CONSTRUCTION

The identities recorded in [3] which are satisfied by η_L, η_R, ε, Δ, c, are easily seen to be precisely the axioms for a *cogroupoid object* in the category of commutative graded rings. Since groupoids have bar constructions, we may expect objects such as BP_*BP to have cobar constructions, and we turn now to this device[12].

Fix a commutative ring K. Let A and Γ be commutative graded K-algebras equipped with K-algebra morphisms $\eta_L, \eta_R: A \to \Gamma$, $\varepsilon: \Gamma \to A$, $\Delta: \Gamma \to \Gamma \otimes_A \Gamma$, $c: \Gamma \to \Gamma$, satisfying the axioms for a cogroupoid object[3]. A Γ-*comodule* is a left A-module M together with a left A-module map $\psi: M \to \Gamma \otimes_A M$ which is associative and unitary. For example, the homotopy analogue of 2.2 makes $BP_*(X)$ à BP_*BP-comodule for any spectrum X.

Let $\Omega^n(\Gamma; M) = \Gamma \otimes_A \cdots \otimes_A \Gamma \otimes_A M$, with n copies of Γ, be the n-th group in the cochain complex $\Omega(\Gamma; M)$ with differential

$$\begin{aligned} d(\gamma_1 \otimes \cdots \otimes \gamma_n \otimes m) &= 1 \otimes \gamma_1 \otimes \cdots \otimes \gamma_n \otimes m \\ &+ \sum_{i=1}^{n} (-1)^{\varepsilon(i)} \gamma_1 \otimes \cdots \otimes \gamma_i' \otimes \gamma_i'' \otimes \cdots \otimes \gamma_n \otimes m \\ &+ (-1)^{\varepsilon(n+1)} \Sigma \gamma_1 \otimes \cdots \otimes \gamma_n \otimes m' \otimes m'' \end{aligned} \qquad 3.1$$

where $\varepsilon(i) = |\gamma_1| + \cdots + |\gamma_{i-1}| + i$ and $\Delta\gamma = \Sigma\gamma' \otimes \gamma''$, $\psi(m) = \Sigma m' \otimes m''$. Write $\Omega\Gamma = \Omega(\Gamma; A)$ where the Γ-comodule structure on A is given by

$$A \xrightarrow{\eta_L} \Gamma \simeq \Gamma \otimes_A A. \qquad 3.2$$

Define a product

$$\Omega\Gamma \otimes_K \Omega(\Gamma; M) \to \Omega(\Gamma; M)$$

by juxtaposition. Then $\Omega\Gamma$ is a *DG* K-algebra and $\Omega(\Gamma; M)$ is a *DG* $\Omega\Gamma$-module.

Define

$$\operatorname{Ext}_\Gamma^{**}(A, M) = H^{**}\Omega(\Gamma; M).$$

It is routine to check that this is the derived functor of $\operatorname{Hom}_\Gamma(A, -)$ as described by Adams [4]. The DG products above make $\operatorname{Ext}_\Gamma(A, A)$ a bigraded K-algebra and $\operatorname{Ext}_\Gamma(A, M)$ an $\operatorname{Ext}_\Gamma(A, A)$-module in a natural way.

In case $n = 0$, 3.1 reads $d(m) = 1 \otimes m - \Sigma m' \otimes m''$. In particular, if $M = A$,

$$d(a) = 1 \otimes a - \eta_L(a) \otimes 1 = \eta_R(a) - \eta_L(a) \qquad 3.3$$

under the identification $\Gamma \otimes_A A \simeq \Gamma$.

We mention some immediate applications of this construction.

An ideal $I \subset A$ is *invariant* iff it is a subΓ-comodule under the coaction 3.2. Denote by Γ/I the K-algebra $(A/I) \otimes_A \Gamma = \Gamma \otimes_A (A/I)$. Then the structure maps η_L, η_R, ε, Δ, c, factor through A/I and Γ/I. If M is a Γ-comodule such that $IM = 0$, then M is naturally a Γ/I-comodule and clearly $\Omega(\Gamma; M) \simeq \Omega(\Gamma/I; M)$. Thus:

LEMMA 3.4. *If $I \subset A$ is invariant and M is a Γ-comodule such that $IM = 0$, then*

$$\mathrm{Ext}_\Gamma(A, M) \simeq \mathrm{Ext}_{\Gamma/I}(A/I, M).$$

It follows that $\mathrm{Ext}_\Gamma(A, A/I)$ has a natural K-algebra structure.

Finally, assume that Γ is flat over A. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of Γ-comodules. Then

$$0 \to \Omega(\Gamma; M') \to \Omega(\Gamma; M) \to \Omega(\Gamma; M'') \to 0$$

is short exact, so we obtain a long exact sequence

$$\begin{array}{ccc} \mathrm{Ext}_\Gamma(A, M') & \rightarrow & \mathrm{Ext}_\Gamma(A, M) \\ {}^{\delta}\nwarrow & & \swarrow \\ & \mathrm{Ext}_\Gamma(A, M''). & \end{array} \qquad 3.5$$

§4. MORE ON BP

We return now to BP_*BP. Let I_n denote the ideal $(p, v_1, \ldots, v_{n-1}) \subset BP_*$ for $0 \le n \le \infty$. By induction using 2.5 and 2.1 it is easy to see that I_n is invariant. We are concerned with $\mathrm{Ext}^{**}(BP_*, BP_*/I_n)$ where we have omitted the subscript BP_*BP.

Since $BP_t = 0 = BP_tBP$ unless $t \equiv 0 \bmod q = 2(p-1)$, the cobar construction and hence also $\mathrm{Ext}^{**}(BP_*, BP_*/I_n)$ exhibits the same sparseness [23] in internal degree.

In homological degree 0 we have by Novikov [18]

$$\mathrm{Ext}^{0,*}(BP_*, BP_*) \simeq \mathbb{Z}_{(p)} \qquad 4.1$$

concentrated in degree zero and by Landweber ([9]; see also [8]) for $0 < n < \infty$

$$\mathrm{Ext}^{0,*}(BP_*, BP_*/I_n) \simeq \mathbb{F}_p[v_n]. \qquad 4.2$$

Clearly

$$\mathrm{Ext}^{0,*}(BP_*, BP_*/I_\infty) \simeq \mathbb{F}_p. \qquad 4.3$$

By 3.5 the short exact sequence

$$0 \to BP_*/I_n \xrightarrow{v_n} BP_*/I_n \to BP_*/I_{n+1} \to 0$$

of BP_*BP-comodules induces a long exact sequence

$$\begin{array}{ccc} \mathrm{Ext}^{**}(BP_*, BP_*/I_n) & \xrightarrow{v_n} & \mathrm{Ext}^{**}(BP_*, BP_*/I_n) \\ {}^{\delta_n}\nwarrow & & \swarrow{}^{\rho_n} \\ & \mathrm{Ext}^{**}(BP_*, BP_*/I_{n+1}). & \end{array} \qquad 4.4$$

These long exact sequences will be the basis of an inductive calculation in §5. For $n = \infty$, we need the following result.

LEMMA 4.5. *In BP_*BP,*

$$\Delta t_n \equiv \sum_{i+j=n} t_i \otimes t_j^{p^i} \bmod I_\infty.$$

Proof. From 2.5 it follows that $pH_*(BP) \cap \pi_*(BP) = I_\infty$. Therefore $\pi_*(BP)/I_\infty \hookrightarrow H_*(BP; \mathbb{Z}/p\mathbb{Z})$, and similarly for $BP \wedge BP$ and $BP \wedge BP \wedge BP$. Hence we calculate in $H_*(BP \wedge BP; \mathbb{Z}/p\mathbb{Z})$.

Clearly $\Delta t_1 = t_1 \otimes 1 + 1 \otimes t_1$. By induction using 2.3,

$$\begin{aligned}\Delta t_n &= \sum_{i+j+k=n} m_i t_j^{p^i} \otimes t_k^{p^{i+j}} - \sum_{\substack{i+j=n\\ i\neq 0}} m_i (\Delta t_j)^{p^i}\\ &\equiv \sum_{i+h=n} m_i \sum_{j+k=h} t_j^{p^i} \otimes t_k^{p^{i+j}} - \sum_{\substack{i+h=n\\ i\neq 0}} m_i \Big(\sum_{j+k=h} t_j \otimes t_k^{p^j}\Big)^{p^i}\\ &\equiv \sum_{j+k=n} t_j \otimes t_k^{p^j}.\end{aligned}$$

From Lemma 3.4, then:

COROLLARY 4.6. *Let* $P_* = \mathbb{F}_p[t_1, t_2, \ldots]$ *with diagonal* $\Delta t_n = \sum_{i+j=n} t_i \otimes t_j^{p^i}$. *Then*

$$\mathrm{Ext}^{**}(BP_*, BP_*/I_\infty) \simeq \mathrm{Ext}^{**}_{P_*}(\mathbb{F}_p, \mathbb{F}_p).$$

Note that P_* is just the Hopf algebra dual to the Steenrod reduced powers. Indeed, in [24] Zahler proved that the Thom map $BP_*BP \to H\mathbb{F}_{p*}H\mathbb{F}_p = A_*$ carries t_n to $c(\xi_n)$ for $p > 2$ and to $c(\xi_n^2)$ for $p = 2$, where c is the antiautomorphism of the dual Steenrod algebra A_*.

The work of Adem as interpreted by Adams[1] and Liulevicius[11] now implies:

COROLLARY 4.7. $\mathrm{Ext}^{1,*}(BP_*, BP_*/I_\infty)$ *is the* $\mathbb{F}_p$*-module generated by the classes* h_i *of* $t_1^{p^i} \in \Omega(BP_*BP; BP_*/I_\infty)$, $i \geq 0$.

§5. THE TORSION SUBMODULE

Our purpose in this section is to exhibit certain cycles in $\Omega^1(BP_*BP; BP_*/I_n)$ and to prove that their classes generate the torsion $\mathbb{F}_p[v_n]$-submodule of $\mathrm{Ext}^{1,*}(BP_*, BP_*/I_n)$. We begin with our basic technical lemma.

LEMMA 5.1. *For* $0 < n < \infty$,

$$\eta_R(v_{n+1}) \equiv v_{n+1} + v_n t_1^{p^n} - v_n^p t_1 \bmod I_n.$$

Proof. Note first by induction using 2.5 that $p^i m_i \in (v_1, v_2, \ldots, v_{n-1})\pi_*(BP)$ for $0 < i < n$. Thus if $x \in (m_1, \ldots, m_{n-1})H_*(BP) \cap \pi_*(BP)$, then $p^{n-1}x \in (v_1, \ldots, v_{n-1})\pi_*(BP)$. Since $\pi_*(BP)/(v_1, \ldots, v_{n-1})$ is free of p-torsion, this implies that $x \in (v_1, \ldots, v_{n-1})\pi_*(BP)$.

So we compute in $H_*(BP \wedge BP)$ modulo $(m_1, \ldots, m_{n-1})H_*(BP)$. From 2.5 it is immediate that $pm_n \equiv v_n$, $pm_{n+1} \equiv v_{n+1} + m_n v_1^{p^n}$, and $v_i \equiv 0$, $0 < i < n$. Now using 2.1, 2.5, and the fact that η_R is a ring-homomorphism,

$$\begin{aligned}\eta_R(v_{n+1}) &= p\eta_R(m_{n+1}) - \sum_{i=1}^{n} \eta_R(m_i)\eta_R(v_{n+1-i}^{p^i})\\ &= p\sum_{j=0}^{n+1} m_j t_{n+1-j}^{p^j} - \sum_{i=1}^{n}\sum_{j=0}^{i} m_j t_{i-j}^{p^j}\eta_R(v_{n+1-i}^{p^i})\\ &\equiv pt_{n+1} + pm_n t_1^{p^n} + pm_{n+1} - m_n\eta_R(v_1^{p^n}) - \sum_{i=1}^{n} t_i\eta_R(v_{n+1-i}^{p^i}).\end{aligned}$$

Now $pm_1 = v_1$ and $\eta_R(v_1) = v_1 + pt_1$, so for $n > 1$,

$$m_n\eta_R(v_1^{p^n}) \equiv v_n p^{p^n-1} t_1^{p^n} + m_n v_1^{p^n}.$$

We leave the easier case $n = 1$ to the reader. Thus for $n > 1$,

$$\eta_R(v_{n+1}) \equiv pt_{n+1} + v_n t_1^{p^n} + v_{n+1} - p^{p^n-1} v_n t_1^{p^n} - \sum_{i=1}^{n} t_i\eta_R(v_{n+1-i}^{p^i}).$$

This holds $\bmod (v_1, \ldots, v_{n-1}) \subset I_n$ by the above remarks; so continuing mod I_n,

$$\eta_R(v_{n+1}) = v_n t_1^{p^n} + v_{n+1} - t_1 v_n^p.$$

Here we have used the fact that $\eta_R(v_k) \equiv v_k \bmod I_k$.

Remark. D. C. Ravenel[20] has proved a beautiful generalization of this lemma, and we expect his formulas to be very useful. For example, they are essential in Moreira's proof of Theorem 1.5.

Our search for cycles representing torsion generators of the $\mathbb{F}_p[v_n]$-module $\mathrm{Ext}^{1,*}(BP_*, BP_*/I_n)$ is governed by the following immediate consequence of 4.2 and 4.4.

LEMMA 5.2. *Let $0 \le n < \infty$. The elements $\delta_n(v_{n+1}^t)$, $t > 0$, are nonzero and generate*

$$\ker v_n \subset \mathrm{Ext}^{1,*}(BP_*, BP_*/I_n)$$

additively.

Since every connected module over $\mathbb{F}_p[v_n]$ is a direct sum of cyclic modules, we must find how far $\delta_n(v_{n+1}^t)$ can be divided by v_n. Notice that since $\Omega^1(BP_*BP; BP_*/I_n)$ is a DG $\mathbb{F}_p[v_n]$-module (if $n = 0$, a DG $\mathbb{Z}_{(p)}$-module) which is free of v_n-torsion, c is a cycle iff $\bar{c} = v_n{}^k c$ is a cycle for some k.

If $p = 2$, the cases $n = 0, 1$ behave peculiarly and will be treated elsewhere in joint work with D. C. Ravenel; see also Remark 5.10 below. The case $p = 2$, $n = 0$ was solved by Novikov [18]. So assume that either $p > 2$ and $0 \le n \le \infty$ or that $p = 2$ and $2 \le n < \infty$.

Let $s = kn + i + 1$ with $0 \le i < n$. For a prime to p and $1 \le j \le k$ let

$$a_j = a_j(n, ap^s) = (ap - 1)p^{jn} + 1$$

$$b_j = b_j(n, ap^s) = \sum_{l=0}^{j-1} p^{(k-l)n+i+1} - \sum_{l=1}^{j} p^{(k-l)n+i}.$$

For $r = ap^s > 0$, a prime to p, define

$$\bar{c}_n(ap^s) = v_{n+1}^{ap^s} \text{ if } n = 0 \text{ or } s = 0 \text{ or } a = 1;$$

$$\bar{c}_n(ap^s) = v_2{}^{ap^s} - av_1{}^{p^s} v_2{}^{(a-1)p^s - p^{s-1}} v_3{}^{p^{s-1}} - av_1{}^{b_1} v_2{}^{a_1 p^{s-2}}$$

$$- av_1{}^{p^s + p^{s-1} - p^{s-2}} v_2{}^{(ap-2)p^{s-1}} v_3{}^{p^{s-2}} - 2a \sum_{j=2}^{s-1} v_1{}^{b_j} v_2{}^{a_j p^{s-1-j}}$$

if $n = 1$, $s > 1$, and $a \ne 1$;

$$\bar{c}_n(ap^s) = v_{n+1}^{ap^s} - av_n{}^{p^s} v_{n+1}^{(a-1)p^s - p^{s-1}} v_{n+2}^{p^{s-1}} - a \sum_{j=1}^{k} v_n{}^{b_j} v_{n+1}^{a_j p^{(k-j)n+i}} \text{ otherwise.}$$

If $n = 0$ let $q(ap^s) = q_0(ap^s) = s + 1$. If $0 < n < \infty$, let

$$q(ap^s) = q_n(ap^s) = \begin{cases} p^s & \text{if } a = 1, \\ p^s + (p-1)\sum_{l=0}^{k-1} p^{ln+i} & \text{otherwise.} \end{cases}$$

PROPOSITION 5.3. *Let $p > 2$, $0 \le n < \infty$, or $p = 2$, $2 \le n < \infty$, and let $r > 0$.*

(a) *There exists a unique cycle*

$$c_n(r) \in \Omega^1(BP_*BP; BP_*/I_n)$$

such that

$$v_n{}^{q(r)} c_n(r) = \mathrm{d}\bar{c}_n(r).$$

(b) *Modulo I_{n+1} we have, for a prime to p,*

$$\begin{aligned} c_n(ap^s) &\equiv av_1{}^{ap^{s-1}} t_1 && \text{if } n = 0 \\ &\equiv t_1{}^{p^{s+n}} && \text{if } a = 1 \\ &\equiv av_{n+1}^{a-1} t_1{}^{p^n} && \text{if } s = 0 \\ &\equiv 2av_2{}^{(ap-1)p^{s-1}} t_1 && \text{if } n = 1,\ s > 1, \text{ and } a \ne 1 \\ &\equiv av_{n+1}^{(ap-1)p^{s-1}} t_1{}^{p^i} && \text{otherwise} \end{aligned}$$

where $s - 1 \equiv i$ mod n, $0 \le i < n$.

Proof. We work mod I_n. Recall that for $x, y \in BP_*/I_n$, $\mathrm{d}(x) = \eta_R(x) - \eta_L(x) = \eta_R(x) - x$ mod I_n by 3.3, and that $\eta_R(xy) = \eta_R(x)\eta_R(y)$. In case $n = 0$ we have $\eta_R(v_1) = v_1 + pt_1$ and the result is straightforward. In case $s = 0$ or $a = 1$, both assertions are clear from the congruence

$$dv_{n+1}^{ap^s} \equiv a v_n^{p^s} v_{n+1}^{(a-1)p^s} t_1^{p^{s+n}} \bmod v_n^{2p^s}. \tag{5.4}$$

which is a consequence of Lemma 5.1.

The other cases result similarly from the following consequences of 5.1:

$$d(v_{n+1}^{(a-1)p^s - p^{s-1}} v_{n+2}^{p^{s-1}}) \equiv v_{n+1}^{(a-1)p^s} t_1^{p^{s+n}} - v_{n+1}^{(ap-1)p^{s-1}} t_1^{p^{s-1}} \bmod v_n^{p^{s-1}} \tag{5.5}$$

$$d(v_{n+1}^{a_j p^{(k-j)n+i}}) \equiv v_{n+1}^{(ap-1)p^{kn+i}} [v_n^{p^{(k-j)n+i}} t_1^{p^{(k-j+1)n+i}} - v_n^{p^{(k-j)n+i+1}} t_1^{p^{(k-j)n+i}}] \bmod v_n^{p^{kn+i}} \tag{5.6}$$

$$\begin{aligned} dv_2^{a_1 p^{s-2}} \equiv\ & v_1^{p^{s-2}} v_2^{(ap-1)p^{s-1}} t_1^{p^{s-1}} - v_1^{p^{s-1}} v_2^{(ap-1)p^{s-1}} t_1^{p^{s-2}} \\ & - v_1^{p^{s-1}} v_2^{(ap-2)p^{s-1}+p^{s-2}} t_1^{p^s} \bmod v_1^{p^{s-1}+p^{s-2}}. \end{aligned} \tag{5.7}$$

$$d(v_2^{(ap-2)p^{s-1}} v_3^{p^{s-2}}) \equiv v_2^{(ap-2)p^{s-1}+p^{s-2}} t_1^{p^s} - v_2^{(ap-1)p^{s-1}} t_1^{p^{s-2}} \bmod v_1^{p^{s-2}}. \tag{5.8}$$

Remark. If $0 < s \le n$, $a > 1$, than $dv_{n+1}^{ap^s}$ and $d\bar{c}_n(ap^s)$ are divisible by the same power of v_n, but $v_n^{-p^s}\, d\bar{c}_n(ap^s)$ has a more convenient value modulo I_{n+1}.

Remark 5.9. Thus

$$\bar{c}_n(r) \in \mathrm{Ext}^{0,*}(BP_*, BP_*/(I_n, v_n^{q(r)})) \subset BP_*/(I_n, v_n^{q(r)}),$$

and $c_n(r) \in \mathrm{Ext}^{1,*}(BP_*, BP_*/I_n)$ is the image of $\bar{c}_n(r)$ under the boundary homomorphism induced by the short exact sequence

$$0 \to BP_*/I_n \xrightarrow{v_n^{q(r)}} BP_*/I_n \to BP_*/(I_n, v_n^{q(r)}) \to 0.$$

By customary abuse of notation let $c_n(r)$ denote the class in $\mathrm{Ext}^{1,*}(BP_*, BP_*/I_n)$ represented by the cycle $c_n(r)$. Recall the class h_s of $t_1^{p^s}$ in $\mathrm{Ext}^{1,2p^s(p-1)}(BP_*, BP_*/I_n)$, $0 < n \le \infty$. We can now prove the main theorem stated in §1.

Proof of Theorem 1.1

First note that properties (i) and (iii) follow from Proposition 5.3 and Lemma 5.2. Property (ii) holds on the cochain level:

$$c_n(p^s) = t_1^{p^{s+n}} - v_n^{p^{s+1}-p^s} t_1^{p^s}$$

by Lemma 5.1.

Suppose now $n > 0$. Let $M(n)$ be the $\mathbb{F}_p[v_n]$-module with generators h'_i, $0 \le i < n$ and $c'_n(r)$, $0 < r$, subject only to $v_n^{q(r)} c'_n(r) = 0$. By (i) there is a map

$$f_n : M(n) \to \mathrm{Ext}^{1,*}(BP_*, BP_*/I_n)$$

carrying h'_i to h_i and $c'_n(r)$ to $c_n(r)$. We must show that f_n is injective and has torsion-free cokernel.

The $(2(p^n - 1) - 1)$-connected map $BP_*/I_n \to BP_*/I_\infty = \mathbb{F}_p$ induces for each $s \ge 0$ a map $\Omega^s(BP_*BP; BP_*/I_n) \to \Omega^s(BP_*BP; BP_*/I_\infty)$ with the same connectivity. Thus

$$\mathrm{Ext}^{s,*}(BP_*, BP_*/I_n) \to \mathrm{Ext}^{s,*}(BP_*, BP_*/I_\infty)$$

is $(2(p^n - 1) - 1)$-connected.

Fix $L > 0$ and let n be such that $2(p^n - 1) > L$. Then in internal degrees $t < L$, Theorem 1.1 is equivalent to Corollary 4.7. We now proceed by downward induction on n.

Recall that for an $\mathbb{F}_p[v_n]$-module M we have an exact sequence

$$0 \to \mathrm{Tor}(\mathbb{F}_p, M) \to M \xrightarrow{v_n} M \to \mathbb{F}_p \otimes_{\mathbb{F}_p[v_n]} M \to 0.$$

Observe that $f_n : M(n) \to \mathrm{Ext}^{1,*}(BP_*, BP_*/I_n)$ is injective with torsion-free cokernel in degrees $t < L$ if simultaneously $\mathrm{Tor}(\mathbb{F}_p, f_n)$ is surjective and $\mathbb{F}_p \otimes_{\mathbb{F}_p[v_n]} f_n$ is injective in degrees $t < L$.

So assume that f_{n+1} injects in degrees $t < L$. Consider

$$\begin{array}{ccc} M(n) & \xrightarrow{f_n} & \mathrm{Ext}^{1,*}(BP_*, BP_*/I_n) \\ \downarrow & & \downarrow \\ \mathbb{F}_p \otimes_{\mathbb{F}_p[v_n]} M(n) & \xrightarrow{\bar{f}_n} & \mathbb{F}_p \otimes_{\mathbb{F}_p[v_n]} \mathrm{Ext}^{1,*}(BP_*, BP_*/I_n) \\ \downarrow \bar{\rho}'_n & & \downarrow \bar{\rho}_n \\ M(n+1) & \xrightarrow{f_{n+1}} & \mathrm{Ext}^{1,*}(BP_*, BP_*/I_{n+1}) \end{array}$$

where $\bar{\rho}_n$ factors ρ_n and $\bar{\rho}'_n$ is the obvious map given by (ii) and (iii). Then $\bar{\rho}'_n$ is injective, so by the inductive assumption $\bar{\rho}_n\bar{f}_n = f_{n+1}\bar{\rho}'_n$ is injective. Thus $\bar{f}_n = \mathbb{F}_p \otimes_{\mathbb{F}_p[v_n]} f_n$ injects. Since

$$\mathrm{Ext}^{0,*}(BP_*, BP_*/I_{n+1}) \xrightarrow{\delta_n} \mathrm{Ext}^{1,*}(BP_*, BP_*/I_n) \xrightarrow{v_n} \mathrm{Ext}^{1,*}(BP_*, BP_*/I_n)$$

is exact and the left group is $\mathbb{F}_p[v_{n+1}]$, (i) implies that $\mathrm{Tor}(\mathbb{F}_p, f_n)$ surjects. So the induction is complete. Since L was arbitrary this finishes the proof for $n > 0$.

For $n = 0$ it suffices to show that $\rho_0 c_0(r) \neq 0$ for $r > 0$. This follows from (iii) and from (a) for $n = 1$. It is elementary that there is no torsion-free part.

Remark. As one can see by the simplicity of the proof, the work is mostly done in the finding of the numbers $q_n(r)$ and the cycles $c_n(r)$. At the request of the referee we comment on how we arrived at these formulas. Our insight came from computing $\mathrm{Ext}^{1,*}(BP_*, BP_*/(p))$ for $p = 3$ up through dimension 7776.

Remark 5.10. In case $p = 2$, $n = 1$, let

$$q(2^s a) = \begin{cases} 2^s & \text{if } s = 0 \text{ or } s = 1 \text{ or } a = 1, \\ 2^s + 2^{s-1} & \text{otherwise} \end{cases}$$

for a odd.

Then Proposition 5.3(a) clearly still holds, and in joint work with D. C. Ravenel we will show that Theorem 1.1(a) and (b) i, ii hold too.

§6. INVARIANT IDEALS

Recall that $I \subset BP_*$ is *invariant* iff it is a subBP_*BP-comodule. Our goal in this section is to classify all invariant ideals of the form (I_n, x, y), $n \geq 0$, for p odd and $n \geq 2$ for $p = 2$. Computations of Ext groups are useful in this direction because of the following observation.

LEMMA 6.1. *If I is invariant and $x \in BP_*$, then (I, x) is invariant iff*

$$\bar{x} \in \mathrm{Ext}^{0,*}(BP_*, BP_*/I) \subset BP_*/I$$

where $\bar{x}$ is x mod I.

Clearly $(I, x) = (I, y)$ iff $x \equiv uy \bmod I$ for some unit u, so ideals of the form (I, x) are classified by $\mathrm{Ext}^{0,*}(BP_*, BP_*/I)$ modulo units.

An easy induction proves:

LEMMA 6.2. *Let $I \subset BP_*$ be any proper ideal containing I_n and minimally generated by $\{x_0, \ldots, x_k\}$ where $|x_0| \leq \cdots \leq |x_k|$. Then $k \geq n-1$ and $I = (p, v_1, \ldots, v_{n-1}, x_n, \ldots, x_k)$.*

Our primary interest is in invariant ideals containing I_n generated by $n+2$ elements and no fewer. By 6.2 they are of the form $(p, v_1, \ldots, v_{n-1}, x, y)$ with $|x| \leq |y|$. For dimensional reasons $(p, v_1, \ldots, v_{n-1}, x)$ is invariant; and by the above discussion,

$$\bar{x} \in \mathrm{Ext}^{0,*}(BP_*, BP_*/I_n) \simeq \mathbb{F}_p[v_n],$$

so we may take x to be v_n^t, for some $t > 0$. Thus the problem of classifying all invariant ideals

(I_n, x, y) is reduced to classifying invariant ideals of the form (I_n, v_n^t, y). Again, by the above discussion, the following computation will allow us to complete our classification.

PROPOSITION 6.3. *Let $p > 2$ and $n \geq 0$ or $p = 2$ and $n \geq 2$. Then* $\mathrm{Ext}^{0,*}(BP_*, BP_*/(I_n, v_n^t))$ *is the sub$\mathbb{F}_p[v_n]/(v_n^t)(\mathbb{Z}_{(p)}/(p^t)$ if $n = 0)$ module of $BP_*/(I_n, v_n^t)$ generated by* 1 *and $v_n^m \bar{c}_n(r)$ where $m = \max\{0, t - q_n(r)\}$, $r > 0$.*

Remark. As will be clear from our proof, these elements in fact generate summands. 1 generates a free $\mathbb{F}_p[v_n]/(v_n^t)(\mathbb{Z}_{(p)}/(p^t)$ if $n = 0)$ submodule, as does $\bar{c}_n(r)$ if $q_n(r) \geq t$. If $q_n(r) < t$ then $v_n^{t-q_n(r)}\bar{c}_n(r)$ generates a submodule of the form $\mathbb{F}_p[v_n]/(v_n^{q_n(r)})(\mathbb{Z}_{(p)}/(p^{q_0(r)})$ if $n = 0)$.

Proof. From the long exact sequence of

$$0 \to BP_*/I_n \xrightarrow{v_n^t} BP_*/I_n \to BP_*/(I_n, v_n^t) \to 0$$

we obtain a 'universal coefficient' short exact sequence

$$0 \to \mathrm{coker}\,(v_n^t|\mathrm{Ext}^{0,*}(BP_*, BP_*/I_n)) \to \mathrm{Ext}^{0,*}(BP_*, BP_*(I_n, v_n^t)) \to \ker\,(v_n^t|\mathrm{Ext}^{1,*}(BP_*, BP_*/I_n)) \to 0.$$

The first term provides the submodule generated by 1. By Theorem 1.1, the last term is generated as an $\mathbb{F}_p[v_n]/(v_n^t)(\mathbb{Z}_{(p)}/(p^t)$ if $n = 0)$ module by

$$\{v_n^k c_n(r) \colon k = \max\{0, q_n(r) - t\}, r > 0\}.$$

Now the result follows from the construction of $c_n(r)$; see Remark 5.9.

These Ext groups now figure in the following classification theorem.

THEOREM 6.4. *Let $n \geq 0$.*

(a) *All invariant ideals containing I_n and generated by $n + 2$ elements and no fewer are of the form*

$$I = (p, v_1, \ldots, v_{n-1}, v_n^t, y)$$

with $t > 0$, $|v_n^t| \leq |y|$, and

$$0 \neq \bar{y} \in \mathrm{Ext}^{0,*}(BP_*, BP_*/(I_n, v_n^t)) \subset BP_*/(I_n, v_n^t),$$

where $\bar{y}$ is y $\mathrm{mod}\,(I_n, v_n^t)$.

(b) *$I = I'$ iff $t = t'$ and $\bar{y} = u\bar{y}'$, u a unit.*

(c) *I is a regular ideal iff $y \not\equiv 0 \bmod I_{n+1}$.*

Proof. The first two parts are proved in discussion throughout the section. Their worth comes from the computation 6.3. Part (c) is straightforward.

The proof of Corollary 1.3 in the introduction follows from 6.3, 6.4, and the construction of the elements $\bar{c}_n(r)$.

See [10] for a more general discussion of invariant regular ideals.

REFERENCES

1. J. F. ADAMS: On the non-existence of elements of Hopf invariant one, *Ann. Math.* **72** (1960), 20–104.
2. J. F. ADAMS: On the groups J(X), IV, *Topology* **5** (1966), 21–71.
3. J. F. ADAMS: Lectures on generalized cohomology, in *Lecture Notes in Mathematics*, Vol. **99**, Springer, Berlin, 1969.
4. J. F. ADAMS: *Stable Homotopy and Generalized Homology*, University of Chicago Press, Chicago, 1974.
5. E. H. BROWN and F. P. PETERSON: A spectrum whose Z_p-cohomology is the algebra of reduced p-th powers, *Topology* **5** (1966), 149–154.
6. M. HAZEWINKEL: Constructing formal groups, *I.* Over $Z_{(p)}$-algebras, *Report of the Econometric Institute No.* 7119, Netherlands School of Economics, 1971.
7. D. C. JOHNSON, H. R. MILLER, W. S. WILSON and R. S. ZAHLER: Boundary homomorphisms in the generalized Adams spectral sequence and the nontriviality of infinitely many γ_t in stable homotopy, Proceedings for the August, 1974, Northwestern University homotopy theory conference, to appear as a monograph of the Mexican Mathematical Society.
8. D. C. JOHNSON and W. S. WILSON: *BP* operations and Morava's extraordinary K-theories, *Math. Zeit.* **144** (1975), 55–75.
9. P. S. LANDWEBER: Annihilator ideals and primitive elements in complex bordism, *Illinois Jour. Math.* **17** (1973), 273–284.
10. P. S. LANDWEBER: Invariant regular ideals in Brown–Peterson homology, *Duke J. Math.* **42** (1975), 499–505.
11. A. LIULEVICIUS: The factorization of cyclic reduced powers by secondary cohomology operations, *Mem. Am. Math. Soc.* **42** (1962).
12. H. R. MILLER: Princeton University thesis, 1974.

13. H. R. MILLER, D. C. RAVENEL and W. S. WILSON: Novikov's Ext^2 and the nontriviality of the gamma family, *Bull. Am. Math. Soc.* **81** (1975), 1073–1075.
14. H. R. MILLER and W. S. WILSON: On Novikov's Ext^1 modulo an invariant prime ideal, Proceedings for the August, 1974, Northwestern University homotopy theory conference, to appear as a monograph of the Mexican Mathematical Society.
15. J. MORAVA: The algebra $\mathrm{Ext}^*_{\mathbb{C}}(U, U)$ of Novikov (to appear).
16. J. MORAVA: Structure theorems for cobordism comodules (to appear).
17. M. R. F. MOREIRA: M.I.T. thesis, 1975.
18. S. P. NOVIKOV: The methods of algebraic topology from the view point of cobordism theories, *Math. U.S.S.R.-Izvetiia* **1** (1967), 827–913 (Izv. Akad. Nauk S.S.S.R., Seriia Matematicheskaia 31 (1967), 855–951).
19. D. QUILLEN: On the formal group laws of unoriented and complex cobordism theory, *Bull. Am. Math. Soc.* **75** (1969), 1293–1298.
20. D. C. RAVENEL: The structure of BP_*BP modulo an invariant prime ideal (to appear).
21. L. SMITH: On realizing complex cobordism modules, *Am. J. Math.* **92** (1970), 793–856.
22. H. TODA: On realizing exterior parts of the Steenrod algebra, *Topology* **10** (1971), 53–65.
23. E. THOMAS and R. S. ZAHLER: Generalized higher order cohomology operations and stable homotopy groups of spheres, *Adv. Math.* (to appear).
24. R. S. ZAHLER: The Adams–Novikov spectral sequence for spheres, *Ann. Math.* **96** (1972), 480–504.
25. R. S. ZAHLER: Fringe families in stable homotopy of spheres (to appear).
26. M. HAZEWINKEL: A universal formal group law and complex cobordism, *Bull. Am. Math. Soc.* **81** (1975), 930–933.

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