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PROJECTIVE DIMENSION AND BROWN–PETERSON HOMOLOGY

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$BP_*(X)$ is the reduced Brown–Peterson homology of a finite complex X for a fixed prime p. We study a sequence of homology theories

$$BP_*(X) \to \cdots \to BP\langle n+1\rangle_*(X) \xrightarrow{\rho(n,n+1)} BP\langle n\rangle_*(X) \xrightarrow{\rho(n-1,n)} BP\langle n-1\rangle_*(X)$$
$$\to \cdots \to BP\langle 0\rangle_*(X) \xrightarrow{\rho(-1,0)} BP\langle -1\rangle_*(X).$$

Our main result states that there is an n such that..., $\rho(n,n+1)$, $\rho(n-1,n)$ are all epimorphisms; each of the remaining homomorphism fails to be onto; and n is the projective dimension of $BP_*(X)$ as a module over the coefficient ring BP_*.

The first two sections are introductory in nature while the third contains the core result. The reader should be pleased to know that §4–6 are independent of each other.

§1. INTRODUCTION

Let $MU_*(\ )$ be the reduced complex bordism theory, the homology theory associated to the unitary Thom spectrum MU. (This is also denoted by $\tilde{\Omega}_*{}^U(\ )$ and $\tilde{\mathscr{U}}_*(\ )$ in the literature.) This is a multiplicative homology theory with a particularly nice coefficient ring, $MU_* \cong Z[y_2, y_4, \ldots, y_{2n}, \ldots]$ (dimension $y_{2n} = 2n$). In general, $MU_*(X)$ is difficult to compute; so it was natural for Conner and Smith to consider the structure of $MU_*(X)$ as a module over MU_*[13]. Define hom $\dim_{MU_*} MU_*(X)$ to be the projective dimension of $MU_*(X)$ as a MU_* module, i.e. the minimal possible length of a projective MU_* resolution. Conner and Smith proved that for a finite complex X, the following statements are equivalent.

(i) hom $\dim_{MU_*} MU_*(X) \leqslant 1$.

(ii) The Thom homomorphism $MU_*(X) \to H_*(X; Z)$ is epic.

(iii) The Thom homomorphism induces an isomorphism $Z \otimes_{MU_*} MU_*(X) \to H_*(X; Z)$. [13, 3.11] $k_*(\ )$ is the connective K homology theory associated to the connective BU spectrum; it is a multiplicative theory with coefficient ring, $k_*(S^0) \cong Z[t]$ (dimension $t = 2$). Again, we have three statements for a finite complex.

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(i′) hom $\dim_{MU_*} MU_*(X) \leqslant 2$.

(ii′) The Thom homomorphism $MU_*(X) \to k_*(X)$ is epic.

(iii′) The Thom homomorphism induces an isomorphism $Z[t] \otimes_{MU_*} MU_*(X) \to k_*(X)$.

Conner and Smith showed (i′) implies (ii′) and (iii′) which are equivalent [13, 11.2]. Johnson and Smith [19] then completed the analogy with the integral homology example by showing that (ii′) implies (i′). Their key step was in demonstrating that the statement,

(iv) Multiplication by t in $k_*(X)$ is monic

is equivalent to hom $\dim_{MU_*} MU_*(X) \leqslant 1$.

Baas [7], using ideas of Sullivan, constructed a tower of homology theories

$$MU_*(\ \ ) \to \cdots \to MU\langle n+1\rangle_*(\ \ ) \to MU\langle n\rangle_*(\ \ ) \to \cdots \to MU\langle 0\rangle_*(\ \ ) = H_*(\ \ ;Z)$$

such that $MU\langle n\rangle_*(S^0) \cong Z[y_2, \ldots, y_{2n}]$. A natural question [6] was whether there exist analogs of the theorems of Conner–Smith and Johnson–Smith for these intermediate theories We show in (4.24) that this is not possible without localization.

Recall $Z_{(p)}$ is the integers localized at the prime p; i.e. the subring of rational numbers represented by fractions a/b where $gcd\ (b, p) = 1$. (Topologists frequently denote $Z_{(p)}$ by Q_p.) Quillen [3; 27] described a multiplicative splitting of $MU_*(X) \otimes Z_{(p)}$ into a direct sum of shifted copies of $BP_*(X)$, the Brown–Peterson homology of X [9]. $BP_*(X)$ is a module over the coefficient ring $BP_* = BP_*(S^0) \cong Z_{(p)}\ [x_1, x_2, \ldots, x_n, \ldots]$ where the dimension of x_n is $2p^n - 2$. We define x_0 to be the fixed prime p. By Quillen's splitting, hom $\dim_{MU_* \otimes Z(p)} MU_*(X) \otimes Z_{(p)} =$ hom $\dim_{BP_*} BP_*(X)$. It is elementary that hom $\dim_{MU_*} MU_*(X)$ is the maximum of the hom $\dim_{BP_*} BP_*(X)$'s for all primes.

For BP, theories with the properties optimists had hoped for do exist. Using Baas's approach to Sullivan's theory of manifolds with singularities [7], BP_* module theories $BP\langle n\rangle_*(\ \ )$ with $BP\langle n\rangle_* = BP\langle n\rangle_*(S^0) \cong Z_{(p)}\ [x_1, \ldots, x_n]$ were constructed in [38]. These also fit into a tower:

$$BP_*(\ \ ) \to \cdots BP\langle n+1\rangle_*(\ \ ) \xrightarrow{\cdot x_n} BP\langle n\rangle_*(\ \ ) \to$$
$$\cdots \to BP\langle 1\rangle_*(\ \ ) \xrightarrow{\rho(0,1)} BP\langle 0\rangle_*(\ \ ) \xrightarrow{\rho(-1,0)} BP\langle -1\rangle_*(\ \ ).$$

They come equipped with exact sequences.

$$\begin{tikzcd} BP\langle n\rangle_*(X) \arrow[r, "\cdot x_n"] & BP\langle n\rangle_*(X) \arrow[r, "\rho(n-1,n)"] & BP\langle n-1\rangle_*(X) \arrow[ll, bend left, "\Delta_n"] \end{tikzcd}$$

The interlocking nature of these theories makes them a powerful tool when they are all used at once. A few of them are already familiar: $BP\langle 0\rangle_*(\ \ )$ is homology with $Z_{(p)}$ coefficients; we define $BP\langle -1\rangle_*(\ \ ) = H_*(\ \ ; Z_p)$, mod p homology; and in (2.7), we prove $BP\langle 1\rangle_*(\ \ )$ is the summand of connective K-theory localized at the prime p [1; 5]. Our main theorem then reads:

THEOREM 1.1. *For X a finite complex, the following conditions are equivalent.*

(i) hom $\dim_{BP_*} BP_*(X) \leqslant n+1$.

(ii) *$BP_*(X) \to BP\langle n\rangle_*(X)$ is epic.*

(iii) *The exact sequence*

$$\begin{tikzcd} \to BP\langle n+1\rangle_*(X) \arrow[r, "\cdot x_{n+1}"] & BP\langle n+1\rangle_*(X) \arrow[r, "\rho(n, n+1)"] & BP\langle n\rangle_*(X) \arrow[ll, bend left, "\Delta_{n+1}"] \end{tikzcd}$$

is short exact in that $\Delta_{n+1} \equiv 0$.

(iv) *There is an isomorphism*

$$BP_*(X) \otimes_{BP_*} BP\langle n\rangle_* \to BP\langle n\rangle_*(X).$$

(v) $\mathrm{Tor}_1^{BP_*}(BP_*(X); BP\langle n\rangle_*) = 0.$

(vi) *There is an isomorphism*

$$BP\langle n+1\rangle_*(X) \otimes_{Z_{(p)}[x_{n+1}]} Z_{(p)} \to BP\langle n\rangle_*(X).$$

(vii) $\mathrm{Tor}_{1,*}^{Z_{(p)}[x_{n+1}]}(BP\langle n+1\rangle_*(X); Z_{(p)}) = 0.$

(viii) hom dim $_{MU_*\otimes Z_{(p)}} MU_*(X) \otimes Z_{(p)} \leqslant n+1.$

The proof is given in Propositions 3.8, 3.13, and 5.12. It is based on the "splitting theorem" of [38] which states that $BP^s(X) \to BP\langle n\rangle^s(X)$ is $Z_{(p)}$-split epic for $s < 2(p^n + p^{n-1} + \cdots + p + 1)$. This is an analog of Conner and Floyd's result that $K^0(X)$ is a direct summand of $MU^0(X)$ [10]. Although we refer often to the fundamental work of Conner and Smith [13], the completeness of our theory allows for elementary proofs which should be easily understood without prior knowledge of Conner–Smith theory.

A corollary to our main theorem and the splitting theorem is:

COROLLARY 4.4. *If X is a q dimensional finite complex and if $q < p^n + \cdots + p + 1$, then* hom dim$_{BP_*} BP_*(X) \leqslant n$.

This seems to be the first upper bound on the projective dimension of BP_* (or complex bordism) modules. The theorem of Adams and Conner–Smith [1; 13] that hom dim$_{MU_*}$ $MU_*(X)$ is finite for a finite complex is an immediate consequence of (4.4).

How does one compute lower bounds of this invariant? Conner and Smith gave a lower estimate of hom dim$_{MU_*} MU_*(X)$ which depends on the structure of possible annihilator ideals of elements of $MU_*(X)$. We replace this with the following ideal annihilator test.

PROPOSITION 4.6. *Let X be a finite complex. If $x_n \cdot y = 0$* for $0 \neq y \in BP_*(X)$, *then* hom dim$_{BP_*} BP_*(X) \geqslant n+1$.

Also Conner and Smith gave a lower bound of hom dim$_{MU_*} MU_*(X)$ involving the non-triviality of certain Steenrod operators [11]. They noted that their new test was not as effective as their annihilator ideal estimate for a stable complex

$$X(\eta, \nu) = S^0 \cup_2 e^1 \cup_{\bar\eta} e^3 \cup_{\bar\nu} e^7$$

which they constructed using the first two elements of Hopf invariant one [11; 12; 14]. This motivated the following improvement of their theorem. Incidentally, our new techniques applied to the study of $X(\eta, \nu)$ show that this complex is a counterexample to the conjecture that the Conner–Smith program could be generalized directly [6].

THEOREM 4.8. *Let X be a finite complex and let the* mod p *Steenrod algebra, $\mathscr{A}(p)$, operate on $H_*(X; Z_p)$ in the obvious way. Let (Q_0) be the two-sided ideal of $\mathscr{A}(p)$ generated by the Bockstein. Suppose there is a selection of operations $b_1, b_2, \ldots, b_k$ in (Q_0) such that the composition $b_1, b_2 \ldots b_k$ acts nontrivially on the image of $BP\langle n\rangle_*(X) \to H_*(X;\ Z_p)$, then* hom $\dim_{BP_*} BP_*(X) \geqslant n + k + 1$ *for* $n = -1, 0, 1, 2, \ldots$.

Conner and Smith use two spectral sequences in their work: one of the universal coefficient type and one of the Kunneth type. We observe that these can be combined to give a spectral sequence

$$E^2_{*,*}(X;\ Y) \cong \mathrm{Tor}^{BP_*}_{*,*}(BP_*(X); BP\langle n\rangle_*(Y)) \Rightarrow BP\langle n\rangle_*(X \wedge Y).$$

Unlike Conner and Smith, our proofs do not use new spectral sequences. For completeness, however, we have included a discussion of some of the spectral sequences which arise from the Brown–Peterson tower of spectra.

Finally, we begin a study of spherical classes in $BP_*(X)$. We raise some questions which do not seem to be tractable by our present methods. However, we prove the following theorem, the $n = 1$ case of which is the Stong–Hattori theorem [4; 15; 16; 17; 28; 34].

THEOREM 6.1. *If X is a finite complex with* hom $\dim_{BP_*} BP_*(X) \leqslant n$, *then the Hurewicz homomorphism*

$$BP_*(X) \to BP\langle n\rangle_*(BP \wedge X)$$

is a monomorphism.

Conventions. We shall work in Boardman's stable category. In particular, we shall operate with two full subcategories. The first has as objects, pointed finite complexes; thus when we speak of a stable map of finite complexes, $f\colon X \to Y$, we mean a stable homotopy class represented by a base-point preserving, continuous function between suspensions of the complexes, e.g. $f\colon S^m X \to S^n Y$. By a "connected" spectrum, we mean one which is n-connected for some (possibly negative) integer n. The second full subcategory of Boardman's which we shall work with is the one whose objects are connected CW spectra with $Z_{(p)}$ homology of finite type. All homology and cohomology theories are reduced theories. This is no restriction, for we can add a discrete base point to a complex X to form X^+ knowing that the reduced homology of X^+ is the unreduced homology of X. Again, we remind topologists that $Z_{(p)}$ is the integers localized at p (i.e. Q_p) and we warn algebraists that Z_p is the integers modulo p (i.e. Z/pZ).

§2. THE *BP* TOWER

In [38], a tower of BP module spectra was constructed using Sullivan's theory of manifolds with singularities [7]:

$$\begin{tikzcd}
 & & & BP = BP\langle\infty\rangle \arrow[dll, "\rho(n,\,\infty)"'] \arrow[dl, "\rho(n-1,\,\infty)"] \arrow[dr, "\rho(0,\,\infty)"'] \arrow[drr, "\rho(-1,\,\infty)"] & & \\
\cdots \arrow[r] & BP\langle n\rangle \arrow[r, "\rho(n-1,\,n)"] & BP\langle n-1\rangle \arrow[r] & \cdots \arrow[r] & BP\langle 0\rangle \arrow[r, "\rho(-1,\,0)"] & BP\langle -1\rangle \\
 & & & & = KZ_{(p)} & = KZ_p
\end{tikzcd}$$

The module structure gives a natural transformation

$$\widetilde{\rho(n, \infty)}: BP_*(X) \otimes BP\langle n\rangle_*(Y) \to BP\langle n\rangle_*(X \wedge Y).$$

Letting $X = S^0$, $BP\langle n\rangle_*(Y)$ is a module over $BP_*(S^0) = BP_* = Z_{(p)}[X_1, x_2, \ldots]$ where the degree of x_k is $2p^k - 2$. It is convenient to define x_0 to be p, the fixed prime in consideration. Multiplication by x_n gives us an exact sequence (2.1) which we shall call the BPBockstein exact sequence

$$\begin{tikzcd} BP\langle n\rangle_*(X) \arrow[r, "\cdot x_n"] & BP\langle n\rangle_*(X) \arrow[r, "\rho(n-1, n)"] & BP\langle n-1\rangle_*(X) \arrow[ll, bend left, "\Delta_n"] \end{tikzcd} \tag{2.1}$$

The degree of $\rho(n-1, n)$ is zero and the degree of Δ_n is $-2p^n + 1$. We have for positive n, $BP\langle n\rangle_* = BP\langle n\rangle_*(S^0) = Z_{(p)}[x_1, \ldots, x_n]$. $BP\langle 0\rangle_* = Z_{(p)}$ and $BP\langle -1\rangle_* = Z_p$. For (-1)-connected spectra (including "honest" complexes), $\rho(n, \infty): BP_s(X) \to BP\langle n\rangle_s(X)$ is an isomorphism for $s < 2p^{n+1} - 2$.

The fountainhead of our work is the following result of the second-named author.

THEOREM 2.2 [38, Corollary 5.6]. *The homomorphisms* $\rho(n, \infty): BP^s(X) \to BP\langle n\rangle^s(X)$ and $\rho(n, n+1): BP\langle n+1\rangle^s(X) \to BP\langle n\rangle^s(X)$ *are epic for* $s \leqslant 2(p^n + \cdots + p + 1)$ *and* $Z_{(p)}$ *– split epic for* $s < 2(p^n + \cdots + p + 1)$.

For most of our purposes, we need to transform this into a statement about homology theories. In this paper, we refer to Theorem 2.2 and its corollary (2.3) as the splitting theorem.

COROLLARY 2.3 *For X a finite complex, the homomorphism* $\rho(n, \infty): BP_*(X) \to BP\langle n\rangle_*(X)$ *and* $\rho(n, n+1): BP\langle n+1\rangle_*(X) \to BP\langle n\rangle_*(X)$ *are* $Z_{(p)}$ *– split epic for all but finitely many dimensions.*

Proof. Let DX be a Spanier–Whitehead dual for X. For some large k we have duality (vertical) isomorphisms in (2.4).

$$\begin{tikzcd} BP^s(DX) \arrow[r, "\rho(n+1, \infty)"] & BP\langle n+1\rangle^s(DX) \arrow[r, "\rho(n, n+1)"] & BP\langle n\rangle^s(DX) \\ BP_{k-s}(X) \arrow[r, "\rho(n+1, \infty)"] \arrow[u, "\cong"] & BP\langle n+1\rangle_{k-s}(X) \arrow[r, "\rho(n, n+1)"] \arrow[u, "\cong"] & BP\langle n\rangle_{k-s}(X). \arrow[u, "\cong"] \end{tikzcd} \tag{2.4}$$

For low s, the top horizontal homomorphisms are $Z_{(p)}$ – split epic by Theorem 2.2.

Let $\mathscr{A}(p)$ be the mod p Steenrod algebra and let Q_i be the Milnor elements [23].

LEMMA 2.5 [38, Proposition 1.7]. $H^*(BP\langle n\rangle, Z_p) = \mathscr{A}(p)/\mathscr{A}(p)(Q_0, \ldots, Q_n)$ *and if* $\Delta_n: BP\langle n-1\rangle \to S^{2p^n-1} BP\langle n\rangle$ *is the map of spectra induced by* (2.1), *then* $\Delta_n^*(1) = \lambda Q_n$, $0 \neq \lambda \in Z_p$.

Remark 2.6. Smith [30] and Toda [35] have studied the existence of stable complexes $V(n)$ satisfying the property that $BP_*(V(n)) \cong BP_*/(p, x_1, \ldots, x_n)$ or, equivalently, $H^*(V(n); Z_p) \cong E[Q_0, Q_1, \ldots, Q_n]$. A third equivalent property is that $\pi_*(BP\langle n\rangle \wedge V(n)) = BP\langle n\rangle_*(V(n)) \cong Z_p$. So we can think of the $V(n)$ existence problem as a factorization problem. For a given prime p, for which n can we factor the Z_p Eilenberg–MacLane spectrum as $KZ_p \simeq BP\langle n\rangle \wedge V(n)$?

The theories $BP\langle n\rangle_*(\ )$ are new and unfamiliar; however, $BP\langle 1\rangle_*(\ )$ does have a familiar interpretation. It is a folk theorem that the spectrum for connective K-theory splits into $p-1$ many shifted copies of a ring spectrum G when localized at p. $G_*(S^0) = G = Z_{(p)}[y]$ where the degree of y is $2p-2$. We shall show that G and $BP\langle 1\rangle$ are homotopy equivalent. We need this identification so that the results of Section 6 give the Stong–Hattori theorem. We shall not prove the whole of the folk theorem, just enough to get a nice grip on G. Let $bu_{(p)}$ denote the spectrum for connective K-theory localized at p.

PROPOSITION 2.7

(i). *There is a ring spectrum G and a map of ring spectra $G \to bu_{(p)}$.*

(ii) *$G_*(-) \to bu_{(p)*}(-)$ maps isomorphically to a direct summand.*

(iii) *There is a homotopy equivalence $G \simeq BP\langle 1\rangle$.*

Proof. We need the following results from the Adams, Anderson–Meiselman splitting of K-theory [1; 5]. K-theory localized at p has a natural, multiplicative, representable direct summand, $K_0(-)$. (Notational warning: the subscript refers to the zero coset in $Z_p = Z/pZ$ not to a dimension. See [1, pp. 90–92].) There is a periodicity isomorphism (2.8) given by the external product with a generator of $K_0(S^{2p-2})$

$$\phi\colon K_0(X) \xrightarrow{\cong} K_0(S^{2p-2} \wedge X). \tag{2.8}$$

$$K_0(S^n) = \begin{cases} Z_{(p)} & \text{if } n = 2i(p-1) \\ 0 & \text{otherwise.} \end{cases} \tag{2.9}$$

Let $G(0)$ be the classifying space for the representable functor $K_0(-)$. The natural direct summand statement says that there is an equivalence $Z_{(p)} \times BU_{(p)} \cong G(0) \times Y$ for some Y. Let $G(n)$ be the $2n(p-1)-1$ connected covering of $G(0)$.

CLAIM 2.10. *There is an equivalence $\Omega^{2(p-1)}G(n+1) \simeq G(n)$.*

Proof of claim. $\Omega^{2(p-1)}G(0) \simeq G(0)$ by (2.8). $G(0) \simeq G(1) \times Z_{(p)}$; so $\Omega^{2(p-1)}G(1) \simeq G(0)$. By (2.9) and the definition of the $G(n)$'s we have a fibration, $G(n+1) \to G(n) \to K(Z_{(p)}, 2n(p-1))$. Applying the functor $\Omega^{2(p-1)}$, we get. $G(n) = \Omega^{2(p-1)}G(n+1) \to G(n-1) \to K(Z_{(p)}, 2(n-1)(p-1))$ by induction.

The property (2.10) allows the spaces $\{G(n)\}$ to define an omega spectrum G. (Note that the $2n(p-1)$ st space of G is $G(n)$.) Similarly, we can represent $bu_{(p)} = \{bu_{(p)}(n)\}$, where $bu_{(p)}(n)$ is the $2n(p-1)-1$ st connected covering of $Z_{(p)} \times BU_{(p)}$. (Compare [2].) Again $\Omega^{2(p-1)}bu_{(p)}(n) \simeq bu_{(p)}(n-1)$ by the same proof. Now by our construction and the splitting of $Z_{(p)} \times BU_{(p)}$, we get $bu_{(p)}(n) \simeq G(n) \times W(n)$, some $W(n)$, giving us (ii).

$$\begin{tikzcd}
G(m) \wedge G(n) \arrow[rr, dashed] \arrow[dr] \arrow[dd] & & G(m+n) \arrow[dr] \arrow[dd] & \\
& bu_{(p)}(m) \wedge bu_{(p)}(n) \arrow[rr, dashed] \arrow[dd] & & bu_{(p)}(m+n) \arrow[dd] \\
G(0) \wedge G(0) \arrow[rr] \arrow[dr] & & G(0) \arrow[dr] & \\
& bu_{(p)}(0) \wedge bu_{(p)}(0) \arrow[rr] & & bu_{(p)}(0)
\end{tikzcd} \tag{2.11}$$

To prove (i), we need to define the dashed arrows in (2.11) to make the diagram homotopy commute. The bottom square of (2.11) is given by the fact that $K_0(-)$ is a multiplicative summand of K-theory. $G(m) \wedge G(n)$ is $2(n+m)(p-1)-1$ connected so the composition $G(m) \wedge G(n) \to G(0) \wedge G(0) \to G(0)$ lifts uniquely to $G(m+n)$. The maps for $bu_{(p)}$ are defined similarly. The top square homotopy commutes since both compositions are the unique lifts of the composition,

$$G(m) \wedge G(n) \to G(0) \wedge G(0) \to bu_{(p)}(0) \wedge bu_{(p)}(0) \to bu_{(p)}(0).$$

We can now move on to (iii). Let $BP\langle 1\rangle(n)$ be the classifying space for the functor $BP\langle 1\rangle^{2n(p-1)}(-)$. From the exact sequence (2.1) and the fact that $BP\langle 0\rangle = KZ_{(p)}$, we see that $\Omega^{2(p-1)}BP\langle 1\rangle(n+1) \simeq BP\langle 1\rangle(n)$ and that $BP\langle 1\rangle(n)$ may be taken to be the $2n(p-1)-1$ connective covering of $BP\langle 1\rangle(0)$. In [25; 38], $BP\langle 1\rangle(0)$ and $G(0)$ are shown to be homotopy equivalent. Thus $BP\langle 1\rangle(n) \simeq G(n)$ as spaces and $BP\langle 1\rangle \simeq G$ as spectra.

We want our map $\rho(1,\infty)$: $BP \to BP\langle 1\rangle$ to be equivalent to the standard map ρ: $BP \to G$. The next proposition will suffice.

PROPOSITION 2.12. *There exists a homotopy equivalence f making diagram* (2.13) *homotopy commute.*

$$\begin{tikzcd} BP \arrow[r, "\rho"] \arrow[d, "f"] & G \arrow[d, "g", "\cong"'] \\ BP \arrow[r, "\rho(1,\infty)"] & BP\langle 1\rangle \end{tikzcd} \tag{2.13}$$

Proof. $\rho(1,\infty)$: $E_2^{*,*}(BP) \cong H^*(BP; BP^*) \to \tilde{E}_2^{*,*}(BP) \cong H^*(BP; BP\langle 1\rangle^*)$ is epic. The two spectral sequences $E_2^{*,*}(BP) \Rightarrow BP^*(BP)$ and $\tilde{E}_2^{*,*}(BP) \Rightarrow BP\langle 1\rangle^*(BP)$ collapse and so $\rho(1,\infty)$: $BP^*(BP) \to BP\langle 1\rangle^*(BP)$ is onto. Thus there is an f such that $[\rho(1,\infty)\circ f] = \rho(1,\infty)$ $[f] = [g\circ\rho]$. Now f must induce an isomorphism on $\pi_0(BP)$ and so f^*: $H^*(BP; Z_p) \to H^*(BP; Z_p) = \mathscr{A}(p)/(Q_0)$ is an isomorphism. Since BP is a localized spectrum, this implies f is a homotopy equivalence.

For certain constructions, we need the module $BP_*(X)$ to be finitely generated over BP_* when X is a finite complex. Originally, this was proved by Novikov and Smith independently [24; 32] using the properties of coherent rings. It is of interest that this work is independent of coherence; so we shall point out a more recent proof. Quillen [26] showed that for a connected complex of dimension k the MU_* generators of $MU_*(X)$ are in dimensions $\leqslant 2k$. $MU_*(X)$ is a finitely generated abelian group in dimensions $\leqslant 2k$; so Quillen's theorem gives the finite generation result. A new proof of Quillen's theorem using (2.2) can be found in [38]. The same proof shows that the $Z_{(p)}[X_1, \ldots, X_n] \subset BP_*$ generators of $BP\langle n\rangle_*(X)$ are in dimensions $\leqslant 2k$. For $BP\langle 1\rangle_*(X)$, the generators are known to be in dimensions $\leqslant k$ [33].

Question 2.14. Is it possible to give a more precise statement about the generators which depends on hom $\dim_{BP_*} BP_*(X)$?

§3. TORSION AND PROJECTIVE DIMENSION

In this section, we prove perhaps the most interesting of the equivalences of the main theorem. We begin with a review of standard facts about localization and torsion (e.g. see [8]).

Fix a graded, integral domain with unit, R_*. It is convenient to assume that R_* is concentrated in even dimensions. Let $T \subseteq R_* \backslash \{0\}$ be a set containing 1 which is closed under multiplication (such is called a multiplicative set). Given a graded R_* module M_*, we define $T^{-1}M_*$ to be the set of equivalence classes of "fractions" m/t, $m \in M_*$ and $t \in T$, under the relation induced by:

$$m/t \sim m'/t' \Leftrightarrow \exists\ t_0 \in T \ni t_0 t' m = t_0 t m'.$$

Abusively, we let m/t stand for the equivalence class of m/t. $T^{-1}M_*$ is an R_* module by the rules for addition and multiplication of fractions recalled from elementary school. A homomorphism of graded R_* modules (of arbitrary degree), $f\colon M_* \to N_*$; induces a homomorphism $T^{-1}f\colon T^{-1}M_* \to T^{-1}N_*$ by the rule $(T^{-1}f)(m/t) = f(m)/t$. We see that T^{-1} is an endofunctor of the category of graded R_* modules and homomorphisms of arbitrary degree. There is a canonical homomorphism of graded R_* modules $\tau\colon M_* \to T^{-1}M_*$ defined by $\tau(m) = m/1$. We call it the localization homomorphism; it induces a natural equivalence of functors, $\tilde{\tau}\colon T^{-1}R_* \otimes_{R_*} M_* \to T^{-1}M_*$ defined on generators by $\tilde{\tau}(r/t \otimes m) = (rm)/t$. T^{-1} is an exact functor and $T^{-1}R_*$ is a flat R_* module. (Right exactness follows from the equivalence $\tilde{\tau}$; left exactness is checked easily.) There is a category of R_* module homology theories: the objects are homology theories $E_*(-)$ with nice natural pairings $R_* \otimes E_*(-) \to E_*(-)$ and the morphisms are natural transformations of homology theories (of arbitrary degree) respecting these natural pairings. The exactness of T^{-1} means that it induces an endofunctor of this category of R_* module homology theories. We call this induced functor T^{-1} also.

Where there is localization, there is torsion. Fix a multiplicative subset $T \subseteq R_* \backslash \{0\}$. An element of the kernel of the localization homomorphism $\tau\colon M_* \to T^{-1}M_*$ is said to be a T-torsion element of the graded R_* module M_*. If $\tau(M_*) = 0$ (or equivalently, $T^{-1}M_* = 0$), we say M_* is a T-torsion module. On the other hand, if τ is a monomorphism, we say M_* is T-torsion free. If $f\colon M_* \to N_*$ is a homomorphism of graded R_* modules, f is monic modulo T torsion (epic modulo T torsion; zero modulo T torsion) provided that $T^{-1}f$ is a monomorphism (epimorphism; zero morphism).

A motivating example concerns complex connective K-theory, $k_*(\ \ )$, which is a module homology theory over the coefficient ring $k_*(S^0) \cong Z[t]$. $T = \{1, t, t^2, \cdots\} \subseteq Z[t] \backslash \{0\}$ is a multiplicative set. $T^{-1}k_*(X)$ may be identified with $K_*(X)$, complex K-homology theory (Notice that localization unconnects a connected homology theory when positive dimensional elements are present in T.) We now interpret the key result of [19] as saying: $k_*(X)$ is T-torsion free if and only if the projective dimension of $MU_*(X)$ as a MU_* module is at most one.

Our attention will center on the case when $R_* = BP_* \cong Z_{(p)}[x_1, x_2, \cdots]$. Define T_n to be the multiplicative subset $\{1, x_n, x_n^{\,2}, \ldots\} \subseteq BP_* \backslash \{0\}$. As a convention, we let x_0 denote p so that $T_0 = \{1, p, p^2, \cdots\}$ and so that T_0^{-1} is equivalent to the functor $- \otimes Q$. Note that an element y of a BP_* module M_* is a T_n torsion element if and only if $x_n{}^s y = 0$ for some power s. A BP_* module M_* is T_n torsion free if and only if multiplication by x_n in M_* is a monomorphism.

LEMMA 3.1. *Let φ_X: $E_*(X)$ be a natural transformation of R_* module homology theories defined on Boardman's stable category* [36] *and let $T \subseteq R_* \backslash \{0\}$ be a multiplicative subset. If φ_X is monic modulo T torsion (epic modulo T torsion; modulo T torsion) for all finite CW complexes, then φ_X is monic modulo T torsion (epic modulo T torsion; zero modulo T torsion) for all CW spectra in Boardman's category.*

Proof. $E_*(X) \cong \varinjlim E_*(X_\alpha)$ and $F_*(X) \cong \varinjlim F_*(X_\alpha)$ where the limits are taken over all finite subspectra of the spectrum X [36, 12.9]. $\varinjlim$ is an exact functor which commutes with T^{-1}.

PROPOSITION 3.2. *Let X be a connected CW spectrum and let $0 \leqslant k \leqslant n$. In the BP–Bockstein sequence* (2.1), *Δ_n is zero modulo T_k torsion (i.e. $T_k^{-1}\,\Delta_n = 0$).*

Proof. By Lemma 3.1, it suffices to prove the proposition for the case when X is finite. For $k = 0$, the result follows the fact that $\Delta_n = 0$ when $X = S^0$ and from the usual Serre modulo-torsion theory. For $k = n$, it is an immediate consequence of the exactness of the sequence (2.1). Now suppose $1 \leqslant k \leqslant n - 1$ and suppose $(T_k^{-1}\Delta_n)(y/x_k{}^s) \neq 0$ for some element $y/x_k{}^s \in T_k^{-1}BP\langle n-1\rangle_*(X)$ where X is a finite complex. Multiplication by x_k is an isomorphism in $T_k^{-1}BP\langle n\rangle_*(X)$; so $0 \neq (x_k{}^t)(T_k^{-1}\Delta_n)(y/x_k^s) = (T_k^{-1}\Delta_n)(x_k{}^t y/x_k^s) =)T_k^{-1}\Delta_n)(x_k^{t-s}y/1)$ for all non-negative t. This implies Δ_n is non-zero for infinitely many dimensions contradicting the splitting theorem (2.3). So $T_k^{-1}\Delta_n \equiv 0$.

PROPOSITION 3.3. *Let X be a connected CW spectrum and let $0 \leqslant k \leqslant n$. If $BP\langle n\rangle_*(X)$ is T_k torsion free, then $BP\langle n+1\rangle_*(X)$ is also T_k torsion free.*

Proof. Let τ' be the composition $BP\langle n\rangle_*(X) \to BP\langle n\rangle_*(X) \xrightarrow{\tau} T_k^{-1}BP\langle n\rangle_*(X)$ where τ is the localization homomorphism which is a monomorphism by the hypothesis. Since $T_k^{-1}\Delta_{n+1} \equiv 0$ (3.2), the bottom row in diagram (3.4) is short exact. The top row is exact and the diagram (of abelian groups) commutes.

$$\begin{tikzcd}
BP\langle n\rangle_{s+1} \arrow[r, "\Delta_{n+1}"] \arrow[d, "\tau''"] & BP\langle n+1\rangle_{s-2p^{n+1}+2}(X) \arrow[r, "\cdot x_{n+1}"] \arrow[d, "\tau_{s-2p^{n+1}+2}"] & BP\langle n+1\rangle_s(X) \arrow[r, "\rho(n,n+1)"] \arrow[d, "\tau_s"] & BP\langle n\rangle_s(X) \arrow[d, "\tau'"] \\
0 \arrow[r] & T_k^{-1}BP\langle n+1\rangle_*(X) \arrow[r] & T_k^{-1}BP\langle n+1\rangle_*(X) \arrow[r] & T_k^{-1}BP\langle n\rangle_*(X)
\end{tikzcd} \tag{3.4}$$

By an induction on dimension, $\tau_{s-2p^{n+1}+2}$ is monic. (We need the hypothesis that X is connected to start induction.) By the strong "five" lemma, τ_s is monic. This being true for all s, the localization homomorphism τ: $BP\langle n+1\rangle_*(X) \to T_k^{-1}BP\langle n+1\rangle_*(X)$ is then a monomorphism.

COROLLARY 3.5. *Let X be a connected CW spectrum and let $0 \leqslant k \leqslant n \leqslant m$. If $BP\langle n\rangle_*(X)$ is T_k torsion free, then $BP\,\langle m\rangle_*(X)$ and $BP_*(X)$ are T_k torsion free.*

Proof. Use the obvious induction to show $BP\langle m\rangle_*(X)$ is T_k torsion free. Observe that $BP_s(X) = BP\langle n\rangle_s(X)$ for some large m.

COROLLARY 3.6. *Let X be a connected CW spectrum. If $BP\langle n\rangle_*(X)$ is T_n torsion free, then $BP\langle n+1\rangle_*(X)$ is T_{n+1} torsion free. (If $\Delta_n \equiv 0$, then $\Delta_{n+1} \equiv 0$ or if $\rho(n-1, n)$ is epic, then $\rho(n, n+1)$(is epic.)*

Proof. $BP\langle n+1\rangle_*(X)$ is T_n torsion free (3.3); so $\Delta_{n+1} \equiv 0$ in the BP Bockstein sequence (3.2) implying multiplication by x_{n+1} is monic in $BP\langle n+1\rangle_*(X)$.

COROLLARY 3.7. *Let X be a connected CW spectrum and let $0 \leqslant k \leqslant n$. $\rho(n, \infty)$: $BP_*(X) \to BP\langle n\rangle_*(X)$ is epic modulo T_k torsion. Furthermore, if $BP\langle n+1\rangle_*(X)$ is T_k torsion free, then $\rho(n, \infty)$ is epic.*

Proof. Implicit in the splitting theorem (2.3) is the fact that for a finite complex X, there is an integer m such that $\rho(m, \infty)$: $BP_*(X) \to BP\langle m\rangle_*(X)$ is epic (namely an m such that a Spanier–Whitehead dual of X has dimension less than $2(p^m + \cdots + p + 1)$.) So the first statement for a finite complex follows from the fact that $\rho(m-1, m), \ldots, \rho(n, n+1)$ are epic modulo T_k torsion (3.2). This is generalized to CW spectra by (3.1). If $BP\langle n + 1_*(X)$ is T_k torsion free, $\Delta_{n+1} \equiv 0$ (3.2) and $\rho(n, n+1)$ is epic. By iteration of Corollary 3.6, $\rho(n, n+1)$, $\rho(n+1, n+2), \cdots$ are epic.

PROPOSITION 3.8. *For a connected CW spectrum X, the following three conditions are equivalent:*

(i) $\rho(n, \infty)$: $BP_*(X) \to BP\langle n\rangle_*(X)$ *is epic;*
(ii) $\rho(n+1, n)$: $BP\langle n+1\rangle_*(X) \to BP\langle n\rangle_*(X)$ *is epic;*
(iii) $BP\langle n+1\rangle_*(X)$ *is* T_{n+1} *torsion free.*

Proof. The equivalence of (ii) and (iii) follows from the BP Bockstein exact sequence (2.1). The implication (iii)$\Rightarrow$(i) is in (3.7). Since $\rho(n, \infty) = \rho(n, n+1) \circ \rho(n+1, \infty)$, (i) implies (ii).

As Larry Smith remarked [31] on the analogous theorem relating integral homology, connective K-theory, and complex bordism, this proposition gives a curious phenomenon. Thinking cohomologically and letting X be a finite complex, this proposition says that if *every* stable map f: $X \to BP\langle n\rangle$ lifts through $\rho(n, n+1)$ to f' (in diagram (3.9), then *every* such f lifts through $\rho(n, \infty)$ to f''.

$$\begin{tikzcd}
 & BP \arrow[d, "\rho(n+1,\infty)"] \arrow[dd, bend left=60, "\rho(n,\infty)"] \\
 & BP\langle n+1\rangle \arrow[d, "\rho(n,n+1)"] \\
X \arrow[r, "f"] \arrow[ur, dashed, "f'"] \arrow[uur, dashed, "f''"] & BP\langle n\rangle
\end{tikzcd} \tag{3.9}$$

The modifier *every* is essential. It is easy to construct examples of complexes X such that there is an $\alpha \in H_*(X; Z_p)$ which is the mod p reduction of an integral homology class which is not Steenrod representable (not in the image of the Thom homomorphism $MU_*(X) \not\twoheadrightarrow H_*(X; Z)$.)

So far in this section, our results have been corollaries to the splitting theorem. Now to prove results about projective dimension, we need more data. Our proofs tend to be inductive; so the following definition of hom $\dim_{BP_*} M_*$, M_* a graded BP_* module, suits our purposes. hom $\dim_{BP_*} M_* = 0$ if and only if M_* is BP_* projective. If M_* is not BP_* projective, let $0 \to N_* \to P_* \to M_* \to 0$ be a short exact sequence of graded BP_* modules with P_*, BP_* projective. We inductively define hom $\dim_{BP_*} M_* = 1 +$ hom $\dim_{BP_*} N_*$. Our inductive step will use a geometric resolution theorem which was proved by Conner and Smith using the basic idea of Atiyah as adapted by Landweber. To begin induction, we need the purely algebraic fact that graded projective $Z_{(p)}[x_1, x_2, \ldots]$ modules are free [13, 3.2]; we omit its proof. Also we need the theorem of Conner and Smith that $BP_*(X)$ being a free (= projective) BP_* module is equivalent to $H_*(X; Z_{(p)})$ begin $Z_{(p)}$ free. We feel the reader will appreciate our inclusion of sketches of Conner and Smith's proof of this last result and of the resolution theorem.

PROPOSITION 3.10 (Conner–Smith). *Let X be a connected CW spectrum with $H_*(X; Z_{(p)})$ of finite type. $H_*(X; Z_{(p)})$ is free $Z_{(p)}$ if and only if $BP_*(X)$ is BP_* free.*

Sketch proof. If $H_*(X; Z_{(p)})$ is $Z_{(p)}$ free, the spectral sequence

$$E^2_{*,*}(X) \cong H_*(X; BP_*) \Rightarrow BP_*(X)$$

collapses as the differentials are torsion valued. Thus the associated graded object $E^\infty_{*,*}(X) = E^2_{*,*}(X) \cong H_*(X; Z_{(p)}) \otimes BP_*$ to the filtration of $BP_*(X)$ is BP_* free implying $BP_*(X)$ is BP_* free. Now assume $BP_*(X)$ is BP_* free. If $H_*(X; Z_{(p)})$ has torsion, it is easy to show using the above spectral sequence that the lowest dimensional torsion element y is in the image of $\rho(0, \infty)$: $BP_*(X) \to H_*(X; Z_{(p)})$ which factors through $\widetilde{\rho(0, \infty)}$: $BP_*(X) \otimes_{BP_*} Z_{(p)} \to H_*(X; Z_{(p)})$. So $\widetilde{\rho(0, \infty)}(z) = y$ for some $z \in BP_*(X) \otimes_{BP_*} Z_{(p)}$ which is a torsion free group. We perceive a contradiction when we note that $\widetilde{\rho(0, \infty)} \otimes Q$ is an isomorphism, $z \otimes 1 \neq 0$, and $y \otimes 1 = 0$.

Definition 3.11. For $-1 \leqslant n \leqslant \infty$, a (geometric) $BP\langle n\rangle_*$ resolution of a connected CW spectrum X is a stable cofibration of connected CW spectra $W \xrightarrow{g} A \xrightarrow{f} X$ such that $H_*(A; Z)$ is free abelian and the induced sequence

$$0 \to BP\langle n\rangle_*(W) \xrightarrow{g_*} BP\langle n\rangle_*(A) \xrightarrow{f_*} BP\langle n\rangle_*(X) \to 0$$

is short exact.

PROPOSITION 3.12 (Conner–Smith). *Every connected CW spectrum has a $BP_* = BP\langle\infty\rangle_*$ resolution.*

Sketch proof for a finite complex. (The extension to connected CW spectra is given in [20] and is credited to Smith.) Given a finite complex X, we choose a Spanier–Whitehead dual DX. $MU^*(DX)$ is finitely generated over MU^*, say by f_i: $DX \to S^{n_i}MU$. Let $(S^{n_i}MU)^{k_i}$ be the k_i skeleton of $S^{n_i}MU$ where the $k_i \gg$ dimension of DX. Then $H^*(\vee_i(S^{n_i}MU)^{k_i};Z)$ is free abelian and $(\vee f_i)^*$: $MU^*(\vee(S^{n_i}MU)^{k_i};Z) \to MU^*(DX;Z)$ is epic. Let A be a dual of $\vee(S^{n_i}MU)^{k_i}$; so we get a stable cofibration. $W \xrightarrow{g} A \xrightarrow{f} X$ with $H_*(A;Z)$ free abelian and $MU_*(f)$ epic. Thus $BP_*(f)$ is epic as required.

PROPOSITION 3.13. *For a connected CW spectrum X with $H_*(X;Z_{(p)})$ of finite type, the following statements are equivalent,* $-1 \leqslant n$.

(i) hom $\dim_{BP_*} BP_*(X) \leqslant n+1$.
(ii) $\rho(n,\infty)$: $BP_*(X) \to BP\langle n\rangle_*(X)$ *is epic.*
(iii) $BP\langle n+1\rangle_*(X)$ *is* T_{n+1} *torsion free.*

Proof. The equivalence of (ii) and (iii) was noted in (3.8). The proof is by induction on n. Proposition 3.10 gives the $n=-1$ case. Use Proposition 3.12 to choose a BP_* resolution of X inducing the commutative diagram (3.14) with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & BP_*(W) \arrow[r, "g_1"] \arrow[d, "\rho_W"] & BP_*(A) \arrow[r, "f_1"] \arrow[d, "\rho_A"] & BP_*(X) \arrow[r] \arrow[d, "\rho_X"] & 0 \\
& BP\langle n\rangle_*(W) \arrow[r, "g_2"] & BP\langle n\rangle_*(A) \arrow[r, "f_2"] & BP\langle n\rangle_*(X) \arrow[ll, bend left] &
\end{tikzcd} \tag{3.14}$$

Note that since $H_*(A;Z)$ is free abelian, $BP\langle n\rangle_*(A)$ is T_n torsion free and $\rho_A = \rho(n,\infty)$ is epic (3.6 and 3.7). ρ_X is epic modulo T_n torsion (3.7); so by commutativity f_2 is epic modulo T_n torsion and by exactness g_2 is monic modulo T_n torsion. We complete the proof:

ρ_X is epic $\Leftrightarrow f_2$ is epic $\Leftrightarrow g_2$ is monic $\Leftrightarrow BP\langle n\rangle_*(W)$ is T_n torsion free $\Leftrightarrow$ hom $\dim_{BP_*} BP_*(W) \leqslant n \Leftrightarrow$ hom $\dim_{BP_*} BP_*(X) \leqslant n+1$.

The next-to-the-last equivalence was by induction.

COROLLARY 3.15. *Let X be a connected CW spectrum with $H_*(X;Z_{(p)})$ of finite type. X has a $BP\langle n\rangle_*$ resolution if and only if* hom $\dim_{BP_*} BP_*(X) \leqslant n+1$.

Proof. Consider diagram (3.14). If hom $\dim_{BP_*} BP_*(X) \leqslant n+1$, ρ_X is epic (3.13) and thus $f_2 = BP\langle n\rangle_*(f)$ is epic. Conversely: if X has a $BP\langle n\rangle_*$ resolution, $BP\langle n\rangle_*(f) \circ \rho_A = f_2 \circ \rho_A$ will be epic forcing $\rho_X = \rho(n,\infty)$ to be epic. Thus hom $\dim_{BP_*} BP_*(X) \leqslant n+1$ (3.13).

§4. ESTIMATES OF hom $\dim_{BP_*} BP_*(X)$

We collect some diverse ways of computing the invariant, hom $\dim_{BP_*} BP_*(X)$. The dimension estimate (4.4) which gives an upper bound of the invariant is new; the other estimates, both of which give lower bounds, are improvements or adaptations of techniques found in the papers of Conner and Smith. The section ends with a discussion of relations between the two lower bound estimating techniques and computation of examples.

PROPOSITION 4.1. *Let X be a finite complex with $H^k(X;Z_{(p)})$ T_0 torsion free for* $k \geqslant 2(p^n + \cdots + p + 1)$, *then* $\rho(n-1,n)$; $BP\langle n\rangle^*(X) \to BP\langle n-1\rangle^*(X)$ *is epic.*

Proof. The usual spectral sequence,

$$E_2^{*,*}(X) \cong H^*(X; BP\langle n\rangle^*) \Rightarrow BP\langle n\rangle^*(X)$$

shows that $BP\langle n\rangle^k(X)$ is T_0 torsion free for $k \geqslant 2(p^n + \cdots + p + 1)$. Now let us consider the cohomology version of the BP Bockstein sequence

$$BP\langle n\rangle^s(X) \xrightarrow{\rho(n-1,n)} BP\langle n-1\rangle^s(X) \xrightarrow{\Delta_n} BP\langle n\rangle^{s+2p^n-1}(X). \tag{4.2}$$

Δ_n is T_0 torsion valued (3.2); so $\rho(n-1, n)$ is epic for $s \geqslant 2(p^n + \cdots + p + 1) - (2p^n - 1)$. By the splitting theorem (2.2), $\rho(n-1, n)$ is epic for $s \leqslant 2(p^{n-1} + \cdots + p + 1)$.

PROPOSITION 4.3. *Let a finite complex $X \subseteq R^t$ have $H_s(X; Z_{(p)})$ T_0 torsion free for $s \leqslant k$. If $t \leqslant k + 2(p^n + \cdots + p + 1)$, then* hom $\dim_{BP_*} BP_*(X) \leqslant n$.

Proof. Let N be a regular neighborhood of $X \subseteq R^t$. $X \subseteq N$ is a homotopy equivalence and N is a t dimensional manifold with boundary. The tangent bundle of N is trivial; so N is BP orientable. Using the BP module structure of $BP\langle m\rangle$, there are duality isomorphisms $BP\langle m\rangle^q(N/\partial N) \cong BP\langle m\rangle_{t-q}(X^+)$. For $q \geqslant 2(p^n + \cdots + p + 1) \geqslant t - k$ (implying $t - q \leqslant k$), the $BP\langle 0\rangle$ duality and the hypothesis imply $H^q(N/\partial N; Z_{(p)})$ is T_0 torsion free. Proposition 4.1 applies to $N/\partial N$ to imply that $\rho(n-1, n)\colon BP\langle n\rangle^*(N/\partial N) \to BP\langle n-1\rangle^*(N/\partial N)$ is epic. Using the $BP\langle n-1\rangle$ duality, we have $\rho(n-1, n)\colon BP\langle n\rangle_*(X^+) \to BP\langle n-1\rangle_*(X^+)$ is epic. By (3.13), hom $\dim_{BP_*} BP_*(X) =$ hom $\dim_{BP_*} BP_*(X^+) \leqslant n$.

COROLLARY 4.4 (Dimension estimate). *If X is a q dimensional finite complex and if $q < p^n + \cdots + p + 1$, then* hom $\dim_{BP_*} BP_*(X) \leqslant n$.

Proof. This follows by letting $k = 0$ and $t = 2q + 1$ in Proposition 4.3. Note that $q < p^n + \cdots p + 1$ implies $2q + 1 \leqslant 2(p^n + \cdots + p + 1)$ as required.

COROLLARY 4.5 (Adams, [1]; Conner and Smith [13]). *If X is a finite complex, then* hom $\dim_{MU_*} MU_*(X)$ *is finite.*

PROPOSITION 4.6 (Ideal annihilator estimate). *Let X be a connected CW spectrum with $H_*(X; Z_{(p)})$ of finite type. If $x_n \cdot y = 0$ for $0 \neq y \in BP_*(X)$, then* hom $\dim_{BP_*} BP_*(X) \geqslant n + 1$.

Proof. Immediate from (3.5) and (3.13).

Let $c\colon \mathscr{A}(p) \to \mathscr{A}(p)$ be the canonical antiautomorphism of the mod p Steenrod algebra. Since the Bockstein Q_0 ($Q_0 = Sq^1$ if $p = 2$) is primitive, $c(Q_0) = -Q_0$. Thus c restricts to an antiautomorphism of (Q_0), the two-sided ideal generated by Q_0. The universal coefficient theorem isomorphism $H^q(X; Z_p) \to \mathrm{Hom}_{Z_p}(H_q(X; Z_p); Z_p)$ defines a dual pairing $\langle\ ,\ \rangle$: $H^q(X; Z_p) \otimes H_q(X; Z_p) \to Z_p$. For $\alpha \in \mathscr{A}(p)$ of dimension r and $x \in H_q(X; Z_p)$, define $\alpha x \in H_{q-r}(X; Z_p)$ to be the homology class defined by (4.7).

$$\langle y, \alpha x\rangle = \langle c(\alpha)y, x\rangle \text{ for all } y \in H^{q-r}(X; Z_p). \tag{4.7}$$

This defines a left $\mathscr{A}(p)$ module structure on $H_*(X; Z_p)$.

THEOREM 4.8 (Steenrod operation estimate). *Let X be a connected CW spectrum with $H_*(X; Z_{(p)})$ of finite type. If there is a selection of operations $b_1, b_2, \ldots, b_k$ in (Q_0) such that the composition $b_1 b_2 \ldots b_k$ acts nontrivially on the image of $\rho(-1, n)\colon BP\langle n\rangle_*(X) \to$*

$H_*(X;Z_p)$, then hom $\dim_{BP_*} BP_*(X) \geqslant n+k+1$. *In particular, if $b\ \rho(-1,n)(x) \neq 0$ for $b \in (Q_0)$ and $x \in BP\langle n\rangle_s(X)$, then x is not in the image of $\rho(n,\infty)$: $BP_s(X) \to BP\langle n\rangle_s(X)$.*

Proof. A BP resolution of X, $W \to A \xrightarrow{f} X$ induces the exact rows of the commutative diagram (4.9).

$$\begin{tikzcd}
BP_*(A) \arrow[r, "f_1"] \arrow[d, "\rho_1"] & BP_*(X) \arrow[r] \arrow[d, "\rho_2"] & 0 \\
BP\langle n\rangle_*(A) \arrow[r, "f_2"] \arrow[d, "\rho_3"] & BP\langle n\rangle_*(X) \arrow[r, "\partial_2"] \arrow[d, "\rho_4"] & BP\langle n\rangle_*(W) \arrow[d, "\rho_5"] \\
H_*(A;Z_p) \arrow[r, "f_3"] & H_*(X;Z_p) \arrow[r, "\partial_3"] & H_*(W;Z_p)
\end{tikzcd} \tag{4.9}$$

Since $H_*(A;Z)$ is free abelian (3.11), ρ_1 and ρ_3 are epic (3.6 and 3.7) and (Q_0) acts trivially on $H_*(A;Z_p)$. By hypothesis, $0 \neq b_1 b_2 \cdots b_k \rho_4(x)$ for some $x \in BP\langle n\rangle_*(X)$.

CLAIM 4.10. $0 \neq \partial_3 b_2 \cdots b_k\ \rho_4(x) = b_2 \cdots b_k\ \partial_3 \rho_4(x) = b_2 \ldots b_k\ \rho_5\ \partial_2(x)$.

Proof of claim. Suppose $\partial_3 b_2 \cdots b_k\ \rho_4(x) = 0$, then $b_2 \cdots b_k\ \rho_2(x) = f_3\rho_3\rho_1\ (a)$ for some $a \in BP_*(A)$. So $0 \neq b_1 b_2 \ldots b_k \rho_4(x) = b_1 f_3 \rho_3 \rho_1(a) = f_3 b_1 \rho_3 \rho_1(a)$ which gives the absurd conclusion that $b_1 \in (Q_0)$ acts non trivially on $H_*(A;Z_p)$.

Conclusion of proof. Thus $x \notin \text{Image}\ \rho_2 = \text{Image}\ \rho_2 \circ f_1 = \text{Image} f_2 \circ \rho_1$, else $\partial_2(x) = 0$ contradicting (4.10). If $k = 1$, the failure of ρ_2 to be epic implies hom $\dim_{BP_*} BP_*(X) \geqslant n+2$ (3.13). If $k > 1$, by an induction on the length of the compositions of operations of (Q_0), $0 \neq b_2 \cdots b_k \rho^5 \partial_2(x)$ implies hom $\dim_{BP_*} BP_*(W) \geqslant n + (k-1) + 1$. This implies hom $\dim_{BP_*} BP_*(X) \geqslant n+k+1$.

By Spanier–Whitehead duality, we can do the entire homological dimension theory for Brown–Peterson cohomology modules over the cohomology coefficient ring BP^*. In general, the resulting invariant hom $\dim_{BP_*} BP^*(X)$ is not equal to hom $\dim_{BP_*} BP_*(X)$ (see (4.29).)

COROLLARY 4.11 (Conner and Smith [11]). *Let X be a finite complex. If there is a selection of operations $b_1, b_2, \ldots, b_k$ in (Q_0) such that the composition $b_1\ b_2 \ldots b_k$ acts non trivially in $H^*(X;Z_p)$, then both* hom $\dim_{BP_*} BP_*(X) \geqslant k$ *and* hom $\dim_{BP_*} BP^*(X) \geqslant k$.

Proof. $b_1\ b_2 \ldots b_k$ acting non trivially in $H^*(X;Z_p)$ implies $c(b_k) \ldots c(b_2) c(b_1)$ acts non trivially in $H_*(X;Z_p)$. Apply (4.8) and its dual.

We now have two techniques for computing lower bounds of hom $\dim_{BP_*} BP_*(X)$. Neither is generally effective. One only has to resolve a complex X having hom $\dim_{BP_*} BP_*(X) = n+1$ with a BP_* resolution $W \to A \to X$ to obtain a complex W with no $BP_*(W)$ annihilators and hom $\dim_{BP_*} BP_*(W) = n$. Conner and Smith show how a stable homotopy class $\gamma \in \pi^s_{2m-1}$ of odd order q which is in the image of the J homomorphism gives a stable complex $X(\gamma) = S^0 \cup_q e^1 \cup e_\gamma^{2m+1}$ with hom $\dim_{MU_*} MU_*(X(\gamma)) = 2$[11; 12, Section 2; 14, 7.2]. If γ is not detected by a mod p Hopf invariant, the Steenrod operation estimate will give the correct, but useless, information that hom $\dim_{BP_*} BP_*(X(\gamma)) \geqslant 1$.

How are the ideal annihilator estimate and the Steenrod operation estimate related? In general, not very well. In the special case when $0 = x_n \cdot y$ for $y \in BP_*(X)$ and $0 \neq \rho(-1, \infty)(y) \in H_*(X; Z_p)$, a relation becomes clear however. The reader has probably recognized that the BP Bockstein sequence (2.1) has a "first differential"

$$BP\langle n-1\rangle_s(X) \xrightarrow{\Delta_n} BP\langle n\rangle_{s-2p^n+1}(X) \xrightarrow{\rho(n-1,n)} BP\langle n-1\rangle_{s-2p^n+1}(X)$$

which gives an operation $\rho(n-1, n) \circ \Delta_n \in BP\langle n-1\rangle^{2p^n-1}(BP\langle n-1\rangle)$. A relation between the two estimates in special cases derives from the fact that this operation "covers" a non-zero class in $H^{2p^n-1}(BP\langle n-1\rangle; Z_p) \cong Z_p$. In the following, we assume a familiarity with the notation of [23].

LEMMA 4.12. *Let* $1 \in H^0(BP\langle n-1\rangle; Z_p) \cong Z_p$ *be a generator, then a generator of* $H^{2p^n-1}(BP\langle n-1\rangle; Z_p) \cong Z_p$ is $Q_n 1 = -c(\mathcal{P}^{p^{n-1}} \cdots \mathcal{P}^p \mathcal{P}^1 Q_0)1$.

Proof. From (2.5), it follows that Q_0 is trivial on $H^i(BP\langle n-1\rangle; Z_p)$ for $i < 2p^n - 2$. For $2 \leqslant m \leqslant n$ and $x \in H^{2n^n-2p^m}(BP\langle n-1\rangle;\ Z_p)$, $Q_0 \quad c(\mathcal{P}^{p^{m-1}})x = 0$. Now we compute inductively that $Q_m x = -c(Q_m)x = -c(\mathcal{P}^{p^{m-1}} Q_{m-1} - Q_{m-1}\mathcal{P}^{p^{m-1}})x = -c(Q_{m-1})$ $c(\mathcal{P}^{p^{m-1}})x + c(\mathcal{P}^{p^{m-1}})c(Q_{m-1})x = -c(Q_0)c(\mathcal{P}^1) \cdots c(\mathcal{P}^{p^{m-2}})c(\mathcal{P}^{p^{m-1}})x + 0$ (induction) $=$ $-c(\mathcal{P}^{p^{m-1}} \cdots \mathcal{P}^1 Q_0)x$.

LEMMA 4.13. $\rho(-1, 0)$: $H^{2p^n-1}(BP\langle n-1\rangle;\ Z_{(p)}) \to H^{2p^n-1}(BP\langle n-1\rangle;\ Z_p)$ *is an isomorphism.*

Proof. It is easy to check that $H^{2p^n-1}(BP\langle n-1\rangle; Z_{(p)})$ is a finite cyclic group (*a* Z_{p^s}) mapping onto the mod p reduction. It suffices to show that $H^{2p^n-1}(BP\langle n-1\rangle; Z_{p^2}) \cong Z_p$ and this is done by using the classical Bockstein sequence (4.14) which defines Q_0.

$$\begin{aligned} H^{2p^n-2}(BP\langle n-1\rangle\ Z_p) \xrightarrow{Q_0} H^{2p^n-1}(BP\langle n-1\rangle; Z_p) \to H^{2p^n-1}(BP\langle n-1\rangle; Z_p{}^2) \\ \xrightarrow{\rho} H^{2p^n-1}(BP\langle n-1\rangle; Z_p) \cdots \quad (4.14) \end{aligned}$$

By Lemma 4.12, Q_0 is an epimorphism. By exactness, ρ is then monic as required.

$$\begin{tikzcd}
BP\langle n-1\rangle \arrow[rrr, "\Delta n"] \arrow[dd, "\rho(-1,\, n-1)"'] & & & S^{2p^n-1}BP\langle n\rangle \arrow[d, "\rho(0,\, n)"] & \\
& & & S^{2p^n-1}KZ_{(p)} \arrow[dr, "\rho(-1,\, 0)"] & \\
KZ_p \arrow[rr, "c(\mathcal{P}^{p^{n-1}} \cdots \mathcal{P}^p \mathcal{P}^1)"] & & S^{2p^n-2}KZ_p \arrow[ur, "\Delta_0"] \arrow[rr, "c(Q_0)"] & & S^{2p^n-1}KZ_p
\end{tikzcd} \quad (4.15)$$

LEMMA 4.16 *Diagram* 4.15 *commutes up to multiplication by a unit of* $Z_{(p)}$.

Proof. Both $\rho(-1, 0) \circ \rho(0, n) \circ \Delta_n = \rho(-1, n) \circ \Delta_n$ and $c(Q_0) \circ c(\mathcal{P}^{p^{n-1}} \cdots \mathcal{P}^n \mathcal{P}^1) \circ \rho(-1,\ n-1) = c(\mathcal{P}^{p^{n-1}} \cdots \mathcal{P}^p \mathcal{P}^1 Q_0) \circ \rho(-1,\ n-1)$ determine non-zero classes in $H^{2p^{n-1}}(BP\langle n-1\rangle; Z_p) \cong Z_p$ by (2.5) and (4.12), respectively. The triangle commutes up to sign. By Lemma 4.13, $\Delta_0 \circ c(\mathcal{P}^{p^{n-1}} \cdots \mathcal{P}^p \mathcal{P}^1) \circ \rho(-1,\ n-1)$ and $\rho(0, n) \circ \Delta_n$ both give non-zero classes in $H^{2p^{n-1}}(BP\langle n-1\rangle; Z_{(p)}) \cong Z_p$.

COROLLARY 4.17. *If* $0 \neq a = c(Q_0(\mathcal{P}^1 Q_0)\cdots(\mathcal{P}^{p^{n-1}}\cdots\mathcal{P}^1 Q_0))b \in H_s(X; Z_p)$, *then* $\lambda a = \rho(-1, n)\Delta_n\cdots\Delta_1\Delta_0(b)$ *for some* $0 \neq \lambda \in Z_p$.

Proof. By hypothesis, $a = c(\mathcal{P}^{p^{n-1}}\cdots\mathcal{P}^1 Q_0)\cdots c(\mathcal{P}^1 Q_0)c(Q_0)b$. Inductively, one proves $\lambda'\ \rho(-1, q)\circ \Delta_q\cdots\Delta_1\circ\Delta_0\ (b) = c(\mathcal{P}^{p^{n-1}}\cdots\mathcal{P}^1 Q_0)\cdots c(\mathcal{P}^1 Q_0)c(Q_0)(b)$ using the fact from (4.16) that $c(\mathcal{P}^{p^q}\cdots\mathcal{P}^1 Q_0)\circ\rho(-1, q) = \lambda''\ \rho(-1, q+1)\circ\Delta_{q+1}$ where λ' and λ'' are non-zero elements of Z_p.

Example 4.18. Let $\iota_n \in H^n(K(Z_p, n); Z_p)$ be the fundamental class and let $f\colon B(Z_p)^n = K(Z_p, 1)\times\cdots\times K(Z_p, 1)\to K(Z_p, n)$ classify the n-fold external product $\iota_1\times\cdots\times\iota_1 \in H^n(B(Z_p)^{n+}; Z_p)$. For $0 \leqslant q \leqslant n-2$, $0 \neq Q_0(\mathcal{P}^1 Q_0)\cdots(\mathcal{P}^{p^q}\cdots\mathcal{P}^1 Q_0)(\iota_1\times\cdots\times\iota_1) = f^*(Q_0(\mathcal{P}^1 Q_0)\cdots(\mathcal{P}^{p^q}\cdots\mathcal{P}^1 Q_0)\iota_n)$. By (4.17), $\Delta_{q+1}\colon BP\langle q\rangle_{n+2p^{q+1}-1}(K(Z_p, n)) \to BP\langle q+1\rangle_n(K(Z_p, n)) \cong Z_p$ is non-zero. Let $\sigma_n \in BP_n(K(Z_p, n)) \cong Z_p$ be a generator. We see that $x_{q+1}\rho(q+1, n)(\sigma_n) = 0$ for $0 \leqslant q \leqslant n+2$. For dimensional reasons, this implies that $x_{q+1}\sigma_n = 0$. So $p, x_1, \ldots, x_{n-1}$ all annihilate σ_n. This fact is essentially Theorem A of [29].

We shall use the following technical proposition for computing examples. It is a special case of Corollary 2.4 of [14] designed to deal with the important class of stable complexes of form $S^0 \cup e^1 \cup e^{2n_1+1} \cup\cdots\cup e^{2n_k+1}$.

PROPOSITION 4.19. *Let* Y *be a connected CW spectrum with* $Z_{(p)}$ *homology of finite type and satisfying the following.*

(i) $H_i(Y; Z_{(p)}) = 0$ *for* $i < k$ *or* $i = k+1$.
(ii) $H_k(Y; Z_{(p)}) \cong Z_{p^s}$ *for* $s > 0$, *with generator* σ_0.
(iii) $H_i(Y; Z_{(p)})$ *is* $Z_{(p)}$ *free for* $i > k+1$.

Let $\sigma_n = \rho(0, n)^{-1}(\sigma_0) \in BP\langle n\rangle_k(Y)$. *Let* $\tau^* \in H^{k+1}(Y; Z_p) \cong \mathrm{Ext}(H_k(Y; Z_{(p)}); Z_p)$ *be a generator. We conclude:* $x_n p^{s-1}\sigma_n = 0$ *if and only if* $\mathcal{P}^{p^{n-1}}\cdots\mathcal{P}^p\mathcal{P}^1\tau^* \neq 0$.

Proof. By the exact sequence (4.20), $\tau = \Delta_0^{-1}(p^{s-1}\sigma_0) \in H_{k+1}(Y; Z_p)$ is a generator.

$$0 = H_{k+1}(Y; Z_{(p)}) \longrightarrow H_{k+1}(Y; Z_p) \xrightarrow{\Delta_0} H_k(Y; Z_{(p)}) \xrightarrow{\cdot p} H_k(Y; Z_{(p)}). \qquad (4.20)$$

Without loss of generality, we may choose τ^* to be dual to τ. In what follows, we shall ignore "up to multiplication by a unit of $Z_{(p)}$". By (4.16), the diagram (4.21) commutes; the sequence is exact.

$$\begin{tikzcd}
BP\langle q-1\rangle_{k+2p-1}(Y) \arrow[r, "\Delta_q"] \arrow[dd, "\rho(-1,q-1)"'] & BP\langle q\rangle_k(Y) \arrow[r, "\cdot x_q"] \arrow[d, "\cong"', "\rho(0,q)"] & BP\langle q\rangle_{k+2p^q-2}(Y) \\
 & H_k(Y;Z_{(p)}) & \\
H_{k+2p^q-1}(Y;Z_p) \arrow[r, "c(\mathcal{P}^{p^{q-1}}\cdots\mathcal{P}^p\mathcal{P}^1)"] & H_{k+1}(Y;Z_p) \arrow[u, "\Delta_0"'] &
\end{tikzcd} \qquad (4.21)$$

If $x_n p^{s-1}\sigma_n = 0$, then $p^{s-1}\sigma_n = \Delta_n(y)$ for some $y \in BP\langle n-1\rangle_{k+2p^n-1}(Y)$. So $p^{s-1}\sigma_0 = \rho(0, n)(p^{s-1}\sigma_n) = \rho(0, n)\Delta_n(y) = \Delta_0 c(\mathscr{P}^{p^{n-1}} \cdots \mathscr{P}^p\mathscr{P}^1)\rho(-1, n-1)(y)$ and so $\mathscr{P}^{p^{n-1}} \cdots \mathscr{P}^p \mathscr{P}^1 \tau^* \neq 0$.

We now assume $\mathscr{P}^{p^{n-1}} \cdots \mathscr{P}^{p^{q-1}} \cdots \mathscr{P}^p\mathscr{P}^1\tau^* \neq 0$. We prove the converse by proving inductively (4.22).

4.22. If $0 \neq y \in BP\langle q\rangle_*(Y)$ and $x_q \cdot y = 0$, then $y = \lambda p^{s-1}\sigma_q \in BP\langle q\rangle_k(Y)$ for some unit λ of $Z_{(p)}$. For $q = 0$, (4.22) is the obvious statement about the T_0 torsion of $H_*(Y; Z_{(p)})$. Assume the statement for $0, 1, \ldots, q-1 < n$. Analysis of the BP Bockstein sequences (2.1) in the light of this assumption shows that in dimension $k + 2p^q - 1$, $\rho(-1, q-1) = \rho(-1, 0) \circ \rho(0, 1) \circ \cdots \circ \rho(q-2, q-1)$ is epic. Thus both paths of (4.21) are non-zero, in dimension k, Image $\Delta_q = \rho(0, q)^{-1}\Delta_0(H_{k+1}(Y; Z_p))$ which is additively generated by $p^{s-1}\sigma_q$.

Suppose $x_q y = 0$ for some $0 \neq y \in BP\langle q\rangle_*(Y)$. We may write $y = x_q{}^t z$ where $0 \neq \rho(q-1, q)(z) \in BP\langle q-1\rangle_*(Y)$. Since y is T_q-torsion, it, z, and $\rho(q-1, q)(z)$ are T_{q-1}-torsion. So $(x_{q-1})^u\rho(q-1, q)(z) = 0$ for some smallest integer u. By (4.22) for $q-1$, $0 \neq (x_{q-1})^u\rho(q-1, q)(z)$ implies $(x_{q-1})^{u-1}\rho(q-1, q)(z) = \lambda p^{s-1}\sigma_{q-1}$, λ is a unit of $Z_{(p)}$. So $u = 1$ and $\rho(q-1, q)(z) = \lambda p^{s-1}\sigma_{q-1}$. $\rho(q-1, q)$ is monic in dimension k; so $z = \lambda p^{s-1}\sigma_q$ which is in the image of Δ_q. Thus $0 = x_q z$ and $t = 0$. We conclude $y = \lambda p^{s-1}\sigma_q$ as required to confirm (4.22) for q.

Example 4.23. Using the first two elements of Hopf invariant one, Conner and Smith construct a stable complex $X(\eta, \nu) = S^0 \cup_2 e^1 \cup_\eta e^3 \cup_{\bar{\nu}} e^7$ [11, p. 480; 12, Section 3; 14, pp. 166–168]. The homology of $X(\eta, \nu)$ satisfies the hypothesis of (4.19). Conner and Smith show that $Sq^4Sq^2Sq^1$: $H^0(X(\eta, \nu); Z_2) \to H^7(X(\eta, \nu); Z_2)$ is an isomorphism. Recalling $\mathscr{P}^2 = Sq^4$ and $\mathscr{P}^1 = Sq^4$ when $p = 2$, (4.19) says that $x_1\sigma_1 = 0$ and $x_2 \cdot \sigma_2 = 0$ where $\sigma_1 \in BP\langle 1\rangle_0(X(\eta, \nu)) \cong Z_2$ and $\sigma_2 \in BP\langle 2\rangle_0(X(\eta, \nu)) \cong Z_2$ are the nonzero elements. By BP Bockstein sequences (2.1), these two facts imply $BP\langle 1\rangle_2(X(\eta, \nu)) = 0$ and $\rho(1, 2)$: $BP\langle 2\rangle_7(X(\eta, \nu)) \to BP\langle 1\rangle_7(X(\eta, \nu))$ is not epic. Thus hom $\dim_{BP_*} BP_*(X(\eta, \nu)) \geqslant 3$ (3.13). (Note that $c(Sq^4Sq^2Sq^1) \cdot \rho(-1, +1) \not\equiv 0$; so we have actually used the Steenrod operation estimate (4.8).) To be gross, $S^{15}X(\eta, \nu))$ is an "honest" 22 dimensional complex. Apply (4.3) with $t = 45$ and $k = 15$. $45 \leqslant 15 + 2(2^3 + 2^2 + 2 + 1)$ implies hom $\dim_{BP_*} BP_*(S^{15}X(\eta, \nu))$ $=$ hom $\dim_{BP_*} BP_*(X(\eta, \nu)) \leqslant 3$; so our lower estimate was the best possible.

Remark 4.24 The only torsion of $H_*(X(\eta, \nu); Z)$ is 2-primary. For $p = 2$, $BP\langle 1\rangle_*(\ \ ) = k_*(\ \ ) \otimes Z_{(2)}$. Our analysis in (4.23) gives that $k_2(X(\eta, \nu)) \otimes Z_{(2)} = 0$ and that ζ: $MU_7(X(\eta, \nu)) \to k_7(X(\eta, \nu))$ fails to be an epimorphism. In Baas' work on bordism with singularies, he constructed a tower of spectra

$$MU \to \cdots \to MU\langle 2\rangle \to MU\langle 1\rangle \to MU\langle 0\rangle = KZ \tag{4.25}$$

and asked whether analogs of the Conner–Smith theorems held [6]. The proof of (2.7) implies $MU\langle 1\rangle_*(\ \ ) \otimes Z_{(2)}$ and $k_*(\ \ ) \otimes Z_{(2)}$ are isomorphic; so $MU\langle 1\rangle_2(X(\eta, \nu)) = 0$ and ζ: $MU_7(X(\eta, \nu)) \to MU\langle 1\rangle_7(X(\eta, \nu))$ is not epic. The rows and columns in (4.26) are MU–Bockstein sequences and $X = X(\eta, \nu)$.

$$\begin{tikzcd}
 & MU_7(X) \arrow[dl, "\psi"'] \arrow[d, "\zeta"] & MU\langle 2\rangle_{-2}(X) = 0 \arrow[d, "\cdot y_2"] \\
MU\langle 2\rangle_7(X) \arrow[r, "\mu"] & MU\langle 1\rangle_7(X) \arrow[r] & MU\langle 2\rangle_2(X) \arrow[d] \\
 & & MU\langle 1\rangle_2(X) = 0
\end{tikzcd} \tag{4.26}$$

μ is epic and ζ is not; so $\psi: MU_7(X) \to MU\langle 2\rangle_7(X)$ fails to be epic even though hom $\dim_{MU_*} MU_*(X(\eta, v)) = 3$. This shows that the Conner–Smith program cannot be generalized by a MU tower of spectra without localization.

Example 4.27. Fix a prime p. There is some least integer s such that a stable map f': $CP(1) = S^2 \to S^2$ of degree $p^s q$, $gcd(p, q) = 1$, extends to a stable map f: $CP(p^n) \to S^2$. Let Y be the cofibre of f. $H_i(Y; Z_{(p)}) = 0$ for $i < 2$ and $i = 3$. $H_2(Y; Z_{(p)}) \cong Z_{p^s}$. For $i > 3$, $H_i(Y; Z_{(p)})$ is $Z_{(p)}$ free. δ: $H^i(CP(p^n); Z_p) \to H^{i+1}(Y; Z_p)$ is an isomorphism for $i \geqslant 2$. Let $t \in H^2(CP(p^n); Z_p)$ be a generator, then $t^{p^n} \neq 0$. $\mathscr{P}^{p^{n-1}} \cdots \mathscr{P}^p \mathscr{P}^1 \delta(t) = \delta(\mathscr{P}^{p^{n-1}} \cdots \mathscr{P}^p \mathscr{P}^1 t) = \delta(\mathscr{P}^{p^{n-1}} \cdots \mathscr{P}^p t^p) = \cdots = \delta(\mathscr{P}^{p^{n-1}} t^{p^{n-1}}) = \delta(t^{p^n}) \neq 0$. By (4.19) and (3.13), hom $\dim_{BP_*} BP_*(Y) \geqslant n + 1$. The stable dimension of Y is $2p^n - 1$; so it can be represented as an "honest" complex of dimension $4p^n - 2$. $4p^n - 2 < p^{n+1} + \cdots + p + 1$ for positive n; thus our dimension estimate (4.4) assures us the above lower bound is the best possible and hom $\dim_{BP_*} BP_*(Y) = n + 1$. This shows that the upper bound estimate of Corollary 4.4 is, in fact, a good one.

Remark 4.28. This computation of hom $\dim_{BP_*} BP_*(Y)$ is the first example known to the authors of a complex with a high hom $\dim_{BP_*} BP_*(\quad)$ which is known precisely.

Remark 4.29. $H^*(Y; Z)$ is concentrated in odd dimensions; so the usual spectral sequence

$$E_2^{*,*}(Y) = H^*(Y; MU^*) \Rightarrow MU^*(Y)$$

collapses and $MU^*(Y) \to H^*(Y; Z)$ is epic. This hom $\dim_{MU_*} MU^*(Y) = 1$ while hom $\dim_{MU_*} MU_*(Y)$ is high.

Remark 4.30. The lower bound of hom $\dim_{BP_*} BP_*(Y)$ could be done using the annihilator ideal data found on p. 173 of [14]. One needs only to localize the data at the prime p and then recall the Milnor manifolds $V^{2p^q - 2}$ represent polynomial generators of $MU_* \otimes Z_{(p)}$.

§5. ASSORTED PRODUCTS AND SPECTRAL SEQUENCES

The aims of this section are twofold. First, we shall complete the proof of Theorem 1.1 and then we shall construct a variety of spectral sequences.

LEMMA 5.1. *Let Y be a connected CW spectrum with $H_*(Y; Z_{(p)})$ a free $Z_{(p)}$ module of finite type, then for any CW spectrum X*

$$\rho(\widetilde{n, \infty}): BP_*(Y) \otimes_{BP_*} BP\langle n\rangle_*(X) \to BP\langle n\rangle_*(Y \wedge X)$$

and

$$\rho(\widehat{n, \infty})\colon BP\langle n\rangle_*(X)\otimes_{BP_*} BP_*(Y) \to BP\langle n\rangle_*(X \wedge Y)$$

are isomorphisms.

Proof. $BP_*(Y)$ is a free BP_* module (3.10); so both $BP_*(Y)\otimes_{BP_*} BP\langle n\rangle_*(-)$ and $BP\langle n\rangle_*(Y \wedge -)$ are homology theories. When $X = S^0$, $\rho(\widetilde{n, \infty})$ is proved to be an isomorphism by bookkeeping arguments. By the uniqueness theorem for homology theories, $\rho(\widetilde{n, \infty})$ is an isomorphism for all X. The proof for $\rho(\widehat{n, \infty})$ is the same.

LEMMA 5.2. *Let* (5.3) *be a commutative diagram with exact rows. If α and β are isomorphisms, then*

$$0 \to C_1 \xrightarrow{\gamma} C_2 \xrightarrow{h^{-1}\alpha^{-1}\partial} T \to 0$$

is exact.

$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r, "h"] & A_1 \arrow[r, "g_1"] \arrow[d, "\alpha"] & B_1 \arrow[r, "f_1"] \arrow[d, "\beta"] & C_1 \arrow[r] \arrow[d, "\gamma"] & 0 \\
 & & A_2 \arrow[r, "g_2"] & B_2 \arrow[r, "f_2"] & C_2 \arrow[ll, bend left, "\partial"] &
\end{tikzcd} \tag{5.3}$$

Proof. Routine.

PROPOSITION 5.4. *Let X and Y be connected CW spectra with $Z_{(p)}$ homology of finite type. If either* (i) hom $\dim_{BP_*} BP_*(X) \leqslant n+2$ *and* hom $\dim_{BP_*} BP_*(Y) = 0$ *or* (ii) hom $\dim_{BP_*} BP_*(X) \leqslant 1$, *hold, then there is a natural short exact sequence* (5.5).

$$0 \to BP_*(X)\otimes_{BP_*} BP\langle n\rangle_*(Y) \xrightarrow{\rho(n,\infty)} BP\langle n\rangle_*(X \wedge Y) \to \mathrm{Tor}_{1,*}^{BP_*}(BP_*(X), BP\langle n\rangle_*(Y)) \to 0. \tag{5.5}$$

If hom $\dim_{BP_*} BP_*(X) \leqslant n+1$ and hom $\dim_{BP_*} BP_*(Y) = 0$, then $\rho(\widetilde{n, \infty})$ is an isomorphism.

Proof. First, we choose a BP_* resolution of X, giving us a short exact sequence

$$0 \to BP_*(W) \xrightarrow{g_*} BP_*(A) \xrightarrow{f_*} BP_*(X) \to 0$$

with $BP_*(A)$ a free BP_* module. This induces the commutative diagram (5.6).

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & T \arrow[r] & BP_*(W)\otimes_{BP} BP\langle n\rangle_*(Y) \arrow[dr, "g_1"] \arrow[dd, "\rho_W"] & & BP_*(X)\otimes_{BP_*} BP\langle n\rangle_*(Y) \arrow[r] \arrow[dd, "\rho_X"] & 0 \\
 & & & BP_*(A)\otimes_{BP_*} BP\langle n\rangle_*(Y) \arrow[ur, "f_1"] \arrow[d, "\rho_A"] & & \\
 & & BP\langle n\rangle_*(W \wedge Y) \arrow[r, "g_2"] & BP\langle n\rangle_*(A \wedge Y) \arrow[r, "f_2"] & BP\langle n\rangle_*(X \wedge Y) \arrow[ll, bend left, "\partial"] &
\end{tikzcd} \tag{5.6}$$

$T = \mathrm{Tor}_1^{BP_*}(BP_*(X), BP\langle n\rangle_*(Y))$.

By (5.1), ρ_A is an isomorphism. If (ii) holds, then $BP_*(W)$ is also BP_* free and so ρ_W is an isomorphism. So (ii) and Lemma 5.2 imply the exactness of (5.5). If hom $\dim_{BP_*} BP_*(Y) = 0$ and if $0 < \text{hom} \dim_{BP_*} BP_*(X) \leqslant n + 1$, $f_2 = BP\langle n\rangle_*(f) \otimes_{BP_*} BP_*(Y)$ is epic by (5.1) and (3.15) So $\partial = 0$ and ρ_W is an isomorphism by induction on hom $\dim_{BP_*} BP_*(\quad)$ implying ρ_X is an isomorphism by the "five" lemma. So if (i) holds, ρ_W and ρ_A are isomorphisms and the exactness of (5.5) follows again from Lemma 5.2.

PROPOSITION 5.7. *Let X be a connected CW spectrum with $Z_{(p)}$ homology of finite type. There is a natural short exact sequence* (5.8)

$$0 \to BP\langle n+1\rangle_*(X) \otimes_{Y_{(p)}[x_{n+1}]} Z_{(p)} \xrightarrow{\widetilde{\rho(n, n+1)}} BP\langle n\rangle_*(X) \to \operatorname{Tor}_{1,*}^{Z_{(p)}[x_{n+1}]}(BP\langle n+1\rangle_*(X);\, Z_{(p)}) \to 0 \tag{5.8}$$

$\widetilde{\rho(n, n+1)}$ is an isomorphism if and only if hom $\dim_{BP_*} BP_*(X) \leqslant n + 1$.

Proof. Apply the functor $BP\langle n+1\rangle_*(X) \otimes_{Z_{(p)}[x_{n+1}]}$ to the $Z_{(p)}[x_{n+1}]$ free resolution of $Z_{(p)}$,

$$0 \to Z_{(p)}[x_{n+1}] \xrightarrow{\cdot x_{n+1}} Z_{(p)}[x_{n+1}] \to Z_{(p)} \to 0$$

to yield:

$$T = \operatorname{Tor}_{1,*}^{Z_{(p)}[x_{n+1}]}(BP\langle n+1\rangle_*(X); Z_{(p)})$$

$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r] & BP\langle n+1\rangle_*(X) \arrow[r, "x_{n+1}"] \arrow[d] & BP\langle n+1\rangle_*(X) \arrow[r] \arrow[d, "="] & BP\langle n+1\rangle_*(X) \otimes_{Z_{(p)}[x_{n+1}]} Z_{(p)} \arrow[r] \arrow[d, "\rho(n, n+1)"] & 0 \\
& & BP\langle n+1\rangle_*(X) \arrow[r, "\cdot x_{n+1}"] & BP\langle n+1\rangle_*(X) \arrow[r, "\rho(n, n+1)"] & BP\langle n\rangle_*(X) \arrow[ll, bend left, "\Delta_{n+1}"] &
\end{tikzcd} \tag{5.9}$$

The exact sequence (5.8) follows from (5.2). Note that $\widetilde{\rho(n, n+1)}$ is epic if and only if $\rho(n, n+1)$ is epic which occurs if and only if hom $\dim_{BP_*} BP_*(X) \leqslant n + 1$ by (3.8; 3.13).

PROPOSITION 5.10. *Let X be a finite complex. If* $\operatorname{Tor}_{1,*}^{BP_*}(BP_*(X), BP\langle n\rangle_*) = 0$, *then* hom $\dim_{BP_*} BP_*(X) \leqslant n + 1$.

Proof. The exact sequence

$$0 \to BP\langle n+1\rangle_* \xrightarrow{\cdot x_{n+1}} BP\langle n+1\rangle_* \to BP\langle n\rangle_* \to 0$$

induces the exact sequence

$$\operatorname{Tor}_{1,j-2p^{n+1}+2}^{BP_*}(BP_*(X), BP\langle n+1\rangle_*) \to \operatorname{Tor}_{1,*}^{BP_*}(BP_*(X), BP\langle n+1\rangle_*) \to \operatorname{Tor}_{1,*}^{BP_*}(BP_*(X), BP\langle n\rangle_*).$$

By an induction on dimensions, $\operatorname{Tor}_{1,*}^{BP_*}(BP_*(X), BP\langle n\rangle_*) = 0$ implies $\operatorname{Tor}_{1,*}^{BP_*}(BP_*(X), BP\langle n+1\rangle_*) = 0$. By iteration, this implies $\operatorname{Tor}_{1,*}^{BP_*}(BP_*(X)\ BP\langle m\rangle_*) = 0$

for $m \geqslant n$. hom $\dim_{BP_*} BP_*(X)$ is a finite number (4.5). Suppose hom $\dim_{BP_*} BP_*(X) = m + 1 > n + 1$. The short exact sequence

$$0 \to BP\langle m\rangle_* \xrightarrow{\cdot x_m} BP\langle m\rangle_* \to BP\langle m-1\rangle_* \to 0$$

induces the top exact row in the commutative diagram:

$$\begin{tikzcd} 0 = \mathrm{Tor}_{1,*}^{BP_*}(BP_*(X), BP\langle m-1\rangle_*) \arrow[r] & BP(X) \otimes_{BP_*} BP\langle m\rangle_* \arrow[r, "\cdot x_m"] \arrow[d, "\widetilde{\rho(m, \infty)}"] & BP_*(X) \otimes_{BP_*} BP\langle m\rangle_* \arrow[d, "\widetilde{\rho(m, \infty)}"] \\ & BP\langle m\rangle_*(X) \arrow[r, "\cdot x_m"] & BP\langle m\rangle_*(X). \end{tikzcd} \tag{5.11}$$

The $\widetilde{\rho(m, \infty)}$'s are isomorphisms (5.4); so multiplication x_m is monic in $BP\langle m\rangle_*(X)$. Thus hom $\dim_{BP_*} BP_*(X) \leqslant m$ (3.13) which is a contradiction. So the supposition hom $\dim_{BP_*} BP_*(X) > n+1$ is false.

PROPOSITION 5.12. *Let X be a finite complex. The following conditions are equivalent:*

(i) hom $\dim_{BP_*} BP_*(X) \leqslant n+1$;
(ii) $\widetilde{\rho(n, \infty)}$: $BP_*(X) \otimes_{BP_*} BP\langle n\rangle_* \to BP\langle n\rangle_*(X)$ *is an isomorphism;*
(iii) $\mathrm{Tor}_{1,*}^{BP_*}(BP_*(X), BP\langle n\rangle_*) = 0$;
(iv) $\mathrm{Tor}_{j,*}^{BP_*}(BP_*(X), BP\langle n\rangle_*) = 0, j > 0$;
(v) $\widetilde{\rho(n, n+1)}$: $BP\langle n+1\rangle_*(X) \otimes_{Z_{(p)}[x_{n+1}]} Z_{(p)} \to BP\langle n\rangle_*(X)$ *is an isomorphism;*
(vi) $\mathrm{Tor}_{1,*}^{Z_{(p)}[x_{n+1}]}(BP\langle n+1\rangle_*(X), Z_{(p)}) = 0$;
(vii) $\mathrm{Tor}_{j,*}^{Z_{(p)}[x_{n+1}]}(BP\langle n+1\rangle_*(X), Z_{(p)}) = 0, j > 0$.

Proof. The following implications have been established: (i) ⇒ (ii) (5.4); (i) ⇒ (iii) (5.4; (iii) ⇒ (i), (5.10); (iv) ⇒ (iii), (trivial); (i) ⇔ (v) ⇔ (vi), (5.7); and (vii) ⇒ (vi), (trivial). We now demonstrate (iii) ⇒ (iv) and (ii) ⇒ (i). The reader may complete the proof of (vi) ⇒ (vii) to finish the proposition.

To prove (ii) ⇒ (i), note that in diagram (5.6) with $Y = S^0$, ρ_X epic implies f_2 is epic and thus g_2 to be monic. Just as in the proof of (3.13), $BP\langle n\rangle_*(W)$ is T_n torsion free and hom $\dim_{BP_*} BP_*(X) \leqslant n+1$. If $W \to A \to X$ is a BP_* resolution of X and if $j > 1$, then there is an isomorphism

$$\partial\colon \mathrm{Tor}_{j,*}^{BP_*}(BP_*(X), BP\langle n\rangle_*) \to \mathrm{Tor}_{j-1,*}^{BP_*}(BP_*(W), BP\langle n\rangle_*).$$

The implication (i) ⇒ (iii) ⇒ (iv) is proved by induction over hom $\dim_{BP_*} BP_*(\ \ )$.

Remark 5.13. For any CW spectrum X, there is a natural isomorphism

$$T_n^{-1}\widetilde{\rho(n, \infty)}\colon BP_*(X) \otimes_{BP_*} T_n^{-1}BP\langle n\rangle_* \to T_n^{-1}BP\langle n\rangle_*(X).$$

Proof. If A is a finite complex with $H_*(A; Z_{(p)})$ $Z_{(p)}$ free, then $T_n^{-1}\rho(n, \infty)$ is an isomorphism by (5.1). If $W \to A \to X$ is a BP_* resolution for a finite complex X, then it is also a $T_n^{-1}BP\langle n\rangle_*$ resolution for X (use (3.7) and the proof of (3.15). Now one forms a diagram like (5.6) and establishes the remark for finite complexes by induction on the homological dimension. It generalizes to arbitrary CW spectra by (3.1).

Now we sketch the constructions of some spectral sequences related to our work. In this paper, our proofs do not employ these, but we have used such spectral sequences for first proofs and we feel these might be of independent interest in future investigations. For example, the original proofs of (3.6) and (6.5) used the following spectral sequence.

PROPOSITION 5.14. *Let X be a connected CW spectrum. There is a natural first quadrant spectral sequence*

$$E^2_{s,t}(X) \cong BP\langle n-1\rangle_s(X) \otimes (Z_{(p)}[x_n])_t \Rightarrow BP\langle n\rangle_*(X).$$

The differentials are T_{n-1} torsion valued and the spectral sequence collapses if and only if $\rho(n-1, n)\colon BP\langle n\rangle_*(X) \to BP\langle n-1\rangle_*(X)$ is epic.

Proof. Apply Corollary 2.5 of [17] to the BP Bockstein sequence (2.1). Each differential of the spectral sequence involves a T_{n-1}-torsion valued Δ_n (3.2) in its definition. (N.B. A key to sanity is to use the indexing conventions of the first derived exact couple of the exact couple on page 337 of [21].)

We shall now establish some spectral sequences which arise from the geometric resolution of a complex. The first such spectral sequence takes the form

$$E^2_{*,*}(X, Y) \cong \mathrm{Tor}^{BP_*}_{*,*}(BP_*(X), BP\langle n\rangle_*(Y)) \Rightarrow BP\langle n\rangle_*(X \wedge Y)$$

for $-1 \leqslant n \leqslant \infty$. For $n = \infty$ or $Y = S^0$, this spectral sequence appears in both [1] and [13]. Our main contribution is to put in the module theories $BP\langle n\rangle_*(\ \ )$ in a more significant way. The proofs are essentially those found in [1] and [13]; so we offer only sketches. We shall assume X and Y are finite complexes, although Lemma 5 of [20] may be used to extend the results to a more general setting.

PROPOSITION 5.15. *There is a natural first quadrant spectral sequence*

$$E^2_{s,t}(X, Y) = \mathrm{Tor}^{BP_*}_{s,t}(BP_*(X), BP\langle n\rangle_*(Y)) \Rightarrow BP\langle n\rangle_*(X \wedge Y)$$

with edge homomorphism

$$BP_*(X) \otimes_{BP_*} BP\langle n\rangle_*(Y) = E^2_{0,*}(X, Y) \to E^\infty_{0,*}(X, Y) \to BP\langle n\rangle_*(X \wedge Y)$$

identified with the external product. BP_* acts on $E^r_{*,*}(X, Y)$ commuting with differentials (which have bidegree $(-r, r-1)$).

Sketch proof. We establish a diagram

$$\begin{tikzcd}
* & & W_{m-1} \arrow[ll] \arrow[dr, "g_{m-1}"] & & W_{m-2} \arrow[ll, "h_{m-1}"'] & \cdots & W_1 \arrow[dr, "g_1"] & & W_0 \arrow[ll, "h_1"'] \arrow[dr, "g_0"] & & W_{-1} = X \arrow[ll, "h_0"'] \\
& A_m \arrow[ul, leftarrow] \arrow[ur, "f_m"'] & & A_{m-1} \arrow[ur, "f_{m-1}"'] & & & & A_1 \arrow[ur, "f_1"'] & & A_0 \arrow[ur, "f_0"'] &
\end{tikzcd} \tag{5.16}$$

where each cofibration

$$W_j \xrightarrow{g_j} A_j \xrightarrow{f_j} W_{j-1}$$

is a BP_* resolution of W_{j-1} in the sense of (3.11) (f_{j*} is epic and $BP_*(A_j)$ is BP_*-free.)

$$0 \to BP_*(A_m) \xrightarrow{(g_{m-1} \circ f_m)_*} BP_*(A_{m-1}) \to \cdots \to BP_*(A_1) \xrightarrow{(g_0 \circ f_1)_*} BP_*(A_0) \xrightarrow{f_{0*}} BP_*(X) \to 0$$

is a free BP_* resolution of the module $BP_*(X)$. Apply the functor $BP\langle n\rangle_*(- \wedge Y)$ to the diagram (5.16) to obtain an exact couple with $E^1_{j,*}(X, Y) = BP\langle n\rangle_*(A_j \wedge Y) \cong BP_*(A_j) \otimes_{BP_*} BP\langle n\rangle_*(Y)$ (5.1) and $d^1_{j,*} = (g_{j-1} \circ f_j)_* \otimes 1$. Identification of $E^*_{2,*}(X, Y)$ and convergence of the resulting spectral sequence are then routine. The proof of naturality is naturally tedious and is done in detail in [13].

PROPOSITION 5.17. *There is a natural spectral sequence* $E_2^{s,t}(X, Y) = \mathrm{Ext}^{s,t}_{BP_*}(BP_*(X), BP\langle n\rangle^*(Y)) \Rightarrow BP\langle n\rangle^*(X \wedge Y)$ *with edge homomorphism* $BP\langle n\rangle^*(X \wedge Y) \to E_\infty^{0,*}(X, Y) \subset E_2^{0,*}(X, Y) = \mathrm{Hom}_{BP_*}(BP_*(X), BP\langle n\rangle^*(Y))$ *induced by the slant product (see pp. 258f of [37]).* $BP_* \cong BP^{-*}$ *acts on* $E_r^{*,*}(X, Y)$ *and commutes with the differentials (which have bidegree* $(r, 1-r)$*).*

Proof. The proof is analogous to that of (5.15) with the following lemma replacing (5.1).

LEMMA 5.18. *Let A be a finite complex with $BP_*(A)$ free BP_*. For any finite complex Y there is a natural isomorphism induced by the slant product*

$$BP\langle n\rangle^*(A \wedge Y) \to \mathrm{Hom}_{BP_*}(BP_*(A), BP\langle n\rangle^*(Y)).$$

Proof. First establish the isomorphism for $Y = S^0$ and then use the uniqueness theorem for cohomology theories.

Remark 5.19. There are dual spectral sequences to (5.15) and (5.17): just change the upper stars to lower stars and vice versa. They can be obtained by applying S-duality to (5.15) and (5.17) since we are using finite complexes; or, one can build a BP^* geometric resolution for $BP^*(X)$ directly. Because $BP_*(X)$ and $BP^*(X)$ can have dramatically different module structure (see 4.29), it may be possible to play one against the other.

Remark 5.20. The spectral sequences obtained above can be done in much greater generality. Using Sullivan's theory of manifolds with singularities [7], we can kill off selected polynomial generators of MU_* to obtain a homology theory $MUS_*(\ )$. MUS is then a module theory over MU. The same proofs give spectral sequences once we replace BP by MU and $BP\langle n\rangle$ by MUS.

§6. SPHERICAL CLASSES IN $BP_*(X)$

Choose generators $\iota_\infty \in BP_m(S^m)$, compatible with suspensions, and let $\iota_n = \rho(n, \infty)\iota_\infty \in BP\langle n\rangle_m(S^m)$. For each $-1 \leqslant n \leqslant \infty$, there is a natural Hurewicz homomorphism η_n: $\pi^S_*(X) \to BP\langle n\rangle_*(X)$ which sends the class of a stable map f: $S^m \to X$ to $f_*\iota_n \in BP\langle n\rangle_m(X)$. A class of $BP\langle n\rangle_*(X)$ is said to be spherical if it is in the image of η_n.

THEOREM 6.1. *Let X be a finite complex. The Hurewicz homomorphism*

$$\eta_n\colon \pi^S_*(BP \wedge X) = BP_*(X) \to BP\langle n\rangle_*(BP \wedge X)$$

is a monomorphism if hom $\dim_{BP_*} BP_*(X) \leqslant n$.

Proof. We do the $n = \infty$ case first. η_∞: $\pi_*^S(BP \wedge X) \to \pi_*(BP \wedge BP \wedge X) = BP_*(BP \wedge X)$ has a left inverse induced by the pairing $BP \wedge BP \to BP$. So η_∞ is split monic.

Now suppose f: $S^m \to BP \wedge X$ is a stable map representing a class in the kernel of η_n. $BP\langle n\rangle_*(f)(\iota_n) = 0$ implies $BP\langle n\rangle_*(f) \equiv 0$. Let $S^m \xrightarrow{f} BP \wedge X \xrightarrow{g} C \xrightarrow{h} S^{m+1}$ be the stable cofibration sequence induced by f. It then induces a short exact sequence:

$$0 \to BP\langle n\rangle_*(BP \wedge X) \xrightarrow{g_1} BP\langle n\rangle_*(C) \to BP\langle n\rangle_*(S^{m+1}) \to 0. \tag{6.2}$$

Now we assume hom $\dim_{BP_*} BP_*(X) \leqslant n$. Since $H_*(BP; Z_{(p)})$ is $Z_{(p)}$-free, $BP\langle n\rangle_*(BP \wedge X) \cong BP_*(BP) \otimes_{BP_*} BP\langle n\rangle_*(X)$ (5.1). x_n acts monomorphically on $BP\langle n\rangle_*(X)$ (3.13) and $BP_*(BP) \otimes_{BP_*}$— is an exact functor; so x_n multiplication in $BP\langle n\rangle_*(BP \wedge X)$ is monic. It is certainly monic in $BP\langle n\rangle_*(S^{m+1})$; so a simple application of the "five" lemma to two copies of (6.2) implies that x_n multiplication is monic in $BP\langle n\rangle_*(C)$. Recall that if x_n multiplication is monic in $BP\langle n\rangle_*(Y)$, then x_{n+1} multiplication is monic in $BP\langle n+1\rangle_*(Y)$ for $Y = BP \wedge X$ or C (3.6). Thus the columns of commutative diagram (6.3) are exact.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
 & BP\langle n+1\rangle_*(BP \wedge X) \arrow[r, "g_3"] \arrow[d, "\cdot x_{n+1}"] & BP\langle n+1\rangle_*(C) \arrow[d, "\cdot x_{n+1}"] \\
 & BP\langle n+1\rangle_*(BP \wedge X) \arrow[r, "g_2"] \arrow[d, "\rho(n,n+1)"] & BP\langle n+1\rangle_*(C) \arrow[d, "\rho(n,n+1)"] \\
0 \arrow[r] & BP\langle n\rangle_*(BP \wedge X) \arrow[r, "g_1"] \arrow[d] & BP\langle n\rangle_*(C). \arrow[d] \\
 & 0 &
\end{tikzcd} \tag{6.3}$$

By an induction on dimensions, beginning in negative dimensions, g_3 is monic. So g_2 is monic by the "five" lemma. So $BP\langle n+1\rangle_*(g)$ is monic. Continuing inductively, we see that $BP\langle N\rangle_*(g)$ is monic for $N \geqslant n$. For N sufficiently large, $BP_m(g) = BP\langle N\rangle_m(g)$. Thus $0 = BP\langle N\rangle_*(f)(\iota_0) = BP_*(f)(\iota_\infty) = \eta_\infty$ $[f]$. Since η_∞ is a monomorphism, $[f] = 0$. We conclude that when hom $\dim_{BP_*} BP_*(X) \leqslant n$, the kernel of η_n is trivial.

Remark 6.4. Recall from (2.7) that $BP\langle 1\rangle$ is equivalent to G, the Adams–Anderson–Meiselman summand of connective k-theory localized at the prime p. It was shown in Proposition 6.2 of [17] that the $n = 1$ case of Theorem 6.1 is equivalent to: the Hurewicz homomorphism

$$\eta_1: BP_*(X) \to G_*(BP \wedge X)$$

is a $Z_{(p)}$ split monomorphism for a finite complex X with hom $\dim_{BP_*} BP_*(X) = 0$. It is well known [4] and elementary to show that this last statement is equivalent to the Hattori form of the Stong–Hattori theorem [4; 15; 16; 28; 34].

COROLLARY 6.5. *Let $f\colon X \to Y$ be a stable map of finite complexes. If* hom $\dim_{BP_*} BP_*(X) \leq n$ *and if $BP\langle n\rangle_*(f)\colon BP\langle n\rangle_*(X) \to BP\langle n\rangle_*(Y)$ is monic, then $BP_*(f)\colon BP_*(X) \to BP_*(Y)$ is also monic.*

Proof. The corollary follows immediately from the commutative diagram (6.6)

$$\begin{array}{ccc}
0 & & \\
\downarrow & & \\
BP_*(X) & \xrightarrow{\quad BP_*(f) \quad} & BP_*(Y) \\
\downarrow{\scriptstyle \eta_n} & & \downarrow{\scriptstyle \eta_n} \\
BP\langle n\rangle_*(BP \wedge X) & & BP\langle n\rangle_*(BP \wedge Y) \\
\cong & & \cong \\
0 \to BP_*(BP) \otimes_{BP_*} BP\langle n\rangle_*(X) & \xrightarrow{BP_*(BP) \otimes_{BP_*} BP\langle n\rangle_*(f)} & BP_*(BP) \otimes_{BP_*} BP\langle n\rangle_*(Y)
\end{array} \tag{6.6}$$

COROLLARY 6.7 (Conner). *Let X be a finite complex. If $0 \neq \gamma \in (x_{n+1}, x_{n+2}, \cdots)\cdot BP_*(X)$* $\subseteq$ kernel $\rho(n, \infty)$ *is spherical, then* hom $\dim_{BP_*} BP_*(X) \geq n + 1$.

Proof. Let $f\colon S^m \to X$ be a stable map representing γ in that $BP_*(f)\iota_\infty = \gamma$. Let $g\colon X \to Y$ be the canonical map to the cofibre of f. Since $0 = \rho(n, \infty)\gamma = \rho(n, \infty) BP_*(f)\iota_\infty = BP\langle n\rangle_*(f)\,\rho(n, \infty)\,\iota_\infty = BP\langle n\rangle_*(f)\iota_n$, $BP\langle n\rangle_*(g)$ is monic. Since $0 \neq \gamma = BP_*(f)\iota_\infty \in$ kernel $BP_*(g)$, hom $\dim_{BP_*} BP_*(X) > n$ by (6.5).

CONJECTURE 6.8. *If $S^m \xrightarrow{f} X \xrightarrow{g} Y$ is a cofibration sequence of finite complexes and if* hom $\dim_{BP_*} BP_*(X) \leq n$, *then* hom $\dim_{BP_*} BP_*(Y) \leq n + 1$.

Remark 6.9. We once advertised (6.8) as being proved. P. E. Conner observed that (6.7) is a corollary of (3.13) and (6.8). Although we found a gap in our argument for (6.8), Conner's observation motivated our proof of Theorem 6.1.

The main result of [18] is that if X is a finite complex with k-skeleton X^k, then hom $\dim_{MU_*} MU_*(X) \leq n$ implies hom $\dim_{MU_*} MU_*(X^k) \leq n$ provided that $n = 0, 1, 2$. Then $n = 2$ version follows from Proposition 6.10 by skeletal induction. The reader is invited to construct his own proof of the $n = 1$ version.

PROPOSITION 6.10. *Let $\vee S^m \xrightarrow{f} X \xrightarrow{g} Y$ be a cofibration of finite complexes. Suppose the cellular dimension of X is at most m. If* hom $\dim_{BP_*} BP_*(Y) \leq 2$, *then* hom $\dim_{BP_*} BP_*(X) \leq 2$.

Proof. Suppose hom $\dim_{BP_*} BP_*(X) > 2$; so there is an element $0 \neq a \in BP\langle 2\rangle_*(X)$ with $x_2 \cdot a = 0$ (3.13). We may write $a = x_2{}^t \cdot b$ with $0 \neq \rho(1, 2)(b) \in BP\langle 1\rangle_*(X)$. b is a T_2

torsion element and thus is a T_1 torsion element (3.2). So $\rho(1, 2)(b)$ is T_1 torsion. Consider the BP Bockstein sequence

$$H_{s+2p-1}(X; Z_{(p)}) \xrightarrow{\Delta_1} BP\langle 1\rangle_s(X) \xrightarrow{\cdot x_1} BP\langle 1\rangle_{s+2p-2}(X).$$

Since $H_{s+2p-1}(X; Z_{(p)}) = 0$ for $s + 2p - 1 > m$, x_1 multiplication is monic in dimensions greater than $m - 2p + 1$. We conclude that the dimension of b is well below m and thus $0 \neq g_*(b) \in BP\langle 2\rangle_*(Y)$. Since b is T_2 torsion, this would give T_2 torsion in $BP\langle 2\rangle_*(Y)$ contradicting the hypothesis that hom $\dim_{BP_*} BP_*(Y) \leq 2$ (3.13).

Question 6.11. If $0 \neq y \in BP\langle n\rangle_s(X)$ is a T_n torsion element, can $s+1$ be greater than the cellular dimension of X?

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