The Skinner–Urban method and the Symmetric cube Bloch–Kato conjecture

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December 5, 2020

Part I: The method in general

Fix:

- p a prime,
- V a finite dimensional $\overline{\mathbb{Q}}_p$-vector space,
- $\rho : G_{\mathbb{Q}} \to \mathrm{Aut}(V)$ a continuous representation.

Assume:

- ρ comes from an automorphic representation, and
- ρ is normalized so that the center of the functional equation for $L(s, V)$ is at $s = 0$.

Then Bloch–Kato predicts:

Conjecture (Consequence of Bloch–Kato)

$$L(0, V) = 0 \quad \implies \quad H^1_f(\mathbb{Q}, V^\vee(1)) \neq 0.$$

Conjecture (Consequence of Bloch–Kato)

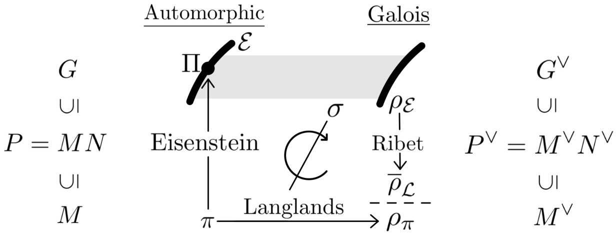
$$L(0, V) = 0 \quad \Longrightarrow \quad H^1_f(\mathbb{Q}, V^\vee(1)) \neq 0.$$

Here, $H^1_f(\mathbb{Q}, V^\vee(1))$ denotes a *Bloch-Kato Selmer group*, consisting of classes of cocycles

$$\sigma : G_{\mathbb{Q}} \to V^\vee(1)$$

which are:

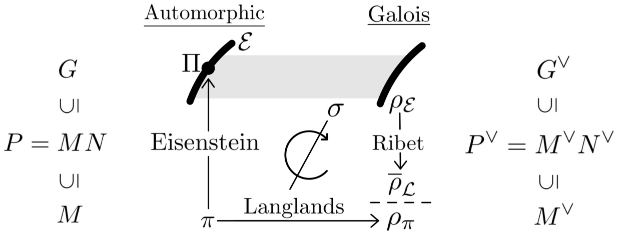
- unramified at $\ell \neq p$, and
- crystalline at p.



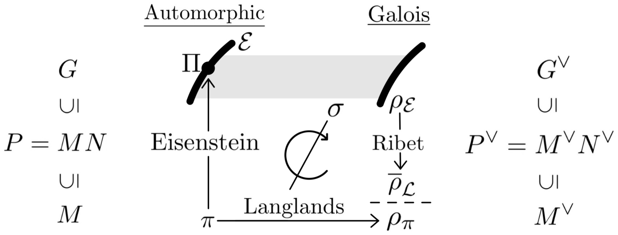
- M is a reductive group over $\mathbb{Q}$,
- π is a "nice" automorphic representation of M,
- $\rho_\pi : G_{\mathbb{Q}} \to M^\vee(\overline{\mathbb{Q}}_p)$ is the Galois representation attached to π,
- R is a representation of $M^\vee$.

Goal

Assuming $L(0, R \circ \rho_\pi) = 0$, construct a cocycle σ giving a nontrivial class in $H^1_f(\mathbb{Q}, (R \circ \rho_\pi)^\vee(1))$.

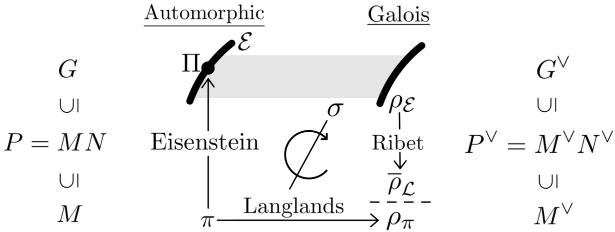


- Embed M in G as the Levi of a parabolic $P = MN$.
- Get a parabolic $P^\vee$ with Levi $M^\vee$ on the dual side.
- We need $N^\vee/[N^\vee, N^\vee] \cong R$ as representations of $M^\vee$.
- Take a functorial lifting Π of π from M to G.
- p-adically deform Π in a generically cuspidal family $\mathcal{E}$.
- Let A be the affinoid $\overline{\mathbb{Q}}_p$-algebra parametrizing $\mathcal{E}$.



- Get a family of Galois representations $\rho_{\mathcal{E}} : G_{\mathbb{Q}} \to G^\vee(\mathrm{Frac}(A))$.
- Choose wisely a lattice $\mathcal{L}$ in $\rho_{\mathcal{E}}$ and specialize at Π to get $\overline{\rho}_{\mathcal{L}}$.
- Then $\overline{\rho}_{\mathcal{L}} : G_{\mathbb{Q}} \to G^\vee(\overline{\mathbb{Q}}_p)$.
- If constructed correctly, $\overline{\rho}_{\mathcal{L}}$ should have image in $P^\vee(\overline{\mathbb{Q}}_p)$.

The diagram *should not commute*, the failure being measured by σ.



Some remarks:

- We need to use the hypothesis that $L(0, \pi, R) = 0$.

This is usually used to show that Π has good properties at ∞ which make it amenable to p-adic deformation.

- Various pieces of "numerology" must hold.

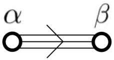
Previous instances of the method:

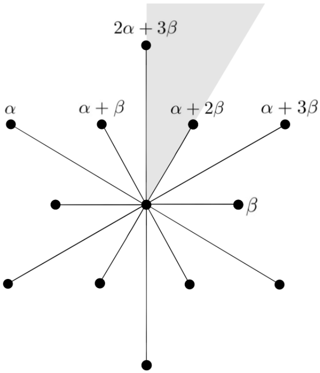
Skinner–Urban (unpublished; ICM 2006):
$M = U(a, b)$ and $G = U(a+1, b+1)$.
They get Selmer classes over an imaginary quadratic field K.

Skinner–Urban (2006):
$M = GL_2$ and $G = GSp_4$.
π comes from a modular form and Π is a Saito–Kurokawa lift.

Part II: The case when $G = G_2$

Let G now be split G_2 over $\mathbb{Q}$.





$G_2 = G_2^\vee$, and passing to dual switches long and short simple roots.

Let $P = MN$ be the long root parabolic; M contains α.
Let $P^\vee = M^\vee N^\vee$ be the short root parabolic; $M^\vee$ contains β.

Then $M \cong GL_2$ and $M^\vee \cong GL_2$.

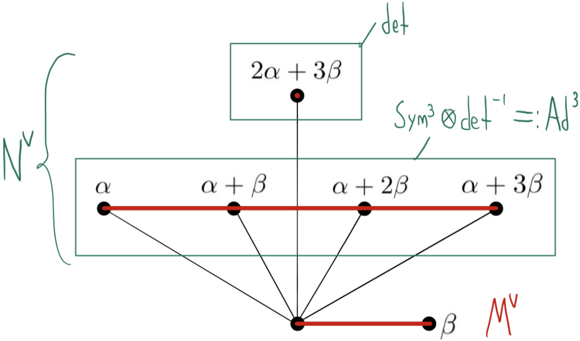
We will apply the Skinner–Urban method to $M \subset G_2$.

Let F be a cuspidal holomorphic eigenform with:

- Trivial nebentypus,
- Weight $k \geq 4$ even,
- Level N with $p \nmid N$.

Let π be the cuspidal automorphic representation of GL_2 associated with F, normalized to be unitary.
It has trivial central character.

Let ρ_F be the Galois representation attached to F by Deligne.
(Note ρ_F corresponds, not directly to π, but to a twist of π.)



We see that $N^\vee/[N^\vee, N^\vee] \cong \mathrm{Ad}^3$, as representations of $M^\vee$.

Goal

Use the Skinner–Urban method to show

$$L(1/2, \pi, \mathrm{Ad}^3) = 0 \quad \Longrightarrow \quad H^1_f(\mathbb{Q}, (\mathrm{Ad}^3\, \rho_F)^\vee(k/2)) \neq 0.$$

Remarks:

- Work in progress.
- Many steps in the Skinner–Urban method for this case will introduce extra hypotheses or dependencies on certain standard conjectures.

Let $\Pi = \mathcal{L}(\pi, 1/10)$, the Langlands quotient of the parabolic induction $\mathrm{Ind}_{P(\mathbb{A})}^{G_2(\mathbb{A})}(\pi \otimes \delta_{P(\mathbb{A})}^{1/10+1/2})$.
This is our functorial lift, which we would like to p-adically deform in a generically cuspidal family.

Numerology: $\delta_{P(\mathbb{A})}^{1/10}$ acts as $|\det|^{1/2}$ through $M(\mathbb{A})$. Therefore the center of the functional equation for $\pi \otimes \delta_{P(\mathbb{A})}^{1/10}$ is at $s = 0$.

Step 1: Locate Π in Eisenstein cohomology.

Write $\Pi = \Pi_f \otimes \Pi_\infty$.
Let Λ be the weight $\frac{k-4}{2}(2\alpha + 3\beta)$.
Let V_Λ be the representation of G_2 of highest weight Λ.

Theorem

Assume $L(1/2, \pi, \mathrm{Ad}^3) = 0$. Then Π_f appears exactly once in the Eisenstein cohomology

$$H^*_{\mathrm{Eis}}(X_{G_2}, V_\Lambda)$$

and it appears in (middle) degree 4.

Step 2: Locate Π in cuspidal cohomology.

Theorem

Assume $L(1/2, \pi, \mathrm{Ad}^3) = 0$. *Under Arthur's conjectures, we have*

$$H^*_{\mathrm{cusp}}(X_{G_2}, V_\Lambda)[\Pi_f] = \begin{cases} \Pi_f & \text{if } \epsilon(1/2, \pi, \mathrm{Ad}^3) = +1, \text{ and } * = 3, 5 \\ & \text{or } \epsilon(1/2, \pi, \mathrm{Ad}^3) = -1, \text{ and } * = 4; \\ 0 & \text{otherwise.} \end{cases}$$

The following steps are in progress:

Step 3: Using Steps 1 and 2, p-adically deform a critical p-stabilization of Π in a generically cuspidal family $\mathcal{E}$.

This uses Urban's eigenvariety, which can be used to p-adically deform cohomological representations under certain circumstances. This requires us to assume $\epsilon(1/2, \pi, \mathrm{Ad}^3) = -1$.

Step 4: Construct the family $\rho_{\mathcal{E}} : G_{\mathbb{Q}} \to G_2(\mathrm{Frac}(A))$ of Galois representations.

Thus uses Lafforgue's pseudo-characters, and assumes a global Langlands correspondence for the cuspidal members of $\mathcal{E}$.

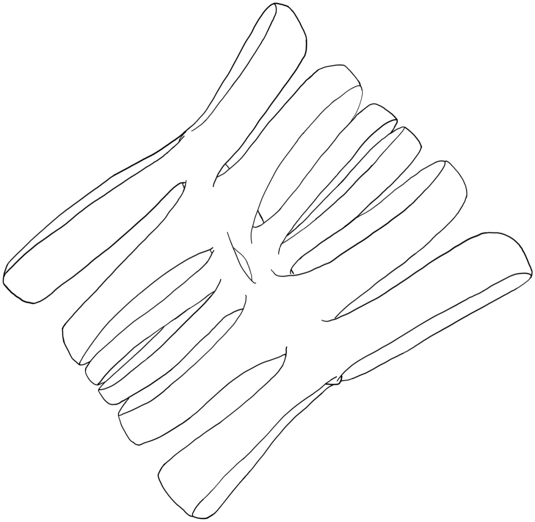
Step 5: Construct the lattice $\rho_{\mathcal{L}}$ and show that the specialization $\overline{\rho}_{\mathcal{L}}$ takes values in $G_2(\overline{\mathbb{Q}}_p)$.

A priori, $\overline{\rho}_{\mathcal{L}}$ only lands in $GL_7(\overline{\mathbb{Q}}_p)$. Matrix coefficient arguments can be used to show that it preserves a certain trilinear form, forcing it to land in $G_2(\overline{\mathbb{Q}}_p)$.

We need F to be non-CM to use a big image argument here.

Step 6: Show that the cocycle coming from $\overline{\rho}_{\mathcal{L}}$ is in the correct Selmer group.

The Selmer conditions come from local properties of $\mathcal{E}$.



Thank you!