DIFFEOMORPHISM TYPE OF THE BERGER SPACE $\mathrm{SO}(5)/\mathrm{SO}(3)$

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Abstract. We compute the Eells-Kuiper invariant of the Berger manifold $\mathrm{SO}(5)/\mathrm{SO}(3)$ and determine that it is diffeomorphic to the total space of an $\mathbf{S}^3$-bundle over $\mathbf{S}^4$. This answers a question raised by K. Grove and W. Ziller.

Introduction. There has been renewed interest in Riemannian structures of nonnegative or positive curvature on the total spaces of $\mathbf{S}^3$-bundles over $\mathbf{S}^4$. These bundles have been of interest to topologists since Milnor showed that if the Euler class of such a bundle is ± 1, then the total space is a homotopy sphere. Until recently there was only one exotic sphere, the so called Gromoll-Meyer sphere (cf. [GM74]), which was known to admit a metric of nonnegative curvature. Then in their paper [GZ00], K. Grove and W. Ziller showed that every $\mathbf{S}^3$-bundle over $\mathbf{S}^4$ admits infinitely many metrics of nonnegative curvature. In particular, all the Milnor spheres i.e., homotopy 7-spheres that arise as $\mathbf{S}^3$-bundles over $\mathbf{S}^4$ admit such metrics. Which of course begs the question: which Milnor spheres, or more generally, which $\mathbf{S}^3$-bundles over $\mathbf{S}^4$ admit metrics of positive sectional curvature?

The Berger space, $M^7 = \mathrm{SO}(5)/\mathrm{SO}(3)$, was first described by M. Berger as a manifold that admits a (normal) homogeneous metric of positive sectional curvature. The embedding of $\mathrm{SO}(3)$ in $\mathrm{SO}(5)$ is maximal and irreducible (cf. [Wo68]), it is a rational homology sphere with $H^4(M, \mathbf{Z}) = \mathbf{Z}_{10}$ (cf. [Ber61]) and it has the cohomology ring of an $\mathbf{S}^3$-bundle over $\mathbf{S}^4$. In [GZ00], K. Grove and W. Ziller asked whether the Berger space is topologically or differentially equivalent (as a manifold) to an $\mathbf{S}^3$-bundle over $\mathbf{S}^4$. Part of this was settled in [KiSh01] where it was shown that the Berger space is *PL*-homeomorphic to such a bundle. To settle the diffeomorphism question requires computing the Eells-Kuiper invariant ek(M). The original definition of ek(M) in [EK62] requires that M^7 be written as the boundary of an eight dimensional spin manifold. Since the cobordism group Ω_7 is known to be trivial, one knows that any closed, 2-connected, 7-manifold admits a spin coboundary. However, an explicit coboundary for the Berger space has not been found.

Instead, we use the analytic formula (2.1) for the Eells-Kuiper invariant due to Donnelly [Don75] and Kreck-Stolz [KS88], which is based on the Atiyah-Patodi-Singer index theorem [APS75]. This formula expresses $\mathrm{ek}(M)$ in terms of the η-invariants of the odd signature operator B and the untwisted Dirac operator D. To determine these η-invariants, we follow the first author's approach in [Go97], [Go99], [Go02]; we first replace the operators B and D that are associated to the Levi-Civita connection on M by operators of the kind $D^{\frac{1}{3}}$. These operators are particularly well adapted to representation theoretic methods, which was first noticed by Slebarski in [Sle87], and later exploited e.g. in [Go97], [Go99] and [Kos99]. In particular, we use the explicit formula for η-invariants from [Go02]. As a consequence of this approach, we can employ the reductive connection on M to compute the secondary Pontrijagin number in (2.1) which is easier than using the Levi-Civita connection. This is accomplished by exploiting the existence of an equivariant G_2 structure on the tangent bundle TM. Our main result is:

THEOREM 1. *Depending on the orientation the Eells-Kuiper invariant of the Berger space $M = \mathrm{SO}(5)/\mathrm{SO}(3)$ is*

$$\mathrm{ek}(M) = \pm\frac{27}{1120}.$$

Note that for our choice of orientation (which descends from a choice of orientation on the octonions) the exact value we obtain is $\mathrm{ek}(M) = -\frac{27}{1120}$. By [KiSh01] we know that the Berger space may be given an orientation so as to have the oriented *PL*-type of some $\mathbf{S}^3$-bundle $M_{m,10}$ over $\mathbf{S}^4$ with Euler class 10 and Pontrijagin class $2(10 + 2m) \in \mathbf{Z} \cong H^4(\mathbf{S}^4)$ with respect to the standard generator (in the notation of [CE03]). So the value of the *PL*-invariant $s_1(M) = 28\,\mathrm{ek}(M) \in \mathbf{Q}/\mathbf{Z}$ of [KS88] equals $\frac{13}{40}$.

Given a pair of 2-connected, 7-manifolds M_1 and M_2, they are *PL*-homeomorphic to each other if and only if there exists an exotic sphere Σ so that $M_1 \# \Sigma = M_2$. This is a consequence of the fact that *PL/O* is 6-connected (cf. [MM79]). Moreover, the *EK*-invariant is additive with respect to connected sums and attains 28 distinct values on the group of exotic 7-spheres. Hence, M_1 and M_2 are oriented diffeomorphic if and only if they are *PL*-homeomorphic and have the same *EK*-invariant. This was used in [CE03] to do the diffeomorphism classification of $\mathbf{S}^3$-bundles over $\mathbf{S}^4$. In fact, by [CE03] one knows that $\mathrm{ek}(M_{m,n}) = -\mathrm{ek}(M_{-m,-n}) = \frac{(n+2m)^2-n}{2^5\cdot 7\cdot n} \in \mathbf{Q}/\mathbf{Z}$ for $n > 0$. We have the following corollary.

COROLLARY 2. *The Berger space is diffeomorphic to $M_{\pm 1,\mp 10}$, the $\mathbf{S}^3$-bundle over $\mathbf{S}^4$ with Euler class ∓ 10 and first Pontrijagin class equal to ∓ 16 times the generator in $H^4(\mathbf{S}^4)$ with respect to the standard choice of orientation on $\mathbf{S}^4$.*

Remark. In general, any $\mathbf{S}^3$-bundle over $\mathbf{S}^4$ with nonvanishing Euler class $n \in \mathbf{Z} \cong H^4(\mathbf{S}^4)$ is diffeomorphic to infinitely many other $\mathbf{S}^3$-bundles over $\mathbf{S}^4$ with the

same Euler class. It follows from Corollary 1.6 in [CE03] that the Berger space with the orientation specified in (2.4) is orientation reversing diffeomorphic to $M_{m,n}$ (orientation preserving diffeomorphic to $M_{-m,-n}$) if and only if $n = 10$ and m is congruent modulo 140 to -1, -9, -29 or 19; this was pointed out to us by C. Escher. Note that there is no space $M_{m,10}$ that is orientation preserving diffeomorphic to $M_{1,-10}$.

We also mention another consequence of Theorem 1. It is a natural question to ask: what is the largest *degree of symmetry* for $\mathbf{S}^3$-bundles over $\mathbf{S}^4$? The degree of symmetry of a Riemannian manifold is the dimension of its isometry group. For instance, it is well known that the maximal degree of symmetry for exotic 7-spheres is 4 (cf. [Str94]). However, some $\mathbf{S}^3$-bundles over $\mathbf{S}^4$ admit actions of larger groups. It follows from [On66, Theorem 4] (see also [Kla88]), that apart from the trivial bundle $\mathbf{S}^4 \times \mathbf{S}^3$, the only seven dimensional homogeneous manifolds that have the cohomology of an $\mathbf{S}^3$-bundle over $\mathbf{S}^4$ are $\mathbf{S}^7 = \mathrm{SO}(8)/\mathrm{SO}(7)$, $T_1\mathbf{S}^4 = \mathrm{Sp}(2)/\Delta\mathrm{Sp}(1)$, the unit tangent bundle of $\mathbf{S}^4$, the Berger space $M = \mathrm{SO}(5)/\mathrm{SO}(3)$, as observed by K. Grove and W. Ziller in [GZ00]. The spaces $\mathbf{S}^4 \times \mathbf{S}^3$, $\mathbf{S}^7$ and $T_1\mathbf{S}^4$ are diffeomorphic to principal $\mathbf{S}^3$-bundles over $\mathbf{S}^4$. On the other hand, it was shown in [GZ00] that the Berger space is not diffeomorphic (or even homeomorphic) to a principal $\mathbf{S}^3$-bundle over $\mathbf{S}^4$ (since its first Pontrijagin class does not vanish), but it is homotopy equivalent to a principal $\mathbf{S}^3$-bundle over $\mathbf{S}^4$.

COROLLARY 3. *Up to diffeomorphism, the only total spaces of* $\mathbf{S}^3$-*bundles over* $\mathbf{S}^4$ *that are homogeneous are the trivial bundle, the Hopf bundle, the unit tangent bundle of* $\mathbf{S}^4$*, and the Berger space* $\mathrm{SO}(5)/\mathrm{SO}(3)$.

The paper is organized as follows: in Section 1 we motivate the question of the diffeomorphism type from problems in the geometry of positive curvature. In Section 2 we compute the Eells-Kuiper invariant for the Berger space using spectral theory. In Section 3 we discuss the existence of independent vector fields on 2-connected 7-manifolds.

1. Motivation. It is a general problem in Riemannian geometry to find and describe closed manifolds that admit a metric of positive sectional curvature. There are few known obstructions and frustratingly, few known examples. The

difficulty stems from the fact that all known examples arise as quotients of Lie groups—as homogeneous spaces or as biquotients (double coset manifolds). Simply connected, homogeneous manifolds with positive curvature were classified by Berger, Wallach and Berard-Bergery in the sixties and seventies. The Berger space evidently appears in the classification of normal homogeneous manifolds of positive curvature due to M. Berger [Ber61]. Other than the homogeneous spaces of positive curvature there are some examples in low dimensions, but in dimensions 25 and up the only known examples are the compact, rank one, symmetric spaces.

The only way we know to construct examples of positively curved manifolds is to look at quotients of compact Lie groups. By the Gray-O'Neill curvature formulas, submersions are curvature nondecreasing. So one looks for positive curvature at the base of a Riemannian submersion. However, all known examples of positively curved manifolds, except the Berger space, fit into a fibration sequence, like the Hopf fibration of spheres over projective spaces. Fibrations may provide us with another way to construct examples of positively curved manifolds by the following method: Given a principal G-bundle, $G \to P \to B$, a *connection metric* on P is a choice of principal connection ω i.e., a choice of horizontal space $\mathcal{H}_G$ in P invariant under G such that the map $P \to B$ is a Riemannian submersion with totally geodesic fibers. The fibers are all isometric to each other and the metric on any fiber is isometric to $(G, \langle, \rangle)$ for some choice of left invariant metric on G. By Hermann [He60], every submersion metric on P with totally geodesic fibers must be of this form. Now we look at associated bundles $G/H \to M = P \times_G G/H \to B$ with fiber G/H. We declare the fibers to be orthogonal to the horizontal spaces $\mathcal{H}$, where $\mathcal{H}$ in M is the image of $\mathcal{H}_g \times \{0\} \subset T(P \times G/H)$. The metric on the total space is taken to be the orthogonal sum of the metrics on the fibers, where each fiber is isometric to G/H with a normal homogeneous metric (or more generally a left invariant metric), and the pullback of the metric on the base.

If we have a fibration with a connection metric, then the fiber G/H, which is totally geodesic, must be a circle or a normal homogeneous space of positive curvature. All known homogeneous spaces of positive curvature fit into fibrations with connection metrics, except the Berger space. Furthermore, Derdzinski and Rigas have shown in [DR81] that for $\mathbf{S}^3$-bundles over $\mathbf{S}^4$, the only bundle that admits a connection metric of positive curvature is the Hopf bundle whose total space is the round sphere. Since we now know that the Berger space is diffeomorphic to the total space of an $\mathbf{S}^3$-bundle over $\mathbf{S}^4$, it follows that its metric is not a connection metric. If one could find an explicit smooth submersion to $\mathbf{S}^4$, then we could check whether the positive curvature metric is a submersion metric. At the very least it makes plausible the suggestion that there are more general metrics of positive curvature on bundles with large degree of symmetry that are not connection metrics.

2. The Eells-Kuiper invariant of $SO(5)/SO(3)$**.** To compute the Eells-Kuiper invariant we use the formula

$$(2.1) \qquad \mathrm{ek}(M) = \frac{\eta(B)}{2^5\,7} + \frac{\eta(D) + h(D)}{2}$$

$$- \frac{1}{2^7\,7} \int_M p_1(M, \nabla^{\mathrm{LC}}) \wedge h(M, \nabla^{\mathrm{LC}}) \qquad \in \mathbf{Q}/\mathbf{Z}$$

due to Donnelly [Don75] and Kreck-Stolz [KS88]. Here B and D are the odd signature operator and the untwisted Dirac operator on M, and $h(M, \nabla^{\mathrm{LC}}) \in \Omega^3(M)$ is a form whose exterior differential is the first Pontrijagin form $p_1(M, \nabla^{\mathrm{LC}})$ with respect to the Levi-Civita connection. Equation (2.1) has the advantage that we do not need to find an explicit zero spin bordism for M.

Now we perform all the computations necessary to determine a numerical value for the Eells-Kuiper invariant $\mathrm{ek}(M)$ for $M = SO(5)/SO(3)$ using the methods of [Go97], [Go99], [Go02]. In Section 2.1 we recall the G_2-structure on TM, which will be important for calculations throughout this chapter. In Section 2.2 we control the spectral flow of the deformation of the odd signature operator to Slebarski's $\frac{1}{3}$-operator. In Section 2.3 we determine the η-invariants of the Dirac operator and the odd signature operator on M up to a local correction. In Section 2.4 we adapt (2.1) to our situation. Finally in Section 2.5 we compute the remaining local correction term and obtain the value of $\mathrm{ek}(M)$.

2.1. The G_2**-structure on** TM**.** Using Schur's lemma, we exhibit a G_2-structure on the tangent bundle of $M = SO(5)/SO(3)$. Using this structure, we will be able to simplify several explicit calculations needed to control both the equivariant spectral flow from the Riemannian signature operator B to its reductive (or "cubical") deformation $\widetilde{B}$, and the Chern-Simons correction term. We will also use some branching rules for $SO(3) \subset G_2$; these can be checked either using the explicit description of these representations in [FH91] or using a suitable computer program like LiE [CLL99].

To facilitate computations, let $e_{ij} \in \mathfrak{so}(n)$ for $i \neq j$ denote the endomorphism that maps the jth vector e_j of the standard orthonormal base of $\mathbf{R}^n$ to the ith vector e_i and e_i to $-e_j$, and vanishes on all other vectors. Then $e_{ji} = -e_{ij}$ and $[e_{ij}, e_{jk}] = e_{jk}$ unless $i = k$. We fix the scalar product $\langle A, B \rangle = -\frac{1}{2}\operatorname{tr}(AB)$ on $\mathfrak{so}(5)$, so that the basis e_{ij} of $\mathfrak{so}(5)$ becomes orthonormal. We fix an embedding $\iota\colon \mathfrak{so}(3) \hookrightarrow \mathfrak{so}(5)$ with

$$(2.2) \qquad \begin{aligned} e_{12} &\mapsto \iota_{12} = 2e_{12} + e_{34}\,, \\ e_{23} &\mapsto \iota_{23} = e_{23} - e_{14} + \sqrt{3}\,e_{45}\,, \\ \text{and} \quad e_{13} &\mapsto \iota_{13} = e_{13} + e_{24} + \sqrt{3}\,e_{35}. \end{aligned}$$

Then the vectors ι_{ij} of $\mathfrak{h} \cong \mathfrak{so}(3)$ are orthogonal and of length $\sqrt{5}$. Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{h} = \iota(\mathfrak{so}(3))$ in $\mathfrak{g} = \mathfrak{so}(5)$.

The embedding of $H = SO(3)$ in $SO(5)$ for the Berger space is given by the conjugation action of $SO(3)$ on real, 3×3 symmetric matrices of trace zero. The isotropy representation π of H on $\mathfrak{p}$ is the seven-dimensional irreducible representation of $SO(3)$ and so the Berger space is isotropy irreducible (cf. [Wo68]). It is well known that the seven-dimensional, irreducible, orthogonal representation of $SO(3)$ factors through the groups G_2 and $Spin(7)$,

$$SO(3) \to G_2 \to Spin(7) \to SO(7).$$

The second factorization is due to the fact that G_2 is simply connected; it implies that M admits an $SO(5)$-equivariant spin structure. The first factorization is more important, since it allows us to identify $\mathfrak{p}$ with the space of imaginary octonions $\mathbf{I}$; here we write Cayley's octonions $\mathbf{O} = \mathbf{R} \oplus \mathbf{I}$ as split into their real and imaginary parts. Let $*$ denote the Cayley product, and let $*_{\mathbf{I}}$ denote its projection onto $\mathbf{I}$. Note that $\mathbf{O}$ carries a natural scalar product $\langle p, q \rangle$ given by the real part of $p\overline{q}$, and that G_2 and H preserve the decomposition $\mathbf{O} = \mathbf{R} \oplus \mathbf{I}$ as well as $*$ and $\langle \cdot, \cdot \rangle$.

LEMMA 2.1. *With a suitable isometric, G_2-equivariant identification of $\mathfrak{p}$ with the imaginary octonions, one has*

$$[v, w]_{\mathfrak{p}} = \frac{1}{\sqrt{5}} \, v *_{\mathbf{I}} w \text{ for all } v, w \in \mathfrak{p}.$$

Proof. Schur's lemma implies that $[v, w]_{\mathfrak{p}} = c \, v *_{\mathbf{I}} w$ for some real constant c, because $\Lambda^2 \mathfrak{p}$ splits $SO(3)$-equivariantly into the irreducible real $SO(3)$-representations κ_1, κ_3 and κ_5 of dimensions 3, 7 and 11, each of multiplicity one. On the other hand, κ_3 is just the restriction of the standard representation of G_2, which leaves "$*_{\mathbf{I}}$" invariant.

To determine c, we pick two orthogonal unit vectors v, $w \in \mathfrak{p}$. Because then $v * w \in \mathbf{I}$, we have $\|v * w\| = 1$ and $\|[v, w]_{\mathfrak{p}}\| = |c|$. For example with $v = \frac{1}{\sqrt{5}}(e_{12} - 2e_{34})$ and $w = e_{25} \in \mathfrak{p}$, we find $[v, w] = \frac{1}{\sqrt{5}} e_{15} \in \mathfrak{p}$, so

$$\pm c = \|[v, w]_{\mathfrak{p}}\| = \frac{1}{\sqrt{5}}.$$

Because $\mathfrak{p}$ is irreducible, an isometric, G_2-equivariant identification $\mathbf{I} \cong \mathfrak{p}$ is unique up to sign, and we may pick the sign so that $c = \frac{1}{\sqrt{5}}$.   □

For later use, we explicitly identify $\mathfrak{p} \cong \mathbf{I}$ as in Lemma 2.1. We are also interested in the decomposition of $\mathfrak{p} \otimes \mathbf{C}$ into weight spaces for the H-representation π. Recall that $\mathbf{I}$ admits an orthonormal base $e_1, \ldots, e_7$ such that

(2.3) $e_i * e_{i+1} = e_{i+3}$

for all $i \in \{1, \ldots, 7\}$, where the indices $i + 1$ and $i + 3$ are understood modulo 7. We identify these imaginary octonions with $\mathfrak{p}$ with an orthonormal basis given by

$$e_1 = \frac{1}{\sqrt{5}} e_{12} - \frac{2}{\sqrt{5}} e_{34}, \qquad e_2 = \frac{\sqrt{2}}{\sqrt{5}} e_{45} - \frac{\sqrt{3}}{\sqrt{10}} (e_{23} - e_{14}), \qquad e_3 = e_{25},$$

$$(2.4) \qquad e_4 = \frac{\sqrt{2}}{\sqrt{5}} e_{35} - \frac{\sqrt{3}}{\sqrt{10}} (e_{13} + e_{24}), \qquad e_5 = \frac{1}{\sqrt{2}} (e_{24} - e_{13}),$$

$$e_6 = -\frac{1}{\sqrt{2}} (e_{23} + e_{14}), \quad \text{and} \quad e_7 = e_{15}.$$

We leave it to the reader to check that indeed $[e_i, e_{i+1}] = \frac{1}{\sqrt{5}} e_{i+3}$ where the indices are taken modulo 7.

Let us also compute the action of $\mathfrak{h}$ on $\mathfrak{p}$. If ι_{12}, ι_{13} and ι_{23} are given by (2.2), then $f_1 = \frac{1}{\sqrt{5}} \iota_{12}$, $f_2 = \frac{1}{\sqrt{5}} \iota_{23}$ and $f_3 = \frac{1}{\sqrt{5}} \iota_{13}$ form an orthonormal base of $\mathfrak{h}$. For $k = 1, \ldots, 3$, we define an element of $\Lambda^2 TM$ by

$$\alpha_k = \langle f_k, [\,\cdot\,, \,\cdot\,] \rangle = \langle \pi_{*f_k}(\,\cdot\,), \,\cdot\, \rangle,$$

so

$$\alpha_1 = \frac{1}{\sqrt{5}} e^2 \wedge e^4 + \frac{2}{\sqrt{5}} e^3 \wedge e^7 - \frac{3}{\sqrt{5}} e^5 \wedge e^6,$$

$$\alpha_2 = \frac{\sqrt{6}}{\sqrt{5}} e^1 \wedge e^4 - \frac{1}{\sqrt{2}} e^2 \wedge e^7 - \frac{1}{\sqrt{2}} e^3 \wedge e^4$$

$$(2.5) \qquad\qquad + \frac{\sqrt{3}}{\sqrt{10}} e^3 \wedge e^5 + \frac{\sqrt{3}}{\sqrt{10}} e^6 \wedge e^7,$$

$$\text{and} \quad \alpha_3 = \frac{\sqrt{6}}{\sqrt{5}} e^1 \wedge e^2 - \frac{1}{\sqrt{2}} e^2 \wedge e^3 + \frac{\sqrt{3}}{\sqrt{10}} e^3 \wedge e^6$$

$$- \frac{1}{\sqrt{2}} e^4 \wedge e^7 - \frac{\sqrt{3}}{\sqrt{10}} e^5 \wedge e^7.$$

Note that the map α has no G_2-symmetry.

With a similar trick as in Lemma 2.1, we can identify Clifford multiplication on spinors with Cayley multiplication. Recall that a quotient $M = G/H$ of compact Lie groups is equivariantly spin if and only if the isotropy representation $\pi \colon H \to SO(\mathfrak{p})$ factors over the spin group $\mathrm{Spin}(\mathfrak{p})$. In this case, the equivariant spinor bundle $\mathcal{S} \to M$ is constructed as the fibered product

$$\mathcal{S} = G \times_{\tilde{\pi}} S \to M,$$

where $\tilde{\pi}$ is the pull-back to H by π of the spin representation of Spin($\mathfrak{p}$) on the spinor module S. Since Clifford multiplication $\mathfrak{p} \times S \to S$ is Spin($\mathfrak{p}$)-equivariant, it is in particular H-equivariant, so there is a fiber-wise Clifford multiplication $TM \times \mathcal{S} \to \mathcal{S}$.

Note that if $\mathfrak{p}$ is odd-dimensional, Clifford multiplication with vectors is uniquely defined only up to sign. To remove this ambiguity, let

$$\omega = i^{\left[\frac{n+1}{2}\right]} e_1 \cdots e_n \in \mathrm{Cl}(\mathfrak{p}) \otimes \mathbf{C}$$

be the *complex Clifford volume element,* which satisfies $\omega^2 = 1 \in \mathrm{Cl}(\mathfrak{p})$. If $\mathfrak{p}$ is odd-dimensional, then ω commutes with Clifford multiplication, and we require that ω acts on S as $+1$.

The spinor module S of Spin(7) is of dimension $2^{\left[\frac{7}{2}\right]} = 8$. Because the smallest representations of G_2 are the trivial representation and the seven-dimensional representation on the imaginary octonions $\mathbf{I}$, and because $\tilde{\pi}$ is nontrivial, it is clear that there is a G_2-equivariant isomorphism $\mathbf{O} \cong \mathbf{R} \oplus \mathbf{I} \cong S$.

LEMMA 2.2. *We identify $\mathfrak{p} \cong \mathbf{I}$ as in Lemma 2.1. With respect to a suitable isometric, G_2-equivariant identification $S \cong \mathbf{O}$ and a suitable orientation of $\mathfrak{p}$, Clifford multiplication $\mathfrak{p} \times S \to S$ equals Cayley multiplication $\mathbf{I} \times \mathbf{O} \to \mathbf{O}$ from the right.*

In other words, the Clifford algebra $\mathrm{Cl}(\mathfrak{p}) \subset \mathrm{End}(S) \cong \mathrm{End}_{\mathbf{R}}(\mathbf{O})$ is generated by the endomorphisms c_v given by right multiplication with some element of $\mathfrak{p} \cong \mathbf{I}$. Since $\mathbf{O}$ is not associative, we do not have the identity $c_v \cdot c_w = c_{-v*w}$ in general. For the same reason, right multiplication on $\mathbf{O}$ does not commute with left multiplication, which agrees with the fact that S is an irreducible $\mathrm{Cl}(\mathfrak{p})$-module.

Proof. We fix a G_2-equivariant orthogonal identification $S \cong \mathbf{O} = \mathbf{R} \oplus \mathbf{I}$. Clifford multiplication $\mathfrak{p} \times (\mathbf{R} \oplus \mathbf{I}) \to (\mathbf{R} \oplus \mathbf{I})$ splits into four components. By Schur's lemma, the component $\mathfrak{p} \times \mathbf{R} \to \mathbf{R}$ vanishes. Because multiplication with a unit vector is an isometry on S, we have $v \cdot 1 = \pm v \in \mathbf{I}$ for $1 \in \mathbf{R} \subset S$, and we choose the identification $S \cong \mathbf{R} \oplus \mathbf{I}$ such that $v \cdot 1 = v$.

Again by Schur's lemma, the component of $v \cdot s$ in $\mathbf{R}$ is $c \langle v, s \rangle$ for some constant c. Since $v \cdot (v \cdot c) = -\|v\|^2 c$, it is easy to see that $v \cdot v = -\|v\|^2 = v * v \in \mathbf{R}$ for $v \in \mathbf{I}$.

Finally, for orthogonal imaginary elements $v, w \in \mathbf{I}$ we must have $\|v \cdot w\| = \|v\| \|w\| = \|w * v\|$, so $v \cdot w = \pm w * v$. To check that the correct sign is $+$, we calculate using (2.3);

$$(2.6) \qquad \omega \cdot s = (\cdots (s * e_7) * \cdots) * e_1 = s. \qquad \square$$

2.2. The spectra of some deformed Dirac operators. We use the explicit formulas for Clifford multiplication and the tangential part of the Lie bracket obtained in the previous section to estimate the spectrum of the family of deformed odd signature operators $B^{\lambda,3\lambda-1}$. We take the orthonormal base $e_1, \ldots, e_7$ of $\mathfrak{p} \cong \mathbf{I}$ as in (2.2). Let c_i and $\hat{c}_i$ denote Clifford multiplication with e_i on the first and second factor of $\Lambda^{ev}\mathfrak{p} \cong S \otimes S \cong \mathbf{O} \otimes \mathbf{O}$. Then the Clifford volume elements

$$\omega = c_1 \cdots c_7 \quad \text{and} \quad \hat{\omega} = \hat{c}_1 \cdots \hat{c}_7$$

act as 1 by (2.6).

We extend $e_1, \ldots, e_7$ to an orthonormal base $e_1, \ldots, e_{10}$ of $\mathfrak{g}$, and let $c_{ijk} = \langle [e_i, e_j]_{\mathfrak{p}}, e_k \rangle$, so for example $c_{124} = \frac{1}{\sqrt{5}}$ by Lemma 2.1 and (2.3). We define two maps $\widetilde{\mathrm{ad}}_{\mathfrak{p}}$ and $\widehat{\mathrm{ad}}_{\mathfrak{p}} \colon \mathfrak{g} \to \mathrm{End}(\Lambda^{ev}\mathfrak{p})$ by

$$\widetilde{\mathrm{ad}}_{\mathfrak{p},X} = \frac{1}{4} \sum_{i,j=1}^{7} \langle [X, e_i], e_j \rangle c_i c_j \quad \text{and} \quad \widehat{\mathrm{ad}}_{\mathfrak{p},X} = \frac{1}{4} \sum_{i,j=1}^{7} \langle [X, e_i], e_j \rangle \hat{c}_i \hat{c}_j,$$

and put

$$\widetilde{\mathrm{ad}}_{\mathfrak{p},i} = \widetilde{\mathrm{ad}}_{\mathfrak{p},e_i} = \frac{1}{4} \sum_{j,k=1}^{7} c_{ijk} \, c_j c_k \quad \text{and} \quad \widehat{\mathrm{ad}}_{\mathfrak{p},i} = \widehat{\mathrm{ad}}_{\mathfrak{p},e_i} = \frac{1}{4} \sum_{j,k=1}^{7} c_{ijk} \, \hat{c}_j \hat{c}_k.$$

Then $\tilde{\pi} = \widetilde{\mathrm{ad}}_{\mathfrak{p}}|_{\mathfrak{h}}$ and $\hat{\pi} = \widehat{\mathrm{ad}}_{\mathfrak{p}}|_{\mathfrak{h}}$ are the differentials of the representations of H on the two factors of $S \otimes S$ that induce the bundle $\Lambda^{ev}TM \to M$.

We consider a family D^λ of G-equivariant deformed Dirac operators on $\Gamma(\mathcal{S})$ and a family $B^{\lambda,\mu}$ of G-equivariant deformed odd signature operators on $\Gamma(\Lambda^{ev}TM)$ as in [Go97]. Using Frobenius reciprocity and the Peter-Weyl theorem, we will write

$$\Gamma(\mathcal{S}) = \overline{\bigoplus_{\gamma \in \widehat{G}} V^\gamma \otimes \mathrm{Hom}_H(V^\gamma, S)}$$

$$\text{and} \quad \Omega^{ev}(M) = \overline{\bigoplus_{\gamma \in \widehat{G}} V^\gamma \otimes \mathrm{Hom}_H(V^\gamma, S \otimes S)}.$$

Since D^λ and $B^{\lambda,\mu}$ are G-equivariant, they preserve these decompositions. Moreover, for each summand above, we may write

$$D^\lambda|_{V^\gamma \otimes \mathrm{Hom}_H(V^\gamma, S)} = id_{V^\gamma} \otimes {}^\gamma D^\lambda$$

$$\text{and} \quad B^{\lambda,\mu}|_{V^\gamma \otimes \mathrm{Hom}_H(V^\gamma, S \otimes S)} = id_{V^\gamma} \otimes {}^\gamma B^{\lambda,\mu}.$$

Let γ_i denote the action of $\gamma_{e_i}^*$ on the dual of the representation space V^γ. With this notation, the operators above take the form

(2.7)
$$\gamma D^\lambda = \sum_{i=1}^{7} c_i \left(\gamma_i + \lambda \, \widetilde{\mathrm{ad}}_{\mathfrak{p},i} \right)$$

$$\text{and} \quad \gamma B^{\lambda,\mu} = \sum_{i=1}^{7} c_i \left(\gamma_i + \lambda \, \widetilde{\mathrm{ad}}_{\mathfrak{p},i} + \mu \, \widehat{\mathrm{ad}}_{\mathfrak{p},i} \right).$$

Note that $D = D^{\frac{1}{2}}$ and $B = B^{\frac{1}{2},\frac{1}{2}}$ are respectively the Dirac operator and the odd signature operator associated to the Levi-Civita connection on M. On the other hand, $\widetilde{D} = D^{\frac{1}{3}}$ and $\widetilde{B} = B^{\frac{1}{3},0}$ are reductive operators in the terminology of [Go97], [Go99] and [Go02].

We now consider the one-parameter family $B^{\lambda,3\lambda-1}$ for $\lambda \in \left[\frac{1}{3}, \frac{1}{2}\right]$. We write

(2.8)
$$\gamma B^{\lambda,3\lambda-1} = \gamma\widetilde{B} + \mu \sum_{i=1}^{7} c_i \left(\frac{1}{3} \widetilde{\mathrm{ad}}_{\mathfrak{p},i} + \widehat{\mathrm{ad}}_{\mathfrak{p},i} \right)$$

for $\mu = 3\lambda - 1 \in \left[0, \frac{1}{2}\right]$. Let us define

(2.9)
$$B_0 = \sum_{i=1}^{7} c_i \left(\frac{1}{3} \widetilde{\mathrm{ad}}_{\mathfrak{p},i} + \widehat{\mathrm{ad}}_{\mathfrak{p},i} \right).$$

The square of $\gamma\widetilde{B}$ has been computed in [Go97], [Go99] as

(2.10)
$$\gamma\widetilde{B}^2 = \|\gamma + \rho_G\|^2 - c_H^{\widehat{\hat\pi}} - \|\rho_H\|^2.$$

Here ρ_H and ρ_G are half sums of positive roots, and $c_H^{\widehat{\hat\pi}}$ is the Casimir operator of H associated to the representation $\widehat{\hat\pi}$, taken with respect to the norm on $\mathfrak{h}$ that is induced by the embedding ι of (2.2) and a fixed Ad-invariant scalar product on $\mathfrak{g}$.

Let $\mathfrak{s} \subset \mathfrak{t}$ be the Cartan subalgebras of $\mathfrak{h} \subset \mathfrak{g}$ spanned by ι_{12} and by e_{12} and e_{34}, respectively. The weights of $\mathfrak{h}$ and $\mathfrak{g}$ are of the form

(2.11)
$$ik\, \iota_{12}^* = \frac{ik}{5}\, (2e_{12}^* + e_{34}^*) \in i\mathfrak{s}^* \quad \text{and} \quad ip\, e_{12}^* + iq\, e_{34}^* \in i\mathfrak{t}^*$$

with $k, p, q \in \mathbf{Z}$. We will pick the Weyl chambers

(2.12) $P_H = \{\, it\, \iota_{12}^* \mid t \geq 0 \,\} \subset i\mathfrak{s}^* \quad \text{and} \quad P_G = \{\, ix\, e_{12}^* + iy\, e_{34}^* \mid x \geq y \geq 0 \,\}.$

With respect to these Weyl chambers, the dominant weights of G and H are the weights in (2.11) with $k \geq 0$ and $p \geq q \geq 0$, respectively. Then we find

$$(2.13) \qquad \rho_H = \frac{i}{2}\, \iota_{12}^* = \frac{i}{10}\, (2e_{12}^* + e_{34}^*) \quad \text{and} \quad \rho_G = \frac{i}{2}\, (3e_{12}^* + e_{34}^*).$$

Let $\gamma_{(p,q)}$ denote the irreducible G-representation with highest weight $ip\, e_{12}^* + iq\, e_{34}$, where $p \geq q \geq 0$ are integers. Let κ_k denote the irreducible H-representation with highest weight $ik\, \iota_{12}^*$, then the dimension of κ_k is $2k+1$. We have seen above that the isotropy representation π on $\mathfrak{p}$ is isomorphic to κ_3, while $\tilde{\pi}$ on $S \cong \mathbf{R} \oplus \mathbf{I}$ is isomorphic to $\kappa_0 \oplus \kappa_3$. We conclude that for $\gamma = \gamma_{(p,q)}$, we have

$$(2.14) \qquad {}^\gamma \widetilde{B}^2 = \begin{cases} \|\gamma_{(p,q)} + \rho_G\|^2 - \|\rho_H\|^2 = p^2 + 3p + q^2 + q + \frac{49}{20} \\ \quad \text{on } \mathrm{Hom}_H\,(V^\gamma, S \otimes \mathbf{R}), \text{ and} \\ \|\gamma_{(p,q)} + \rho_G\|^2 - \|\kappa_3 + \rho_H\|^2 = p^2 + 3p + q^2 + q + \frac{1}{20} \\ \quad \text{on } \mathrm{Hom}_H\,(V^\gamma, S \otimes \mathbf{I}). \end{cases}$$

We now calculate the spectral radii of the various components of the operator B_0. Since the operator B_0 evidently commutes with the action of G_2 on $S \otimes S$ by its definition in (2.9), we can restrict our attention to the G_2-isotypic components of B_0. Let $\mathfrak{u} \subset \mathfrak{g}_2$ be a Cartan subalgebra containing $\mathfrak{s}$. We introduce a basis of $i\mathfrak{u}^* \subset \mathfrak{u}^* \otimes_{\mathbf{R}} \mathbf{C}$ such that $(1,0)$ and $(0,1)$ describe a long and a short root of $\mathfrak{g}_2$ respectively, which belong to the closure of a fixed Weyl chamber in $i\mathfrak{u}^*$. In this basis, the dominant weights of $\mathfrak{g}_2$ are given precisely by pairs of nonnegative integer coordinates. Let $\varphi_{(a,b)}$ denote the irreducible G_2-representation with highest weight (a, b) for $a, b \in \mathbf{Z}$ with $a, b \geq 0$. It is easy to check that $\varphi_{(0,1)}$ denotes the standard representation of $\mathfrak{g}_2$ on $\mathbf{I}$, that $\varphi_{(1,0)}$ is the adjoint representation, and that $\varphi_{(0,2)}$ is the 27-dimensional nontrivial part of the symmetric product $S^2\mathbf{I}$.

Using section 22.3 in [FH91] or the computer program LiE, we see that $\Lambda^{ev}\mathfrak{p}^* \cong S \otimes S$ splits into G_2- and H-isotypical components as:

$$(2.15) \qquad \begin{aligned} \mathbf{R} \otimes \mathbf{R} \quad &\cong_{G_2} \quad \varphi_{(0,0)} \quad \cong_H \quad \kappa_0, \\ \mathbf{I} \otimes \mathbf{R} \quad &\cong_{G_2} \quad \varphi_{(0,1)} \quad \cong_H \quad \kappa_3, \\ \mathbf{R} \otimes \mathbf{I} \quad &\cong_{G_2} \quad \varphi_{(0,1)} \quad \cong_H \quad \kappa_3, \end{aligned}$$

$$\text{and} \qquad \mathbf{I} \otimes \mathbf{I} \quad \cong_{G_2} \quad \begin{matrix} \varphi_{(0,0)} \oplus \varphi_{(0,1)} \oplus \varphi_{(1,0)} \\ \oplus \varphi_{(0,2)} \end{matrix} \quad \cong_H \quad \begin{matrix} \kappa_0 \oplus \kappa_3 \oplus (\kappa_1 \oplus \kappa_5) \\ \oplus (\kappa_2 \oplus \kappa_4 \oplus \kappa_6). \end{matrix}$$

Note that no two G_2-representations involved have a common isomorphic H-subrepresentation.

Lemmas 2.1 and 2.2 give us an explicit formula for B_0. Using a computer program, we can calculate the eigenvalues of B_0 on each G_2-isotypical component

$B_0^{(p,q)}$. A basis for the trivial component is given by

$$1 \otimes 1 \quad \text{and} \quad \frac{1}{\sqrt{7}} \sum_{i=1}^{7} e_i \otimes e_i.$$

With respect to this basis, one has

$$(2.16) \qquad\qquad B_0^{(0,0)} = \frac{1}{2\sqrt{5}} \begin{pmatrix} 7 & -3\sqrt{7} \\ -3\sqrt{7} & 5 \end{pmatrix}.$$

In particular, the eigenvalues of $B_0^{(0,0)}$ are $\frac{7}{\sqrt{5}}$ and $-\frac{1}{\sqrt{5}}$.

The representation $\varphi_{(0,1)}$ has multiplicity 3 in $\Lambda^{ev}\mathfrak{p}$. We pick three vectors that equivariantly span the isotypical component, and that correspond to $e_1 \in \mathbf{I}$, namely

$$e_1 \otimes 1, \qquad 1 \otimes e_1, \quad \text{and} \quad \frac{1}{\sqrt{6}} \sum_{i=2}^{7} e_i \otimes (e_1 * e_i).$$

By equivariance, $B_0^{(0,1)}$ preserves the 3-dimensional subspace $V \subset \Lambda^{ev}\mathfrak{p}$ spanned by these vectors. We find

$$(2.17) \qquad\qquad B_0^{(0,1)}|_V = \frac{1}{2\sqrt{5}} \begin{pmatrix} -1 & 3 & 3\sqrt{6} \\ 3 & 7 & \sqrt{6} \\ 3\sqrt{6} & \sqrt{6} & -4 \end{pmatrix}.$$

The eigenvalues of $B_0^{(0,1)}$ are readily computed to be $\frac{1}{\sqrt{5}}$ and $\pm\sqrt{5}$.

Finally, the G_2-isotypical components isomorphic to $\varphi_{(1,0)}$ and $\varphi_{(0,2)}$ both have multiplicity 1, and we have

$$(2.18) \qquad\qquad B_0^{(1,0)} = \frac{1}{\sqrt{5}} \quad \text{and} \quad B_0^{(0,2)} = -\frac{1}{\sqrt{5}}.$$

The calculations above lead to the following proposition.

PROPOSITION 2.3. *The operator D^λ has no kernel for $\lambda \in \left[\frac{1}{3}, \frac{1}{2}\right]$. For $\lambda \in \left[\frac{1}{3}, \frac{1}{2}\right]$ and $\gamma \in \widehat{G}$, the operator ${}^\gamma B^{\lambda, 3\lambda-1}$ has a non-zero kernel only if $\lambda = \frac{1}{2}$ and $\gamma = \gamma_{(0,0)}$ is the trivial representation. For $\gamma = \gamma_{(0,0)}$, the operator ${}^\gamma B^{\lambda, 3\lambda-1}$ has a positive and a negative eigenvalue if $\lambda \in \left[\frac{1}{3}, \frac{1}{2}\right)$, and only the negative eigenvalue vanishes at $\lambda = \frac{1}{2}$.*

Proof. The claim about D^λ follows from the proof of Lemma 4.6 in [Go99] (see also Bemerkung 1.20 in [Go97]).

Let us now check that for no nontrivial representation γ of G and no $\lambda \in \left[\frac{1}{3}, \frac{1}{2}\right]$, the operator ${}^\gamma B^{\lambda, 3\lambda-1}$ can have a kernel. This is because by (2.14), all

eigenvalues of $^\gamma\widetilde{B}$ belong to $\mathbf{R} \setminus \left(-\frac{9}{2\sqrt{5}}, \frac{9}{2\sqrt{5}}\right)$, where $\pm\frac{9}{2\sqrt{5}}$ is attained on Hom_H $(V^\gamma, S \otimes \mathfrak{p})$ for $\gamma = \gamma_{(1,0)}$. On the other hand, the spectral radius of μB_0 is $\frac{7\mu}{\sqrt{5}}$, which is smaller than $\frac{9}{2\sqrt{5}}$ for $\mu \in [0, \frac{1}{2}]$.

Now, consider the operator $^\gamma\widetilde{B}$ for the trivial representation $\gamma = \gamma_{(0,0)}$. Clearly, $\mathrm{Hom}_H(V^\gamma, S \otimes S)$ is isomorphic to the trivial G_2-isotypical component of $\Lambda^{ev}\mathfrak{p}$. Another machine computation shows that in the basis of (2.16), the operator $^\gamma\widetilde{B}$ takes the form

$$^\gamma\widetilde{B} = \frac{1}{2\sqrt{5}} \begin{pmatrix} 7 & \\ & -1 \end{pmatrix}.$$

The eigenvalues of the operator

$$\gamma B^{\frac{\mu+1}{3},\mu} = {}^\gamma\widetilde{B} + \mu B_0 = \frac{1}{2\sqrt{5}} \begin{pmatrix} 7 + 7\mu & -3\mu\sqrt{7} \\ -3\mu\sqrt{7} & 5\mu - 1 \end{pmatrix}$$

are precisely $\frac{1}{2\sqrt{5}} \left(6\mu + 3 \pm \sqrt{64\mu^2 + 8\mu + 16}\right)$. Since $6\mu + 3 < \sqrt{64\mu^2 + 8\mu + 16}$ except at $\lambda = \frac{1}{2}$ where one gets equality, the claims in the proposition follow. $\square$

2.3. Computing the η-invariants. Next we compute the η-invariants $\eta(B)$ and $\eta(D)$ for the Dirac operators considered in the previous subsection, up to a local correction term. We will use the formula of [Go02].

We fix Weyl chambers P_G and P_H as in (2.12). Then ρ_G and ρ_H are given by (2.13). The choices of P_G and P_H also determine orientations on $\mathfrak{g}/\mathfrak{t}$ and $\mathfrak{h}/\mathfrak{s}$. If $\alpha_1, \ldots, \alpha_l \in i\mathfrak{t}^*$ are the positive roots of $\mathfrak{g}$ with respect to P_G, then we can choose a complex structure on $\mathfrak{g}/\mathfrak{t}$ and a complex basis $z_1, \ldots, z_n$ such that $\mathrm{ad}|_{\mathfrak{t}\times(\mathfrak{g}/\mathfrak{t})}$ takes the form

$$\mathrm{ad}_X = \begin{pmatrix} \alpha_1(X) & & \\ & \ddots & \\ & & \alpha_l(X) \end{pmatrix} \quad \text{for all } X \in \mathfrak{t}.$$

Then we declare the real basis $z_1, iz_1, z_2, \ldots, iz_l$ to be positively oriented.

Having fixed orientations on $\mathfrak{p} = \mathfrak{g}/\mathfrak{h}$ by a choice of an orthonormal base in (2.4) and orientations on $\mathfrak{g}/\mathfrak{t}$ and $\mathfrak{h}/\mathfrak{s}$ as above, there is a unique orientation on $\mathfrak{t}/\mathfrak{s}$ such that the orientations on

$$\mathfrak{g}/\mathfrak{s} \cong \mathfrak{p} \oplus (\mathfrak{h}/\mathfrak{s}) \cong (\mathfrak{g}/\mathfrak{t}) \oplus (\mathfrak{t}/\mathfrak{s})$$

agree. Let $E \in \mathfrak{t}/\mathfrak{s} \cong \mathfrak{s}^\perp \subset \mathfrak{t}$ be the positive unit vector, and let $\delta \in i\mathfrak{t}^*$ be the unique weight such that

$$-i\delta(E) > 0 \quad \text{and} \quad \delta(X) \in 2\pi i\mathbf{Z} \iff e^X \in S$$

for all $X \in \mathfrak{t}$. Then one can check that

$$(2.19) \qquad E = \frac{1}{\sqrt{5}}\,(e_{12} - 2e_{34}) \quad \text{and} \quad \delta = i\,(e_{12}^* - 2e_{34}^*)$$

are compatible with the orientations fixed above.

Let $\widetilde{D}^{(k)}$ be the reductive Dirac operator acting on $\Gamma(\mathcal{S} \otimes V^\kappa M)$, where κ is the H-representation with highest weight $\kappa_k = ik\,\iota_{12}^*$. Then we note that

$$\widetilde{D} = \widetilde{D}^{(0)} \quad \text{and} \quad \widetilde{B} = \widetilde{D}^{(0)} \oplus \widetilde{D}^{(3)}.$$

We have to find the unique weights $\alpha_k \in i\mathfrak{t}^*$ of $\mathfrak{g}$ such that

$$\alpha_k|_{\mathfrak{s}} = ik\,\iota_{12}^* + \rho_H \quad \text{and} \quad -i(\alpha_k - \delta)(E) < 0 \le -i\alpha_k(E).$$

By (2.19) we have

$$\alpha_0 = i\left(\frac{1}{2}\,e_{12}^* - \frac{1}{2}\,e_{34}^*\right) \quad \text{and} \quad \alpha_3 = i\left(\frac{3}{2}\,e_{12}^* + \frac{1}{2}\,e_{34}^*\right).$$

Note that $\alpha_0(E)$, $\alpha_3(E) \ne 0$.

Let $\Delta_+ = \{ie_{12}^* + ie_{34}^*, ie_{12}^* - ie_{34}^*, ie_{12}^*, ie_{34}^*\}$ denote the set of positive roots with respect to P_G. Let $\hat{A}$ denote the map, $z \mapsto \frac{z}{2\sinh(z/2)}$.

We also need some equivariant characteristic differential forms. Note that we will eventually evaluate these forms only at $X = 0$, so that we may actually forget the equivariant formalism in a moment. Let $\hat{A}_X(M, \nabla)$ be the total equivariant $\hat{A}$-form, and $\hat{L}_X(M, \nabla) = 2\,\hat{A}_X(M, \nabla) \wedge \mathrm{ch}_X(\mathcal{S}, \nabla)$ be a rescaled equivariant L-form, both taken with respect to a connection ∇ on TM and the induced connection on $\mathcal{S}$. If $p_k = p_k(M, \nabla) \in \Omega_{\mathfrak{g}}^{4k}(M)$ denotes the kth equivariant Pontrijagin form of M, then

$$\widehat{A}_X(M, \nabla) = 1 - \frac{p_1}{24} + \frac{7p_1^2 - 4p_2}{2^7\,3^2\,5} + \cdots \quad \text{and}$$

$$\widehat{L}_X(M, \nabla) = 16 + \frac{4p_1}{3} + \frac{7p_2 - p_1^2}{45} + \cdots.$$

So in particular,

$$(2.20) \qquad \widehat{A}_X(M, \nabla) + \frac{\widehat{L}_X(M, \nabla)}{2^5\,7} = \frac{15}{14} - \frac{p_1(M, \nabla)}{28} + \frac{p_1(M, \nabla)^2}{2^7\,7} + \cdots.$$

For different connections ∇, ∇' let $\widetilde{\widehat{A}}(M,\nabla,\nabla')$ and $\widetilde{\widehat{L}}(M,\nabla,\nabla') \in \Omega_{\mathfrak{g}}^*(M)/d_{\mathfrak{g}}$ $\Omega_{\mathfrak{g}}^*(M)$ denote the corresponding equivariant Chern-Simons classes satisfying

$$d\widetilde{\widehat{A}}(M,\nabla,\nabla') = \widehat{A}_X(M,\nabla') - \widehat{A}_X(M,\nabla) \quad \text{and}$$

$$d\widetilde{\widehat{L}}(M,\nabla,\nabla') = \widehat{L}_X(M,\nabla') - \widehat{L}_X(M,\nabla).$$

There exist similar classes $\tilde{p}_k(M,\nabla,\nabla')$ with $d\tilde{p}_k(M,\nabla,\nabla') = p_k(M,\nabla') - p_k(M,\nabla)$, and

$$(2.21) \quad \widetilde{\widehat{A}}_X(M,\nabla,\nabla') + \frac{1}{2^5 7}\widetilde{\widehat{L}}_X(M,\nabla,\nabla') = -\frac{1}{28}\tilde{p}_1(M,\nabla,\nabla')$$

$$+ \frac{1}{2^7 7}\left(p_1(M,\nabla)p_1(M,\nabla,\nabla') + p_1(M,\nabla,\nabla')p_1(M,\nabla')\right) + \cdots.$$

We will work with the reductive connection ∇^0 and the Levi-Civita connection ∇^{LC} on TM.

We can now compute the eta-invariants of D and B using the formula for infinitesimal equivariant η-invariants computed in [Go02].

THEOREM 2.4. *The η-invariants of D and B are the values at $X = 0 \in \mathfrak{t}$ of the following:*

$$\eta_X(D) = 2\sum_{w\in W_G}\frac{\mathrm{sign}(w)}{\delta(wX)}\left(\prod_{\beta\in\Delta_+}\widehat{A}\left(\beta(wX)\right)\cdot\widehat{A}\left(\delta(wX)\right)e^{\left(\alpha_0-\frac{\delta}{2}\right)(wX)}\right.$$

$$(1) \qquad\qquad \left. -\prod_{\beta\in\Delta_+}\widehat{A}\left(\beta(wX|_{\mathfrak{s}})\right)\cdot e^{\rho_H(wX|_{\mathfrak{s}})}\right)\cdot\prod_{\beta\in\Delta_+}\frac{1}{\beta(X)}$$

$$+\int_M\widetilde{\widehat{A}}_X(TM,\nabla^0,\nabla^{\mathrm{LC}}),$$

$$\eta_X(B) = 2\sum_{w\in W_G}\frac{\mathrm{sign}(w)}{\delta(wX)}\left(\prod_{\beta\in\Delta_+}\widehat{A}\left(\beta(wX)\right)\cdot\widehat{A}(\delta(wX))\left(e^{\left(\alpha_0-\frac{\delta}{2}\right)(wX)}+e^{\left(\alpha_3-\frac{\delta}{2}\right)(wX)}\right)\right.$$

$$(2) \qquad\qquad \left. -\prod_{\beta\in\Delta_+}\widehat{A}\left(\beta(wX|_{\mathfrak{s}})\right)\cdot\left(e^{\rho_H(wX|_{\mathfrak{s}})}+e^{(\kappa_3+\rho_H)(wX|_{\mathfrak{s}})}\right)\right)$$

$$\cdot\prod_{\beta\in\Delta_+}\frac{1}{\beta(X)}+1+\int_M\widetilde{\widehat{L}}_X(TM,\nabla^0,\nabla^{\mathrm{LC}}).$$

Proof. This follows immediately from Theorem 2.33 and Corollary 2.34 in [Go02], and from Proposition 2.3 above. $\square$

A machine calculation now gives numerical values up to the local correction term.

COROLLARY 2.5. *We have the formulae*

$$\eta(D) = -\frac{12923}{2\ 3^2\ 5^6} + \int_M \widetilde{\widehat{A}}_X(TM, \nabla^0, \nabla^{\mathrm{LC}}),$$

$$and \quad \eta(B) = 1 - \frac{12923}{2\ 3^2\ 5^6} - \frac{277961}{2\ 3^2\ 5^6} + \int_M \widetilde{\widehat{L}}_X(TM, \nabla^0, \nabla^{\mathrm{LC}})$$

$$= -\frac{4817}{3^2\ 5^6} + \int_M \widetilde{\widehat{L}}_X(TM, \nabla^0, \nabla^{\mathrm{LC}}).$$

Remark 2.6. One might be tempted to conjecture that $\eta(\widetilde{D})$ has the value $-\frac{12923}{2\ 3^2\ 5^6}$ (from above) because $\widetilde{D}^2$ involves the Laplacian on $\mathcal{S}$ with respect to the reductive connection ∇^0. However, the equivariant η-invariant $\eta_G(\widetilde{D})$ has been calculated in [Go97] and in particular $\eta(\widetilde{D}) = \frac{207479}{2^5\ 3^2\ 5^6} \neq -\frac{12923}{2\ 3^2\ 5^6} = -\frac{206768}{2^5\ 3^2\ 5^6}$.

2.4. η-invariants and the Eells-Kuiper invariant. We compute the Eells-Kuiper invariant of $M = \mathrm{SO}(5)/\mathrm{SO}(3)$ using Donnelly's formula [Don75], see also [KS88], which involves the nonequivariant η-invariants of the Dirac operator D and the signature operator B on M, and a secondary Pontrijagin number. We show that in our situation, we may work with a secondary Pontrijagin number associated to the reductive connection.

Recall that the Eells-Kuiper invariant is defined as

$$\mathrm{ek}(M) = \frac{\eta(B)}{2^5\ 7} + \frac{\eta(D) + h(D)}{2} - \frac{1}{2^7\ 7} \int_M p_1(M, \nabla^{\mathrm{LC}}) \wedge h(M, \nabla^{\mathrm{LC}}) \qquad \in \mathbf{Q}/\mathbf{Z},$$

(see (2.1)). Note that the form $h(M, \nabla^{\mathrm{LC}})$ exists because $H^4(M, \mathbf{R}) = 0$, and is unique up to exact forms because $H^3(M, \mathbf{R}) = 0$.

As above, let $\widetilde{\widehat{A}}(M, \nabla, \nabla') \in \Omega^*(M)/d\Omega^*(M)$ denote the Chern-Simons class that interpolates between the $\widehat{A}$-forms constructed from two connections ∇ and ∇'. If $h = h(\nabla) \in \Omega^*(M)/d\Omega^*(M)$ is a class such that $dh = p_1(M, \nabla)$ is the first Pontrijagin form of TM, then the class $h(\nabla') = h(\nabla) + \tilde{p}_1(M, \nabla, \nabla')$ satisfies $dh(\nabla') = p_1(M, \nabla')$. In particular

$$(2.22) \quad \frac{1}{2^7\ 7} \int_M \left(p_1(M, \nabla') \, h(M, \nabla') - p_1(M, \nabla) \, h(M, \nabla) \right)$$

$$= \frac{1}{2^7\ 7} \int_M (\tilde{p}_1(M, \nabla, \nabla') \, p_1(M, \nabla) + p_1(M, \nabla') \, \tilde{p}_1(M, \nabla, \nabla'))$$

$$= \int_M \left(\widetilde{\widehat{A}}(M, \nabla, \nabla') + \frac{1}{2^5\ 7} \widetilde{\widehat{L}}(M, \nabla, \nabla') \right)$$

by (2.21). As an immediate consequence of Corollary 2.5, (2.1) and (2.22), we get

$$
\begin{aligned}
(2.23) \qquad \mathrm{ek}(M) &= -\frac{12923}{2^2\,3^2\,5^6} - \frac{4817}{2^5\,7\cdot3^2\,5^6} \\
&\quad + \int_M \left(\widetilde{\widehat{A}}(M,\nabla^0,\nabla^{\mathrm{LC}}) + \frac{1}{2^5\,7}\,\widetilde{\widehat{L}}(M,\nabla^0,\nabla^{\mathrm{LC}}) \right) \\
&\quad - \frac{1}{2^7\,7}\int_M p_1(M,\nabla^{\mathrm{LC}})\,h(M,\nabla^{\mathrm{LC}}) \\
&= -\frac{16189}{2^5\,5^5\,7} - \frac{1}{2^7\,7}\int_M p_1(M,\nabla^0)\,h(M,\nabla^0).
\end{aligned}
$$

Here we have used that D has no kernel by Proposition 2.3, so $h(D) = 0$.

2.5. Computing the Eells-Kuiper invariant. It remains to evaluate the integral of the secondary class $p_1(M,\nabla^0)\,h(M,\nabla^0)$ over M. This is again done with the help of the results of Section 2.1.

Let V, W be vector fields on M. Then there exist H-equivariant functions $\widehat{V}$, $\widehat{W}\colon G \to \mathfrak{p}$ such that

$$
V(gH) = \left[g, \widehat{V}(g)\right] \quad \text{and} \quad W(gH) = \left[g, \widehat{W}(g)\right] \in TM = G \times_\pi \mathfrak{p}.
$$

The reductive connection ∇^0 and its curvature R^0 satisfy

$$
\widehat{\nabla^0_V W} = \widehat{V}\left(\widehat{W}\right) \quad \text{and} \quad \widehat{R^0_{V,W}} = -\pi_*\big[\widehat{V},\widehat{W}\big]_{\mathfrak{h}}.
$$

Because $p_1(M,\nabla^0)$ is G-invariant, it must be given by an H-invariant $\hat{p}_1(M,\nabla^0) \in \Lambda^4\mathfrak{p}^*$. Then $\hat{p}_1(M,\nabla^0)$ is in fact G_2-invariant by (2.15), and hence, it must be a multiple the Poincaré dual λ_4 of the three form λ_3 where

$$
\lambda_3 = \langle \cdot *_{\mathbf{I}} \cdot, \cdot \rangle = \sum_{i=1}^{7} e^i \wedge e^{i+1} \wedge e^{i+3},
$$

(2.24)

$$
\text{so} \quad \lambda_4 = \sum_{i=1}^{7} e^i \wedge e^{i+1} \wedge e^{i+2} \wedge e^{i+5}.
$$

It is thus sufficient to compute $\hat{p}_1(M,\nabla^0)(e_2,e_4,e_5,e_6)$. Using (2.5), one can check that $[e_2,e_5]$, $[e_2,e_6]$, $[e_4,e_5]$, $[e_4,e_6] \in \mathfrak{p}$, and

$$
\begin{aligned}
\hat{p}_1(M,\nabla^0)(e_2,e_4,e_5,e_6) &= -\frac{1}{8\pi^2}\,\mathrm{tr}\big((R^0)^2\big)(e_2,e_4,e_5,e_6) \\
&= -\frac{1}{4\pi^2}\,\mathrm{tr}\left(\pi_{*[e_2,e_4]_{\mathfrak{h}}}\,\pi_{*[e_5,e_6]_{\mathfrak{h}}}\right) \\
&= \frac{3}{20\pi^2}\,\mathrm{tr}\left(\pi_{*f_1}^2\right) = -\frac{21}{25\pi^2}.
\end{aligned}
$$

This implies that

$$(2.25) \qquad \hat{p}_1(M, \nabla^0) = -\hat{p}_1(M, \nabla^0)(e_2, e_4, e_5, e_6)\,\lambda_4$$

$$= \frac{21}{25\pi^2} \sum_{i=1}^{7} e^i \wedge e^{i+1} \wedge e^{i+2} \wedge e^{i+5}.$$

Clearly, (2.25) gives an H-invariant element of $\Lambda^4\mathfrak{p}^*$. By equivariance, we can write $p_1(M, \nabla^0) = dh(M, \nabla^0)$ for some G-invariant form $h(M, \nabla^0)$, which is again given by an H-invariant $\widehat{h(M, \nabla^0)} \in \Lambda^3\mathfrak{p}^*$. Thus, $h(M, \nabla^0)$ must be a multiple of the form λ_3 of (2.24). By Cartan's formula for the exterior derivative and Lemma 2.1, we have

$$\widehat{d\lambda_3}\left(\widehat{V}_0, \ldots, \widehat{V}_3\right) = -\widehat{\lambda}_3\left([\widehat{V}_0, \widehat{V}_1], \widehat{V}_2, \widehat{V}_3\right) + \widehat{\lambda}_3\left([\widehat{V}_0, \widehat{V}_2], \widehat{V}_1, \widehat{V}_3\right) \mp \cdots$$

$$- \widehat{\lambda}_3\left([\widehat{V}_2, \widehat{V}_3], \widehat{V}_0, \widehat{V}_1\right)$$

$$= \frac{6}{\sqrt{5}} \sum_{i=1}^{7} e^i \wedge e^{i+1} \wedge e^{i+2} \wedge e^{i+5},$$

so $\hat{h}(M, \nabla^0) = -\frac{7}{10\sqrt{5}\,\pi^2}\,\lambda_3$. In particular, $p_1(M, \nabla^0)\,h(M, \nabla^0)$ is given by

$$\frac{7}{10\sqrt{5}\pi^2}\,\lambda_3 \,\frac{21}{25\pi^2}\,\lambda_4 = \frac{3\,7^3}{2\,5^{\frac{7}{2}}\,\pi^4}\,e^1 \wedge \cdots \wedge e^7.$$

Because $\mathrm{vol}(\mathrm{SO}(3)) = 8\pi^2$, $\mathrm{vol}(\mathrm{SO}(5)) = \frac{2^7\pi^6}{3}$, $\mathrm{vol}(H) = 5^{\frac{3}{2}}\,\mathrm{vol}(\mathrm{SO}(3))$, the volume of M is $\frac{16\pi^4}{3\,5^{\frac{3}{2}}}$. We can now calculate the last contribution to $\mathrm{ek}(M)$ as

$$(2.26) \quad -\frac{1}{2^7\,7} \int_M p_1(M, \nabla^0)\,h(M, \nabla^0) = -\frac{1}{2^7\,7} \cdot \frac{3\,7^3}{2\,5^{\frac{7}{2}}\,\pi^4} \cdot \frac{16\pi^4}{3\,5^{\frac{3}{2}}} = -\frac{49}{50\,000}.$$

Together with (2.23), this completes the proof of Theorem 1. $\qquad\square$

3. Vector fields on 2-connected 7-manifolds. In this section we prove some general results about smooth, oriented, 2-connected 7-manifolds. In particular, we determine the maximal number of independent vector fields on such a manifold in terms of the first spin characteristic class. We will describe these results after some general remarks about the homotopy type of 2-connected 7-manifolds.

Let M be a closed, oriented, 2-connected 7-manifold. From the structure of $H^*(M)$, we see that M is homotopy equivalent to a CW-complex with cells in dimension $0, 3, 4$ and 7. Furthermore, it follows from Poincaré duality that the number of cells in dimension 3 equals the number of cells in dimension 4,

and that there is a unique cell in dimensions 0 and 7. Let M_k denote the k-skeleton of M. We will denote by M_k/M_{k-1} the space obtained by pinching off the $(k-1)$-skeleton from M_k. Hence, M_k/M_{k-1} is equivalent to a one-point union of k-spheres.

PROPOSITION 3.1. *The following composite map is null homotopic*

$$\mathbf{S}^6 \longrightarrow M_4 \longrightarrow M_4/M_3,$$

where the first map is the attaching map for the 7-cell and the next map is the pinch map.

Proof. Since M is 2-connected, it admits a spin structure. The argument given on page 32 of [MM79] shows that the above map is trivial for any seven dimensional spin manifold. □

Any oriented, 2-connected manifold admits a unique compatible spin structure. Let $\beta \in H^4(M)$ be the first spin characteristic class. The class β is related to the first Pontrijagin class by the relation $2\beta = p_1$. The relation to the Stiefel Whitney classes is given by $\beta \equiv w_4 \pmod 2$. We now recall the definition of the Wu classes.

Definition 3.2. For an n-manifold M, the Wu classes $\mathrm{Wu}_i \in H^i(M, \mathbf{F}_2)$ are defined by the property $\mathrm{Wu}_i \cup x = Sq^i(x)$ for all $x \in H^{n-i}(M, \mathbf{F}_2)$, where Sq^i denotes the ith Steenrod operation.

Let $\mathrm{Wu}_t = 1 + \mathrm{Wu}_1 t + \mathrm{Wu}_2 t^2 + \ldots$, be the total Wu class. Then the total Wu class is related to the total Stiefel Whitney class by the relation $\mathrm{Wu}_t \cup SW_t = 1$.

PROPOSITION 3.3. *If M is a 2-connected 7-manifold, then all of its Stiefel Whitney classes are trivial.*

Proof. By the relation between the Stiefel Whitney classes and the Wu classes, it is sufficient to show that the total Wu class is trivial. The cohomology of a space is an unstable algebra over the Steenrod algebra. In particular we have the relation $Sq^i(x) = 0$ if $i > \deg(x)$. It follows that $\mathrm{Wu}_i = 0$ for $i > 3$. By the sparseness of $H^*(M, \mathbf{F}_2)$, it follows that $\mathrm{Wu}_1 = \mathrm{Wu}_2 = 0$. Finally, one uses the Adem relation $Sq^3 = Sq^1 Sq^2$ to see that $\mathrm{Wu}_3 = 0$. □

COROLLARY 3.4. *The class $\beta \in H^4(M)$ is 2-divisible. In other words, there is some (not necessarily unique) class γ such that $2\gamma = \beta$.*

Proof. We know that $\beta \equiv w_4 \pmod 2$. Since $w_4 = 0$, $\beta \equiv 0 \pmod 2$ which says that β is 2 divisible. □

We now proceed to use the above facts to study the vector fields on M.

THEOREM 3.5. *Any smooth, oriented,* 2-*connected* 7-*manifold is parallelizable if and only if* $\beta = 0$.

Proof. Let $f : M \to B_{\text{Spin}(7)}$ be the map that classifies the tangent bundle of M. We would like to show that f is homotopic to the trivial map. The obstructions to constructing the null homotopy lie in the groups $H^{i+1}(M, \pi_i(\text{Spin}(7)))$. The primary obstruction lies in $H^4(M, \pi_3(\text{Spin}(7))) = H^4(M)$ and is none other than the class β which we assumed to be trivial. Since $\pi_6(\text{Spin}(7)) = 0$, there are no further obstructions to constructing the null homotopy. □

THEOREM 3.6. *Any non-parallelizable smooth, oriented,* 2-*connected* 7-*manifold admits exactly* 4 *independent vector fields. In other words, the structure group of M may be reduced to* Spin(3) *and no further.*

Proof. Let us first see that the structure group cannot be reduced further than Spin(3). Suppose we could reduce the structure group to Spin(2). Since $\text{Spin}(2) = S^1$, any map from M to $B_{\text{Spin}(2)}$ is classified by $H^2(M)$. But M is 2-connected so any such map is trivial. Hence, a reduction of the structure group to Spin(2) would mean that the manifold is parallelizable.

It remains to show that we can always reduce the structure group to Spin(3). This corresponds to lifting the map $f : M \to B_{\text{Spin}(7)}$ to the space $B_{\text{Spin}(3)}$. The obstructions to constructing this lift lie in the groups $H^{i+1}(M, \pi_i(\text{Spin}(7)/\text{Spin}(3)))$. The primary obstruction lies in the group $H^4(M, \mathbf{F}_2)$ and is, by naturality, the element $\beta \,(\text{mod}\, 2)$. Since β is 2 divisible, this element is zero. Hence, we can construct the lift on M_4, the 4-skeleton of M. One now has the following commutative diagram:

$$
\begin{array}{ccc}
\mathbf{S}^6 & \xrightarrow{\;g\;} & \text{Spin}(7)/\text{Spin}(3) \\
\downarrow & & \downarrow \\
M_4 & \longrightarrow & B_{\text{Spin}(3)} \\
\downarrow & & \downarrow \\
M & \xrightarrow[\;f\;]{} & B_{\text{Spin}(7)}
\end{array}
$$

where the vertical maps on the left form a cofibration sequence and those on the right form a fibration sequence. Let g denote the map $\mathbf{S}^6 \to \text{Spin}(7)/\text{Spin}(3)$. In order to complete the lift to all of M, we require that the map g is null homotopic. Consider the composite $\mathbf{S}^6 \to \text{Spin}(7)/\text{Spin}(3) \to B_{\text{Spin}(3)}$. Since $B_{\text{Spin}(3)}$ is 3-connected, this composite factors through the map $\mathbf{S}^6 \to M_4/M_3$, which we know is trivial. Hence, we know that g lifts to Spin(7). But $\pi_6(\text{Spin}(7)) = 0$, and so g is null homotopic. □

Remark 3.7. From the above discussion, the Berger space admits precisely four independent vector fields.

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