Complete minimal surfaces in S^3

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Introduction

It is valuable when dealing with a non-linear theory, such as the study of minimal submanifolds, to have available a large collection of examples for reference and insight. One purpose of this paper is to develop a simple but fruitful procedure for constructing such examples for the study of minimal surfaces in spheres.

The procedure is of particular interest because it shows that even the simplest class of objects, minimal surfaces in the euclidean 3-sphere, is richly endowed. It will be proven that *every compact surface but the projective plane (which is prohibited) can be minimally immersed into S^3. Moreover, every compact, orientable surface can be minimally imbedded in S^3, and if the genus of the surface is not prime the imbedding is not unique. It will furthermore be shown that there exist algebraic minimal surfaces in S^3 of arbitrary degree.*

Minimal surfaces in spheres are related by means of the tangent cone construction to the study of isolated singularities on 3-dimensional minimal varieties in euclidean space. Forming the cones in $\mathbf{R}^4$ over the surfaces in S^3 mentioned above shows that *isolated singularities of every topological type but one can occur on minimal hypersurfaces of $\mathbf{R}^4$.* The exception is that the link of the tangent cone cannot be an immersed projective plane.

Each of the surfaces constructed has a large group of symmetries. This makes it possible to prove that *for each set of integers* $m, k, n, r \geqq 1$ *where* $(n, r) = 1$, *there exists a compact orientable surface of genus* $(m-1)(k-1) + (n-1)mk$ *minimally imbedded in the lens space* $L_{n,r}$, *and for each set of integers* $m, n, r \geqq 1$ *where* m *is odd and* $(2n, r) = 1$, *there exists a compact, non-orientable surface of Euler characteristic* $1 - m(n-1)$ *minimally imbedded in* $L_{2n,r}$. *Moreover, for each pair of integers* m, n *where* $2/mn$ *there exists a compact surface of Euler characteristic* $m(1 - (1/2)mn)$ *minimally imbedded in* $S^3/\mathbf{D}_n^*$ *where* $\mathbf{D}_n^*$ *denotes the binary dihedral group of order* $4n$.

To each complete minimal surface in S^3 there is associated a 1-parameter family of complete, locally isometric surfaces of constant mean curvature in each of the simply-connected, 3-dimensional space forms of curvature $\leqq 1$ (§ 12). Using this theorem and a principle of reflection duality we give a procedure for explicitly building complete constant mean curvature surfaces in $\mathbf{R}^3$. In particular we construct two complete, doubly periodic ones without self-intersections.

The second part of the paper is devoted to the development of a theory for compact minimal surfaces in S^3. This theory is disjoint from that of E. Calabi [3], [4] since his pseudo-holomorphic immersions lie essentially in even-dimensional spheres.

A compact minimal surface is viewed as a conformal immersion $\psi: \mathcal{R} \rightarrow S^3$ where $\mathcal{R}$ is a compact Riemann surface. Fundamental equations for ψ are derived, and the Hopf-Almgren holomorphic form is defined and interpreted geometrically (Prop. 1.5). The metrics on minimal surfaces in S^3 are characterized (Th. 8), and ruled minimal surfaces are classified (Prop. 7.2). Associated polar and bipolar minimal immersions are defined, and their relationships to the geometry of the surface and the nullity of the immersion are discussed. Two reflection principles for minimal surfaces in S^3 are established. A conjugate minimal surface is defined, and a principle of reflection duality is proved.

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1. Differential geometric preliminaries

Let $S^3 = \{x \in \mathbf{R}^4 : |x| = 1\}$. By a 2-dimensional submanifold of S^3 we shall mean a conformal immersion

$$\psi: \mathcal{R} \longrightarrow S^3$$

of some Riemann surface $\mathcal{R}$. The existence of conformal (isothermal) coordinates and a 2-sheeted orientable covering surface makes this definition completely general. The function ψ will always be considered as $\mathbf{R}^4$-valued with $|\psi|^2 = 1$.

Let $z = x_1 + ix_2$ be a local complex coordinate on $\mathcal{R}$ and set $\partial = \frac{1}{2}(\partial/\partial x_1 - i\partial/\partial x_2)$. Then the metric induced by ψ has the form

$$ds^2 = 2F\,|dz|^2 \tag{1.1}$$

and its Gauss curvature K is given by

$$K = -\frac{1}{F}\partial\bar{\partial}\log F\,. \tag{1.2}$$

The vector-valued second fundamental form can be expressed as

$$B_{ij} = \psi_{ij} - \sum\nolimits_k \frac{1}{2F}\langle\psi_{ij}, \psi_k\rangle\psi_k - 2F\delta_{ij}\psi \tag{1.3}$$

where $\langle\cdot, \cdot\rangle$ is the inner product in $\mathbf{R}^4$. Choosing a unit normal vector field η (tangent to S^3) we set

$$\beta_{ij} = \langle B_{ij}, \eta\rangle = \frac{1}{2F}\psi \wedge \psi_1 \wedge \psi_2 \wedge \psi_{ij} \tag{1.4}$$

and recall that β satisfies the *Gauss curvature equation*

$$4F^2(1 - K) = \beta_{12}^2 - \beta_{11}\beta_{22} \tag{1.5}$$

and the *Weingarten equations*

$$\eta_i = -\frac{1}{2F}\sum\nolimits_k \beta_{ik}\psi_k\,. \tag{1.6}$$

(The field η is considered $\mathbf{R}^4$-valued in the natural way.)

The immersion ψ is called *minimal* if trace$(B) \equiv 0$. This condition is equivalent to the equation

$$\partial\bar{\partial}\psi = -F\psi \tag{1.7}$$

and to the fact that ψ represents an extremal of the area integral.

Our first observation is

LEMMA 1.1. *If ψ is minimal, then ψ is a real analytic mapping.*

PROOF. Since ψ is conformal it is also an extremal of the spherical Dirichlet integral (4.3). Hence the representation of ψ in stereographic coordinates for S^3 satisfies equation (4.4) which fulfills the necesssary conditions (1.10.8″) for [15, Th. 1.10.4 p. 34]. The result follows.

The value of conformal parameterizations for minimal surfaces is the following.

LEMMA 1.2. *If ψ is minimal, then the differential form $\omega = \varphi dz^2$ where*

$$\varphi = \frac{1}{2}(\beta_{22} - i\beta_{12}) = \frac{1}{iF}\psi \wedge \partial\psi \wedge \bar{\partial}\psi \wedge \partial^2\psi \tag{1.8}$$

is holomorphic on $\mathcal{R}$.

PROOF. Since ψ is conformal and $\langle\psi, \psi\rangle = 1$ we have

$$\begin{aligned} &\langle\partial^k\psi, \partial^i\psi\rangle = \langle\bar{\partial}^k\psi, \bar{\partial}_i\psi\rangle = 0 \qquad \text{for } 1 \leqq k + i \leqq 3 \\ &\langle\partial\psi, \bar{\partial}\psi\rangle = F\ , \end{aligned} \tag{1.9}$$

where $\langle\cdot, \cdot\rangle$ is extended complex linearly. It follows that

$$\frac{1}{iF}(\psi \wedge \partial\psi \wedge \bar{\partial}\psi \wedge \partial^2\psi)^2 = \frac{-1}{F^2}\begin{vmatrix} \langle\psi, \psi\rangle & \langle\psi, \partial\psi\rangle & \langle\psi, \bar{\partial}\psi\rangle & \langle\psi, \partial^2\psi\rangle \\ \langle\partial\psi, \psi\rangle & \langle\partial\psi, \partial\psi\rangle & \langle\partial\psi, \bar{\partial}\psi\rangle & \langle\partial\psi, \partial^2\psi\rangle \\ \langle\bar{\partial}\psi, \psi\rangle & \langle\bar{\partial}\psi, \partial\psi\rangle & \langle\bar{\partial}\psi, \bar{\partial}\psi\rangle & \langle\bar{\partial}\psi, \partial^2\psi\rangle \\ \langle\partial^2\psi, \psi\rangle & \langle\partial^2\psi, \partial\psi\rangle & \langle\partial^2\psi, \bar{\partial}\psi\rangle & \langle\partial^2\psi, \partial^2\psi\rangle \end{vmatrix}$$
$$= \langle\partial^2\psi, \partial^2\psi\rangle\ .$$

From (1.7) and (1.9) above we then have $\bar{\partial}\langle\partial^2\psi, \partial^2\psi\rangle = 2\langle\partial(\partial\bar{\partial}\psi), \partial^2\psi\rangle = -2\langle\partial(F\psi), \partial^2\psi\rangle = 0$, and the lemma is proved.

Remark 1.3. Corresponding to the vector-valued second fundamental form (1.3) we define $\Phi = \frac{1}{2}(B_{11} - iB_{12})$. Then

$$\Phi = F\partial\left(\frac{1}{F}\partial\psi\right) \tag{1.10}$$

and it can be shown [13] that

$$\bar{\partial}\Phi = -(1 - K)F\partial\psi\ . \tag{1.11}$$

Equations (1.10) and (1.11) generalize to minimal surfaces in S^n and give a holomorphic form $\Omega = \langle\Phi, \Phi\rangle dz^4$ on any such surface.

Observe now that if ψ is minimal, the Gauss curvature equation becomes

$$F^2(1 - K) = |\varphi|^2\ . \tag{1.12}$$

This immediately gives

LEMMA 1.4. *The Gauss curvature K of a minimal surface in S^3 satisfies $K \leqq 1$, and $K = 1$ precisely at the isolated zeros of the holomorphic differential ω.*

Let $\mathcal{R}$ be compact and of genus g. If $g = 0$, then $\omega = 0$ and ψ must be totally geodesic. If $g \geqq 1$, then ω has exactly $4g - 4$ zeros to multiplicity. Using (1.12) we can give a geometric interpretation of these zeros. For each

$p \in \mathcal{R}$ we let $\mathbf{S}_p$ denote the geodesic 2-sphere which is tangent to the immersed surface at $\psi(p)$ (i.e., tangent to the image of a small neighborhood of p), and we let P_p denote the linear subspace of $\mathbf{R}^4$ such that $\mathbf{S}_p = P_p \cap S^3$. The *order of contact* $\mathcal{O}_p$ of ψ with $\mathbf{S}_p$ at p is the largest integer k such that P_p contains the k-jet of ψ at p. Of course, $\mathcal{O}_p \geqq 1$. We define the *degree of spherical flatness* of ψ at p to be $d_p = \mathcal{O}_p - 1$.

PROPOSITION 1.5. *Let $\psi: \mathcal{R} \to S^3$ be a minimal immersion where $\mathcal{R}$ is compact and of genus g. Then*

(a) (F. Almgren) *If $g = 0$, the immersion is totally geodesic.*

(b) *If $g \geqq 1$, then $\sum_{p \in \mathcal{R}} d_p = 4g - 4$.*

In particular if $g > 1$, there must be points where $K = 1$.

PROOF. Part (a) was proved above. For part (b) we assert that $d_p =$ order of the zero of ω at p. Observe that P_p is spanned by the vectors $\psi(p)$, $\psi_1(p)$ and $\psi_2(p)$, and therefore $\mathcal{O}_p = k$ if and only if

$$\psi(p) \wedge \psi_1(p) \wedge \psi_2(p) \wedge \frac{\partial^{i+j}\psi}{\partial x_1^i \partial x_2^j}(p) = 0$$

for $0 \leqq i + j \leqq k$. From the fact that $\beta_{11} = -\beta_{22}$ the assertion and the Proposition follow easily.

We now make an observation which will be relevant later on.

COROLLARY 1.6. *It is impossible to immerse minimally the real projective plane into S^3.*

Remark 1.7. It was shown in [4] that small neighborhoods of p on the surface are divided by $\mathbf{S}_p$ like a pie into exactly $(2d_p + 4)$ wedge-like regions. For the surfaces constructed later this will be a useful means of calculation.

2. Algebraic surfaces

Associated to every minimal surface $\psi: \mathcal{R} \to S^3$ is the *cone* over that surface in $\mathbf{R}^4$ given by

$$C\psi(\mathcal{R}) = \{t\psi(p): p \in \mathcal{R} \text{ and } t \geqq 0\} .$$

It is not difficult to see that ψ is minimal in S^3 if and only if $C\psi(\mathcal{R})$ is an immersed minimal submanifold away from the origin. The surface ψ is called *algebraic* if $C\psi(\mathcal{R})$ is a homogeneous polynomial variety in $\mathbf{R}^4$. In what follows, algebraic surfaces in S^3 will be designated by the defining homogeneous polynomial. As shown in [9], an algebraic surface $p = 0$ is minimal if and only if

$$\Delta p \, |\nabla p|^2 - \nabla p^t \mathbf{H} \nabla p \equiv 0 \qquad (\text{mod } p) \tag{2.1}$$

where $\mathbf{H}$ is the hessian matrix of second derivatives of p.

One important example of an algebraic minimal surface is the flat *Clifford torus* given by

$$X_1X_2 + X_3X_4 = 0 \ . \tag{2.2}$$

This is the unique algebraic minimal surface of degree 2 and is characterized even locally as the only (non-totally-geodesic) minimal surface of constant curvature in S^3 [12].

3. The reflection principle

Let γ be the geodesic in $S^3 = \{(x_1, x_2, x_3, x_4) \in \mathbf{R}^4 \colon |x| = 1\}$ given by $x_3 = x_4 = 0$, and let S be the great 2-sphere given by $x_4 = 0$.

Definition. By *geodesic reflection across* γ we mean the map $r_\gamma \colon S^3 \to S^3$ where

$$r_\gamma(x_1, x_2, x_3, x_4) = (x_1, x_2, -x_3, -x_4) \ .$$

By *geodesic reflection across* S we mean the map $r_{\mathrm{S}} \colon S^3 \to S^3$ where

$$r_{\mathrm{S}}(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, -x_4) \ .$$

These maps can be interpreted as sending a point p to its "opposite" point on a geodesic through p which meets γ (or S) orthogonally.

PROPOSITION 3.1. *Let M be a minimal surface which is of class C^2 at the boundary ∂M. Then*

(a) *If ∂M contains a geodesic arc γ, M can be continued as an analytic minimal surface across each non-trivial component of $\partial M \cap \gamma$ by geodesic reflection.*

(b) *If part of ∂M lies in a geodesic 2-sphere* S *and if M is orthogonal to* S *there, then M can be continued as an analytic minimal surface across each non-trivial component of $\partial M \cap$ S by geodesic reflection.*

PROOF. For part (a) let γ be given by $x_3 = x_4 = 0$ and choose p in the interior of $\partial M \cap \gamma$. There is a conformal map $\Psi \colon \Delta^+ \to S^3$ where $\Delta^+ = \{(x, y) \in \mathbf{R}^2 \colon x^2 + y^2 \leqq 1 \text{ and } y \geqq 0\}$ such that: Ψ is a regular representation of M (and of ∂M) in a neighborhood of p, $\Psi(0, 0) = p$ and $\Psi_3(x, 0) = \Psi_4(x, 0) = 0$ [15, p. 366]. Since Ψ is minimal we have

$$\Delta\Psi = -\langle\nabla\Psi, \nabla\Psi\rangle\Psi \tag{3.1}$$

over Δ^+.

We now extend Ψ to the entire unit disk Δ by setting $\Psi_k(x, y) = (-1)^{[k/3]}\Psi_k(x, -y)$; $k = 1, \cdots, 4$. Clearly $\Psi_k \in C(\Delta)$ for each k, and using the minimal surface equation quickly shows that $\Psi_k \in C^2(\Delta)$ for $k = 3, 4$. It is also immediate that $\Psi_{k,x}$, $\Psi_{k,xx}$ and $\Psi_{k,yy} \in C(\Delta)$ for $k = 1, 2$. We now assert that $\Psi_{1,y} = \Psi_{2,y} = 0$ on $y = 0$. Since $\langle\Psi, \Psi\rangle \equiv 1$ we have

$$\langle\Psi, \Psi_x\rangle = \Psi_1\Psi_{1,x} + \Psi_2\Psi_{2,x} = \langle\Psi, \Psi_y\rangle = \Psi_1\Psi_{1,y} + \Psi_2\Psi_{2,y} = 0 .$$

Furthermore, $\Psi_{1,x}^2 + \Psi_{2,x}^2 = |\Psi_x|^2 > 0$ for almost all x when $y = 0$. Hence, on $y = 0$ we have $\Psi_{k,y} = \Psi_{k,xy} = 0$ for $k = 1, 2$, and therefore $\Psi \in C^2(\Delta; S^3)$. It follows that Ψ satisfies (1.7) in Δ, and therefore by Lemma 1.1 it is analytic in Δ.

Part (b) is immediate.

Note. Analogous reflection principles can be formulated and proved for minimal surfaces in $\mathbf{R}^3$ and in hyperbolic 3-space.

4. The construction procedure

We shall now discuss a general method of constructing complete, non-singular minimal surfaces in S^3 and then use the procedure (in §§ 6, 7, and 8) to generate specific families of compact surfaces.

Let Γ be a geodesic polygon in S^3 having vertices $v_0, v_1, \cdots, v_n = v_0$ and edges $\gamma_0, \gamma_1, \cdots, \gamma_n = \gamma_0$ such that for each i, γ_i meets γ_{i-1} in v_i at an angle of the form $\pi/(k_i + 1)$ where k_i is a positive integer.

Before proceeding we shall need some terminology. If γ and δ are distinct geodesics which meet in S^3, we denote by $\mathrm{S}(\gamma, \delta)$ the unique geodesic 2-sphere containing $\gamma \cup \delta$. $\mathrm{S}(\gamma, \delta)$ is said to *bound* a subset X of S^3 if X is contained in one of the two closed hemispheres determined by $\mathrm{S}(\gamma, \delta)$. For each i we denote by N_i the geodesic perpendicular to $\mathrm{S}(\gamma_{i-1}, \gamma_i)$ at v_i. Γ is then called *proper* if for each i, it is bounded either by $\mathrm{S}(\gamma_{i-1}, N_i)$ or by $\mathrm{S}(\gamma_i, N_i)$.

By the *convex hull* of Γ (cf. [14]) we mean the set

$$\mathcal{C}(\Gamma) = \bigcap \{H : H \text{ is a closed hemisphere containing } \Gamma\} .$$

If $\Gamma \subset \partial\mathcal{C}(\Gamma)$, Γ is called *convex*. We then set

$$\mathfrak{S}_\Gamma = \{\mathrm{S} : \mathrm{S} \text{ is a geodesic 2-sphere in } S^3 \text{ such that } \mathrm{S} \cap \Gamma \text{ has at least four components.}\} .$$

Finally, we denote by Δ the closed unit disk in the plane.

Throughout the following the polygon Γ is assumed to be a proper, convex curve satisfying the following:

(A) Γ lies in an open hemisphere of S^3.

(B) For each $p \in \mathcal{C}(\Gamma)^0$ there is a geodesic 2-sphere S_p containing p such that $\mathrm{S}_p \notin \mathfrak{S}_\Gamma$.

(C) Whenever one of the pair $\mathrm{S}(\gamma_{i-1}, N_i)$, $\mathrm{S}(\gamma_i, N_i)$ fails to bound Γ, we have $k_i = 1$.

(D) There exists a continuous map $\pi : \mathcal{C}(\Gamma) \to \Delta$ which is differentiable in

$\mathcal{C}(\Gamma)^0$ and carries Γ monotonically onto $\partial\Delta$ such that for each $S \in \mathcal{S}_\Gamma$ the differential of the map $\pi \mid S \cap \mathcal{C}(\Gamma)^0$ is everywhere of rank 2.

Remark 4.1. Since Γ is convex we have that for each i there is a geodesic 2-sphere S containing γ_i which bounds Γ.

Let $\Psi: \Delta \rightarrow S^3$ represent Morrey's solution to the Plateau problem for Γ [15, p. 389] and set $\mathfrak{M}_\Gamma = \Psi(\Delta)$. Ψ is continuous on Δ, analytic and almost conformal in Δ^0, represents Γ (in the sense of Fréchet) on $\partial\Delta$, and minimizes the Dirichlet and area integral among all maps in $C(\Delta; S^3) \cap H_2^1(\Delta; S^3)$ which represent Γ on $\partial\Delta$.

By [14, Th. 2] we have that $\Psi(\Delta^0) \subset \mathcal{C}(\Gamma)^0$. Condition B together with [14, Th. 4] shows that Ψ is non-singular (i.e., free of branch points) in Δ^0. Condition D, Theorem 4, and a standard monodromy argument show that Ψ is one-to-one in Δ^0. Hence Ψ *conformally imbeds* Δ^0 *into* $\mathcal{C}(\Gamma)^0 \subset S^3$.

It is well known that Ψ must be one-to-one on $\partial\Delta$ (by arguments similar to [5, pp. 63–64]). Moreover, the recent results of S. Hildebrandt [8] show that Ψ *is analytic* (*in two variables*) *at each point of the boundary which is mapped to the interior of an analytic sub-arc of* Γ. Hence Ψ is analytic on $\partial\Delta$ except possibly at the points corresponding to the vertices of Γ.

The idea now is to extend this surface by reflection across its geodesic boundary arcs. Fix i, $1 \leqq i \leqq n$, and let δ_i be the pre-image of γ_i in $\partial\Delta$. By a conformal mapping carry Ψ into the upper half disk Δ^+ such that δ_i corresponds to the arc $y = 0$. Since Ψ is analytic and one-to-one on δ_i, Proposition 3.1 shows that reflection across δ_i defines an analytic continuation of Ψ throughout all of Δ.

We need now to check that there are no points in δ_i where $|\nabla\Psi| = 0$ (i.e., no branching takes place on δ_i). Fix $(x, 0) \in \delta_i^0$ and choose a small disk $B \subset \Delta^0$ centered at $(x, 0)$. By Remark 4.1 above we can find a geodesic 2-sphere $S \supset \gamma_i$ which divides S^3 into hemispheres H^+ and H^- such that $\Psi(\Delta^+) \subset H^+$. By [14, Th. 2] we have that $\Psi(\text{int}(\Delta^+)) \subset \text{int}(H^+)$ and $\Psi(\text{int}(\Delta^-)) \subset \text{int}(H^-)$ $(\Delta^- = \Delta \sim \Delta^+)$. It follows that $\Psi(\partial B) \cap S$ consists of exactly two points, and therefore applying [14, Th. 4a] to the surface $\Psi \mid B: B \rightarrow S^3$ shows that $|\nabla\Psi| \neq 0$ at $(x, 0)$.

It is now clear that the surface $\mathfrak{M}_\Gamma$, constructed above, can be analytically continued as a non-singular minimal surface across each of its boundary arcs $\gamma_1, \cdots, \gamma_n$ by geodesic reflection.

Each reflection map $r_{\gamma_k}: S^3 \rightarrow S^3$ is an isometry. Hence any reflected image of $\mathfrak{M}_\Gamma$ can itself be reflected about its boundary arcs, and those images in turn reflected, etc. If we successively reflect $2k_i + 2$ times at the vertex v_i

we return to the original surface $\mathfrak{M}_\Gamma$. Hence the indefinite reflection process generates a nice analytic surface near v_i with a possible singularity at v_i itself.

We shall show that the surface so produced is in fact non-singular at v_i. Reflecting $\mathfrak{M}_\Gamma$ k_i-times at v_i produces a surface $\mathfrak{M}^+$ which is bounded near v_i by an unbroken geodesic arc γ, and the total surface near v_i is obtained by reflecting $\mathfrak{M}^+$ across γ. Let $\Psi^+: \Delta^+ \to S^3$ be a conformal, analytic parameterization of $\mathfrak{M}^+$ which maps the interval $[-1, 1]$ of the x-axis onto γ with $(0, 0)$ going to v_i. Ψ^+ is analytic on $[-1, 1]$ except possibly at $(0, 0)$ and is at least continuous there. Extend Ψ^+ to a map $\Psi^*: \Delta \to S^3$ by reflection across γ.

LEMMA 4.2. *Ψ^* is analytic at* $(0, 0)$.

PROOF. The surface $\mathfrak{M}^* = \Psi^*(\Delta)$ consists of $2k_i + 2$ reflected images of $\mathfrak{M}_\Gamma$. Since $\mathfrak{M}_\Gamma$ was minimizing we have an isoperimetric inequality for $\mathfrak{M}_\Gamma$ which can be extended to $\mathfrak{M}$ by raising the constant. That is, there exists a constant C such that

$$\text{Area}(D) \leqq C[\text{length}(\partial D)]^2 \tag{4.1}$$

for any domain D on $\mathfrak{M}^*$.

Let $\mathbf{R}^3$ be a system of local coordinates for S^3 obtained by stereographic projection from the point $-v_i$. The metric in these coordinates has the form

$$ds^2 = \frac{4}{(1 + |X|^2)^2} |dX|^2 , \tag{4.2}$$

where $X = (X^1, X^2, X^3)$ and $|X|$ denotes the euclidean norm. The Dirichlet integral for any S^3-valued function Φ defined in a plane domain D and represented in these coordinates is written

$$\mathfrak{D}(\Phi, D) = \iint_D \frac{4}{(1 + |\Phi|^2)^2} |\nabla\Phi|^2 \, dxdy . \tag{4.3}$$

Note that if Φ is almost conformal, then $\mathfrak{D}(\Phi, D) = 2 \times$ (area integral of Φ over D).

Represent Ψ^* in these coordinates. Then using (4.1) above and replacing the length, area and Dirichlet integrals by those corresponding to the metric (4.2), we can follow precisely the argument in [7, § 4] to show that there exist constants K and μ, independent of r and R, with $0 < \mu < 1$, such that

(1) Ψ^* is μ-Hölder continuous in Δ (in particular, at $(0, 0)$).

(2) For any $p \in \Delta$ and any r, R with $0 < r \leqq R$ we have

$$\mathfrak{D}(\Psi^*, B_r(p)) \leqq K\left(\frac{r}{R}\right)^\mu \mathfrak{D}(\Psi^*, B_R(p))$$

where $B_\rho(p) = \{q \in \Delta : |p - q| < \rho\}$.

Over domains in Δ which parameterize domains on $\mathfrak{M}_\Gamma$ or one of its images, Ψ^* minimizes the integral $\mathfrak{D}$. Since Ψ^* is analytic except possibly at (0, 0), it represents a weak solution in Δ to the equations

$$(4.4) \qquad \frac{\partial}{\partial x}\Big(\frac{\partial\Psi^*/\partial x}{(1 + |\Psi^*|^2)^2}\Big) + \frac{\partial}{\partial y}\Big(\frac{\partial\Psi^*/\partial y}{(1 + |\Psi^*|^2)^2}\Big) + 2\frac{|\nabla\Psi^*|^2}{(1 + |\Psi^*|^2)^3}\Psi^* = 0 .$$

However, this system satisfies the conditions (1.10.8″) in [15, p. 33]. Combining this fact with (1) and (2) above we can use the work of Ladyzenskaya and Ural'tseva [15, p. 34] to conclude that Ψ^* is analytic at (0, 0).

Note. Similar methods can be used to show that if the angle of Γ at v_i is a rational multiple of π, indefinite reflection produces an analytically parameterized branch point at v_i.

We are now in a position to apply the results of [14, § 4].

LEMMA 4.3. $|\nabla\Psi^*(0, 0)| \neq 0$.

PROOF. Choose a small circular disk $B \subset \Delta$ centered at (0, 0) such that $\Psi^*(B)$ is contained in an open hemisphere. By [14, Th. 3] we know that if $|\nabla\Psi^*(0, 0)| = 0$, then for every geodesic 2-sphere S containing v_i we have that $(\Psi^* \mid \partial B)^{-1}(\mathrm{S})$ consists of at least four components. To prove the lemma we shall find an S for which this set has only two points.

Suppose Γ is bounded by both $\mathrm{S}(\gamma_{i-1}, N_i)$ and $\mathrm{S}(\gamma_i, N_i)$. Then Γ lies in a narrow lens-like region L determined by these hyperspheres. Moreover, $\mathfrak{M}_\Gamma^0 \subset L^0$. Observe now that there is a tessellation of S^3 by $2k_i + 2$ regions congruent to L each of which meets N_i. When $\mathfrak{M}_\Gamma$ is reflected at v_i, each distinct image lies in a different one of these regions (with its interior in the interior of the region). The surface $\mathfrak{M}^*$ meets the interfaces of the regions in great circles which are parameterized one-to-one. It follows that if $\mathrm{S} = \mathrm{S}(\gamma_i, N_i)$, then $(\Psi^* \mid \partial B)^{-1}(\mathrm{S})$ consists of exactly two points.

Suppose, on the other hand, that $\mathrm{S}(\gamma_i, N_i) \overset{\text{def.}}{=} \mathrm{S}_0$ bounds Γ and $k_i = 1$. Since Γ is convex, there exists a geodesic 2-sphere $\mathrm{S}_1 \supset \gamma_{i-1}$ which also bounds Γ. S_0 and S_1 are perpendicular and separate S^3 into four disjoint, congruent domains. It is not difficult to see that the interiors of each of the four images of $\mathfrak{M}_\Gamma$ reflected at v_i lie in different domains and that $\Psi^*(B)$ meets $\mathrm{S}_0 \cup \mathrm{S}_1$ in nicely parameterized great circular arcs. It follows that S_0 has precisely two pre-images in ∂B and the lemma is proved.

We have now shown that indefinite reflection of $\mathfrak{M}_\Gamma$ produces a complete, non-singular submanifold in S^3 which we shall denote by M_Γ. In fact, if we let G_Γ be the subgroup of O(4) generated by the reflections $\{r_{\gamma_k}\}_{k=1}^n$, then

$$M_\Gamma = \bigcup_{g \in G_\Gamma} g[\mathfrak{M}_\Gamma] .$$

In particular, if G_Γ is finite, then M_Γ is compact.

Conversely, let $H_\Gamma = \{g \in G_\Gamma: g(\mathfrak{M}_\Gamma) = \mathfrak{M}_\Gamma\}$. H_Γ is a subgroup of the group of symmetries of Γ and hence is finite. Moreover, each coset of H_Γ in G_Γ corresponds to a distinct image of $\mathfrak{M}_\Gamma$ under G_Γ. (Distinct images may intersect but may not coincide.) Hence if M_Γ is compact, the volume of M_Γ (=Hausdorff 2-measure of M_Γ) = [ord (G_Γ)/ord (H_Γ)] $\times$ (volume of $\mathfrak{M}_\Gamma$), and G_Γ is finite.

Summing up we have the following.

THEOREM 1. *To each proper convex polygon Γ in S^3 having vertex angles of the type $\pi/(k+1)$, where k is a positive integer which depends on the vertex, and satisfying conditions* (A), (B), *and* (C) *we have associated a complete, non-singular minimal submanifold M_Γ of S^3 which contains Γ. The surface M_Γ is compact if and only if the group G_Γ generated by reflections across the geodesic sub-arcs of Γ is finite. If Γ further satisfies condition D, then the fundamental region $\mathfrak{M}_\Gamma$, which has boundary Γ and generates M_Γ under G_Γ, has no self-intersections.*

Suppose now that M_Γ is compact and let K denote the Gauss curvature function on $\mathfrak{M}_\Gamma$. By the Gauss-Bonnet formula we have that

$$\int_{\mathfrak{M}_\Gamma} K d\mathbf{H}_2 = \pi\left[2 - \sum_{i=1}^n \frac{k_i - 1}{k_i}\right]$$

where $\mathbf{H}_2$ = Hausdorff 2-measure in S^3. Consider M_Γ now as a point set in S^3 rather than as an immersed manifold. Then the Gauss curvature K is well defined $\mathbf{H}_2$-almost everywhere on M_Γ, and we see that

$$\int_{M_\Gamma} K d\mathbf{H}_2 = \frac{\text{ord }(G_\Gamma)}{\text{ord }(H_\Gamma)}\pi\left[2 - \sum_{i=1}^n \frac{k_i - 1}{k_i}\right] .$$

However, M_Γ can be considered as the image of an immersion of a compact manifold M_Γ^* (perhaps non-orientable) which is *one-to-one almost everywhere*. It is not difficult to see that the Euler characteristic $\chi(M_\Gamma^*)$ is given by

$$2\pi\chi(M_\Gamma^*) = \int_{M_\Gamma} K d\mathbf{H}_2 .$$

This proves

PROPOSITION 4.4.

$$\chi(M_\Gamma^*) = \frac{\text{ord }(G_\Gamma)}{\text{ord }(H_\Gamma)}\left(1 - \sum_{i=1}^n \frac{k_i - 1}{2k_i}\right) . \tag{4.5}$$

Remark 4.5. To simplify the group G_Γ it is sometimes useful to produce a larger fundamental domain $\widetilde{\mathfrak{M}}_\Gamma$ by making several reflections of $\mathfrak{M}_\Gamma$. We still have

$$M_\Gamma = \{g(\widetilde{\mathfrak{M}}_\Gamma): g \in G_\Gamma\} ,$$

and if we choose $\mathfrak{M}_\Gamma$ carefully the group

$$\widetilde{H}_\Gamma = \{g \in G_\Gamma: g(\widetilde{\mathfrak{M}}_\Gamma) = \widetilde{\mathfrak{M}}_\Gamma\}$$

may contain a large normal subgroup of G_Γ. Such observations can lead to drastic simplifications in applying formula (4.5).

In this regard we point out that two successive reflections at v_i constitute a rotation of $2\pi/(k_i + 1)$ about N_i.

Remark 4.6. Let $\mathcal{G}_\Gamma$ be the full group of self-congruences of M_Γ in O(4), and let $H \subset \mathcal{G}_\Gamma$ be a finite subgroup which acts freely on S^3. Then *if M_Γ^* is imbedded in S^3, M_Γ^*/H will be imbedded in S^3/H.*

5. Models for S^3

Choose coordinates (z, w) for $\mathbf{C}^2$ ($\mathbf{C}$ = complex numbers) and set

$$S^3 = \{(z, w) \in \mathbf{C}^2: |z|^2 + |w|^2 = 1\} .$$

If we view S^3 in this context, the formulas for many interesting algebraic surfaces can be simplified. For example, the Clifford torus can be written

$$\operatorname{Im}(z^2 + w^2) = 0 .$$

We now consider $\mathbf{R}^3$ with a distinguished set of coordinates (X_1, X_2, X_3) as a coordinate system for S^3 obtained by stereographic projection from the point $(z, w) = (0, -1)$. The metric in these coordinates is given by (4.2).

The origin $\mathcal{O} = (0, 0, 0)$ corresponds to the "south pole" of the projection, and the unit sphere S centered at $\mathcal{O}$ corresponds to the equatorial hypersphere. The geodesics of S^3 correspond to all straight lines through $\mathcal{O}$, all great circles of S, and all plane circles meeting S in antipodal points. The geodesic 2-spheres of S^3 correspond to all planes through $\mathcal{O}$, the sphere S and every euclidean sphere which meets S in a great circle of S^3.

We shall be particularly concerned with the distinguished great circles $C_1 = X_3$-axis and $C_2 = \{(X_1, X_2, 0): X_1^2 + X_2^2 = 1\}$. We shall assume that

$$(5.1) \qquad \begin{aligned} C_1 &\sim \{(0, w) \in \mathbf{C}^2: |w| = 1\} \\ C_2 &\sim \{(z, 0) \in \mathbf{C}^2: |z| = 1\} . \end{aligned}$$

6. The surfaces $\xi_{m,k}$

Let k and m be non-negative integers and choose points $P_1, P_2 \in C_1$ and $Q_1, Q_2 \in C_2$ such that distance$(P_1, P_2) = \pi/(k + 1)$ and distance$(Q_1, Q_2) = \pi/(m + 1)$. We define $\Gamma_{m,k}$ to be the polygon $P_1Q_1P_2Q_2$. (See Fig. 1.)

The convex hull of $\Gamma_{m,k}$ is easily seen to be a geodesic tetrahedron

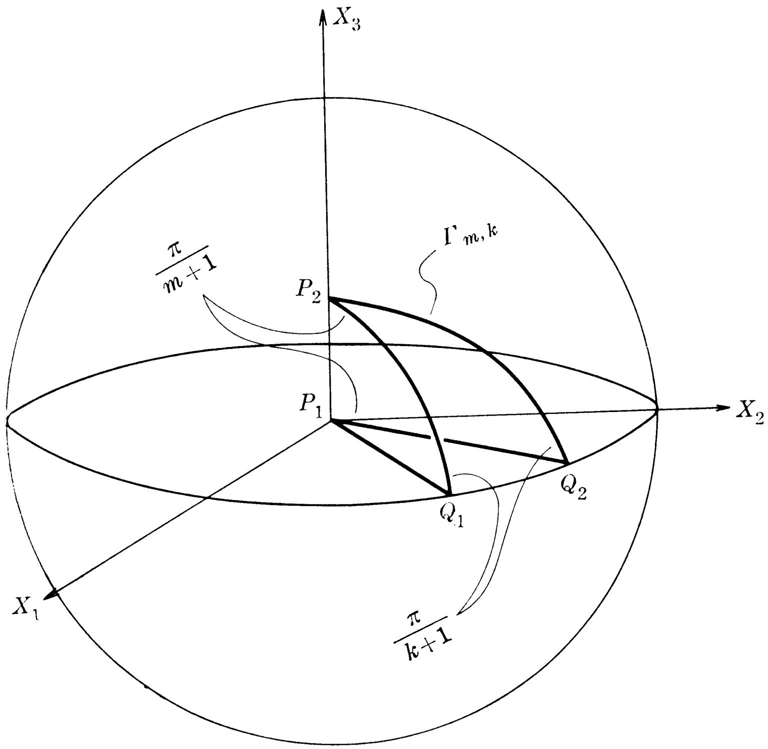

Figure 1.

(bounded by three planes and a sphere in Figure 1) whose tesselations give a simplicial decomposition of S^3, the 1-skeleton of which can be described as follows. Continue the subdivisions of C_1 and C_2 into equally spaced points $P_1, \cdots, P_{2k+2}$ and $Q_1, \cdots, Q_{2m+2}$ respectively. Let C_{ij} be the great circle containing P_i and Q_j. Then the 1-skeleton of the geodesic triangulation is

$$Sk_{m,k} = C_1 \cup C_2 \cup \left(\bigcup_{i,j} C_{ij}\right) .$$

All the polygons but $\Gamma_{m,0}$; $m \geqq 0$, are contained in an open hemisphere. However, under the above procedure each $\Gamma_{m,0}$ is seen to produce a geodesic 2-sphere without recourse to involved arguments. In view of this and the symmetry of $\Gamma_{m,k}$ in m and k we shall henceforth assume that $m \geqq k \geqq 1$.

We first observe that $\Gamma_{m,k}$ is proper, convex, and satisfies conditions (A) and (C) of § 4. What may not be evident is that $\Gamma_{m,k}$ also satisfies conditions (B) and (D).

To check condition (B) consider the family of great spheres passing through C_1, i.e., the family of planes passing through the X_3-axis in Fig. 1. For condition (D) we rotate $\Gamma_{m,k}$ to a position where the X_3-axis becomes the center line of symmetry of $\Gamma_{m,k}$, Q_1 and Q_2 lie in the plane $X_3 = 0$, and P_1, P_2

lie in the set

$$\{(0, X_2, X_3): X_2^2 + X_3^2 = 1 \quad \text{and} \quad X_3 > 0\} .$$

$\Gamma_{m,k}$ now lies on the union of two planes whose intersection is the X_1-axis. (See Fig. 3.) Let $\pi: \mathbf{R}^3 \to \mathbf{R}^2$ be orthogonal projection onto the (X_1, X_2)-plane. Then $\pi \mid \mathcal{C}(\Gamma_{m,k})^0$ is a projection having the properties necessary for condition (D).

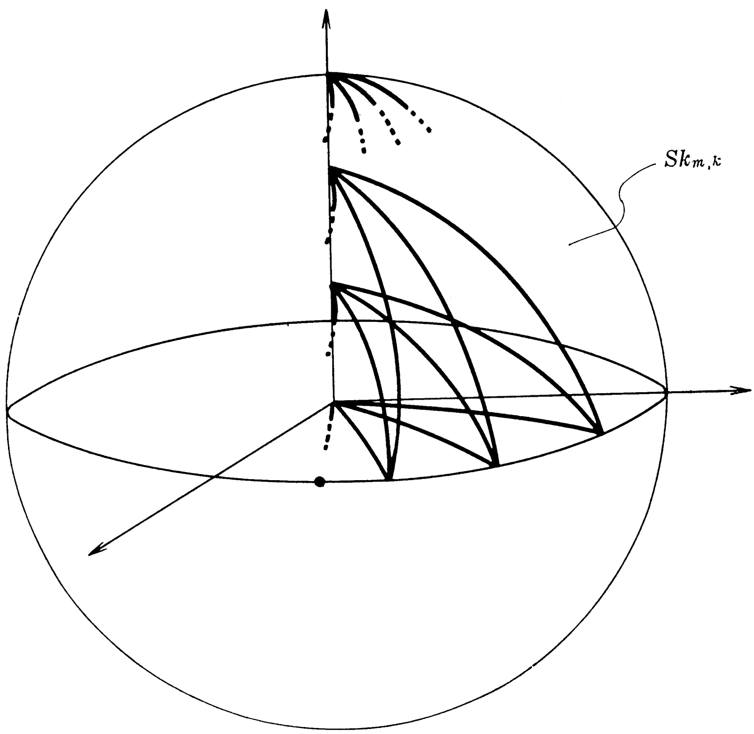

Figure 2.

Hence, by Theorem 1, $\Gamma_{m,k}$ is associated to a complete, non-singular minimal surface in S^3 which we shall denote by $\xi_{m,k}$.

PROPOSITION 6.1. *The surface $\xi_{m,k}$ is a compact orientable surface of genus mk imbedded in S^3.*

PROOF. Since $G_{\gamma_{m,k}}$ must leave $Sk_{m,k}$ invariant it is finite and $\xi_{m,k}$ is compact. $\xi_{m,k}$ can be explicitly constructed by first reflecting the imbedded surface $\mathfrak{M}_{\Gamma_{m,k}}$ $(2m + 2)$-times at P_1 and then reflecting the resulting configuration $(k + 1)$-times at Q_1. The result is a compact, imbedded (hence orientable) surface passing through a checkered array of tetrahedra comprising half the simplices in the above triangulation of S^3. The Euler characteristic $\chi(\xi_{m,k})$ can be computed by (4.5) as

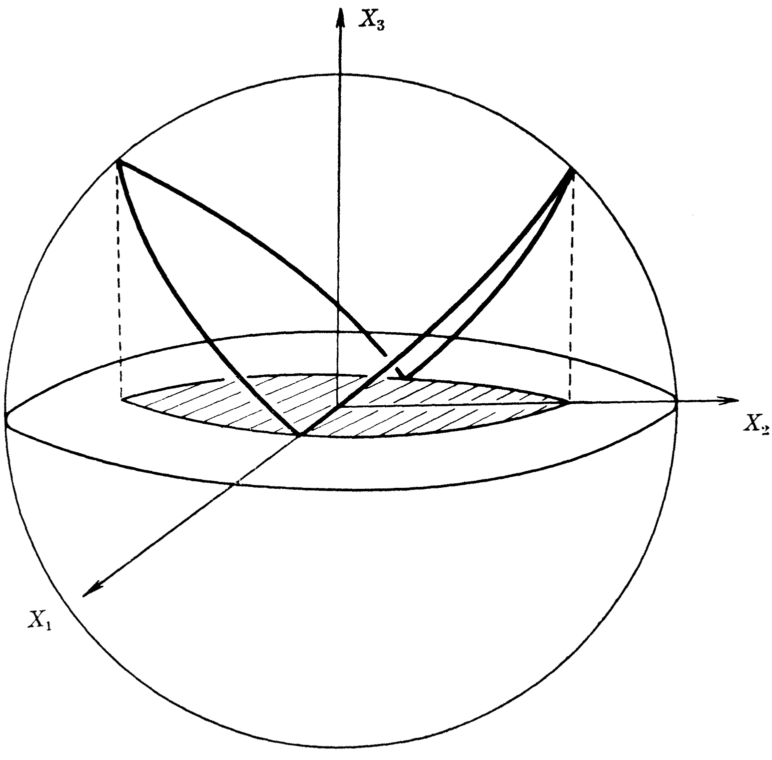

Figure 3.

$$\begin{aligned} 2\pi\chi(\xi_{m,k}) &= 2(k+1)(m+1)\int_{\mathfrak{M}_{\Gamma_{m,k}}} K d\mathbf{H}_2 \\ &= 4\pi(1-mk) . \end{aligned}$$

Alternatively, one could begin with the fundamental surface $\mathfrak{M}^*_{\Gamma_{m,k}}$ produced by reflecting $\mathfrak{M}_{\Gamma_{m,k}}$ across P_1Q_1. $\xi_{m,k}$ is then produced by acting on $\mathfrak{M}^*_{\Gamma_{m,k}}$ with the group $Z_{m+1} \times Z_{k+1}$ generated by $a: S^3 \to S^3$ and $b: S^3 \to S^3$ where

$$a(z, w) = (e^{\frac{2\pi i}{m+1}} z, w) \tag{6.1}$$

$$b(z, w) = (z, e^{\frac{2\pi i}{k+1}} w) . \tag{6.2}$$

Either way the proposition follows.

The surface $\xi_{m,0}$ (any m) is the geodesic 2-sphere, and $\xi_{1,1}$ is the Clifford torus (2.2). In fact $\xi_{g,1}$ is just the surface Σ_g described in [11].

Using techniques developed in [13] one can show that $\xi_{m,k}$ is the *unique* surface arising from this construction; that is, the solution to the Plateau problem for $\Gamma_{m,k}$ is unique even among surfaces of varying topological type. Hence, if $k > 0$ (recall that $m \geqq k$), we have that $\xi_{m,k} = \xi_{m',k'}$ *if and only if*

$m = m'$ and $k = k'$.

The uniqueness of $\mathfrak{M}_{\Gamma_{m,k}}$ further implies that $\mathfrak{M}_{\Gamma_{m,k}}$ must have the symmetries of $\Gamma_{m,k}$. These, of course, extend to all of $\xi_{m,k}$. Hence, the group of congruences of $\xi_{m,k}$ contains a subgroup generated by reflections across the edges of $\Gamma_{m,k}$ and reflections across the "planes" of symmetry of $\Gamma_{m,k}$.

A useful tool for computing this group of congruences of $\xi_{m,k}$ is the algebraic model surface $\Xi_{m,k}$ given by the equation (cf. § 5)

$$\text{(6.3)} \qquad \operatorname{Im}(z^{m+1}) + |w|^{m-k} \operatorname{Im}(w^{k+1}) = 0 .$$

Each surface $\Xi_{m,k}$ contains $Sk_{m,k} \sim \{C_1, C_2\}$ and can be generated by reflecting the piece $\Xi_{m,k} \cap \mathcal{C}(\Gamma_{m,k})$ about these geodesics. Furthermore $\Xi_{m,k}$ has the symmetries of $\Gamma_{m,k}$.

It is easy to see that $\Xi_{0,0} = \xi_{0,0}$ and $\Xi_{1,1} = \xi_{1,1}$. However, $\Xi_{2,2}$ is not minimal, and the author conjectures that $\xi_{2,2}$ is not algebraic.

Observe that by [14, Th. 4c] the Gauss curvature K of $\mathfrak{M}^0_{\Gamma_{m,k}}$ is everywhere <1. Applying the theorem to $\mathfrak{M}^*_{\Gamma_{m,k}}$ shows that $K < 1$ on smooth subarcs of $\mathfrak{M}_{\Gamma_{m,k}}$. Hence, the zeros of the holomorphic form ω defined by (1.8) appear precisely at the points $P_1, \cdots, P_{2k+2}$ and $Q_1, \cdots, Q_{2m+2}$. By Remark 1.7 the order of the zero at P_i must be $k - 1$ and, at Q_j, $m - 1$. It follows that ω has $4mk - 4$ zeros to multiplicity as required. The locus of these zeros, the spherically flat points, is the set where the linked great circles C_1 and C_2 meet $\xi_{m,k}$ orthogonally.

One important conclusion from all this is

THEOREM 2. *For each non-negative integer g there is a minimal imbedding of a compact orientable surface of genus g into S^3. If g is not prime, the imbedding is not unique.*

7. The surfaces $\tau_{m,k}$

By taking different Hamilton circuits on the same family of geodesic tetrahedra we can produce an entirely different family of minimal surfaces.

Fix positive integers m and k and let P_1, P_2, Q_1, Q_2 be chosen as in § 6. We denote by $\Gamma'_{m,k}$ the geodesic polygon $P_1P_2Q_2Q_1$. (Note. $\mathcal{C}(\Gamma'_{m,k}) = \mathcal{C}(\Gamma_{m,k})$. See Figure 4.)

Theorem 1 associates to $\Gamma'_{m,k}$ a compact, non-singular minimal submanifold $\tau_{m,k} = M_{\Gamma'_{m,k}}$ which by formula (4.5) must have Euler characteristic zero. Verification of the hypotheses of Theorem 1 will be omitted, however, because $\tau_{m,k}$ can be explicitly described by the doubly periodic immersion $\Psi_{m,k}: \mathbf{R}^2 \to S^3$ given by

$$\text{(7.1)} \qquad \Psi_{m,k}(x, y) = (\cos mx \cos y, \sin mx \cos y, \cos kx \sin y, \sin kx \sin y) .$$

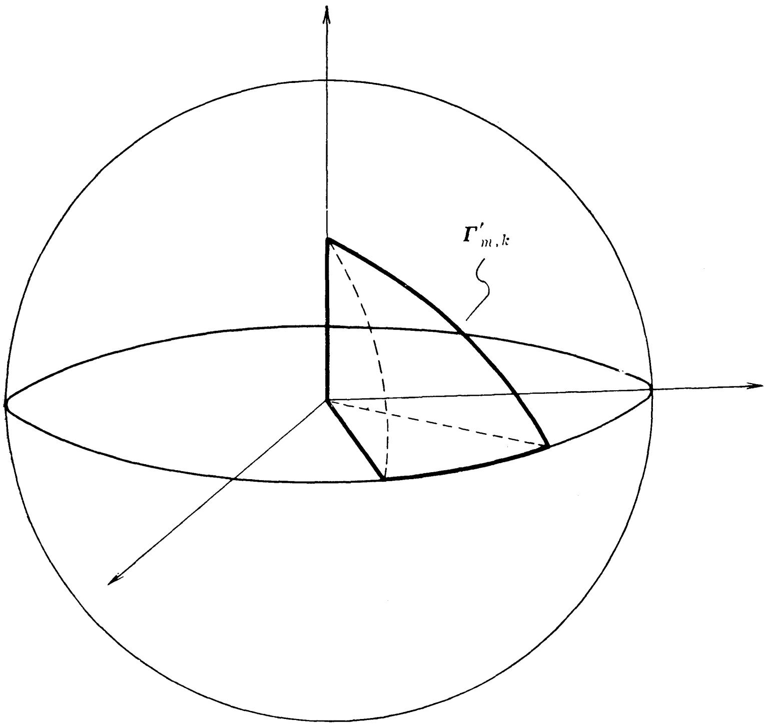

Figure 4.

It is clear from this representation that $\tau_{m,k} = \tau_{m',k'}$ (up to congruences) if and only if there is some integer p such that $m \equiv m' \pmod p$ and $k \equiv k' \pmod p$. Hence we have a countable family of compact surfaces with the following properties.

THEOREM 3. *To each unordered pair of positive integers* $\{m, k\}$ *with* $(m, k) = 1$ *there corresponds a distinct, compact minimal surface* $\tau_{m,k}$ *of Euler characteristic zero in* S^3 *given by* (7.1). *Moreover*

(a) $\tau_{m,k}$ *is non-orientable* (*an immersed Klein bottle*) *if and only if* $2/mk$.

(b) $\tau_{m,k}$ *is real algebraic of degree* $m + k$ *and satisfies the equation*

$$\operatorname{Im}\{z^k \bar{w}^m\} = 0$$

(cf. §§ 2 and 5).

(c) *Each* $\tau_{m,k}$ *admits a distinct one-parameter group of self-congruences.*

(d) *Each* $\tau_{m,k}$ *is geodesically ruled.*

(e) Area $(\tau_{m,k}) \geqq 2\pi^2 \min\{m, k\}$.

(f) $\tau_{1,1}$ *is the Clifford torus and the only surface* $\tau_{m,k}$ *without self-intersections.*

With this family we are able to answer a question of Wu-Yi Hsiang [9].

COROLLARY 7.1. *There exist algebraic minimal hypersurfaces of arbitrary degree in* S^3.

In fact the hypersurfaces

$$\text{Re}\,\{Z_1^{k_1}\cdots Z_n^{k_n}\} \cap S^{2n-1} \subset S^{2n-1} \subset \mathbf{C}^n$$

with $(k_1, \cdots, k_n) = 1$ are always minimal in S^{2n-1} (although not always immersions of non-singular varieties).

We note that the tori $\tau_{m,k}$ are not the ones known to T. Otsuki [16] and E. Calabi [11] because the self-congruence groups are inequivalent.

PROOF OF THEOREM 3. Suppose that $2/mk$. Then either $2/m$ and we have $\Psi_{m,k}(x, y) = \Psi_{m,k}(x + \pi, 2\pi - y)$ or $2/k$ and $\Psi_{m,k}(x, y) = \Psi_{m,k}(x + \pi, \pi - y)$. In either case $\Psi_{m,k}$ immerses a Klein bottle.

To see the converse of this we first observe that $\tau_{m,k}$ is invariant under the action $\Phi_t\colon S^3 \to S^3$ of S^1 on S^3 given by

$$\Phi_t(z, w) = (e^{imt}z, e^{ikt}w) \qquad t \in \mathbf{R}\,. \tag{7.2}$$

In fact $\tau_{m,k}$ is just the union of the orbits meeting the curve $y \to (\cos y, 0, \sin y, 0)$. These orbits are mutually distinct for $y \in [0, \pi)$, and if $2 \nmid mk$ they are distinct for $y \in [0, 2\pi)$. Parts (a) and (c) follow.

Observe that $\Psi_{1,1}$ is the only imbedding. In all other cases the immersion cuts itself m and k times respectively in the distinguished great circles C_1 and C_2, (5.1), which are the exceptional orbits of the action (7.2).

From the immersion (7.1) we see that the curves $y \to \Psi_{m,k}(x, y)$ for any x are geodesics on S^3. This gives part (d). Parts (b) and (f) are straightforward. Part (e) follows from the fact that the metric induced by $\Psi_{m,k}$ has the form $ds^2 = (m^2 \cos^2 y + k^2 \sin^2 y)dx^2 + dy^2$. This completes Theorem 3.

We now show that the fact that $\tau_{m,k}$ is ruled characterizes the surface.

PROPOSITION 7.2. *Every ruled minimal surface in* S^3 *is an open submanifold of one of the surfaces* $\mathfrak{M}_\alpha$ *given by*

$$\Psi(x, y) = (\cos \alpha x \cos y, \sin \alpha x \cos y, \cos x \sin y, \sin x \sin y)$$

for some $\alpha > 0$.

PROOF. Let $\mathfrak{M}$ be a ruled minimal surface in $S^3 \subset \mathbf{R}^4$ and choose a curve γ on $\mathfrak{M}$ which cuts the family of great circles on $\mathfrak{M}$ orthogonally. Consider γ as a curve in $\mathbf{R}^4$ with $|\gamma| = 1$. Then the great circle of $\mathfrak{M}$ passing through $\gamma(t)$ is given by $\theta \to \gamma(t)\cos\theta + \nu(t)\sin\theta$ where $\nu(t)$ is a curve in $\mathbf{R}^4$ satisfying

(i) $|\nu| = 1$,

(ii) $\langle \nu, \gamma \rangle = 0$,

(iii) $\langle \nu, \gamma' \rangle = 0$ (and, therefore, $\langle \nu', \gamma \rangle = 0$).

Hence $\mathfrak{M}$ is described locally by $\psi(t, \theta) = \gamma(t)\cos\theta + \nu(t)\sin\theta$ and the induced metric is $ds^2 = A\,dt^2 + d\theta^2$ where

$$A = |\gamma'|^2 \cos^2\theta + 2\langle \gamma', \nu' \rangle \cos\theta \sin\theta + |\nu'|^2 \sin^2\theta .$$

In these coordinates the minimal surface equation (1.7) has the form

$$\Big(\frac{1}{A}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial \theta^2} + \frac{A_\theta}{2A}\frac{\partial}{\partial\theta}\Big)\psi = -2\psi ,$$

which reduces to the following pair of equations

$$\gamma'' + |\gamma'|^2 \gamma + \langle \gamma', \nu' \rangle \nu = 0 \tag{7.3}$$

$$\nu'' + |\nu'|^2 \nu + \langle \gamma', \nu' \rangle \gamma = 0 . \tag{7.4}$$

It follows immediately that $\langle \gamma'', \gamma' \rangle = \langle \gamma'', \nu' \rangle = \langle \nu'', \gamma' \rangle = \langle \nu'', \nu' \rangle = 0$ and therefore each of the functions $|\gamma'|^2$, $|\nu'|^2$ and $\langle \gamma', \nu' \rangle$ are constant.

If we replace θ by $\theta_0 + \theta'$ and write $\psi_1(t, \theta') = \gamma_1(t)\cos\theta' + \nu_1(t)\sin\theta'$, then the functions $\gamma_1 = \gamma\cos\theta_0 + \nu\sin\theta_0$ and $\nu_1 = \gamma\sin\theta_0 + \nu\cos\theta_0$ also satisfy equations (7.3) and (7.4) and conditions (i), (ii), and (iii). For the proper choice of θ_0 we have $\langle \nu_1', \gamma_1' \rangle \equiv 0$. By a linear change of the variable t we get $|\gamma_1'|^2 \equiv 1$ and $|\nu_1'|^2 \equiv \alpha^2 =$ constant.

It is now straightforward to show that in a properly chosen orthogonal basis for $\mathbf{R}^4$ $\gamma_1(t) = (\cos t, \sin t, 0, 0)$ and $\nu_1(t) = (0, 0, \cos\alpha t, \sin\alpha t)$, and the proposition is proved.

Remark 7.3. It has been noted that the surface $\tau_{m,k}$ is invariant under a compact group of isometries of S^3. In a forthcoming paper by Wu-Yi Hsiang and the author [10] the compact minimal surfaces in S^3 which are invariant under non-trivial, connected groups of isometries of S^3 are classified. For groups of dimension greater than one the only possibilities are the geodesic 2-sphere and the Clifford torus. However, for each compact group of dimension one, there exists a countably infinite family of surfaces.

8. The surfaces $\eta_{m,k}$

We shall now construct a family of non-orientable surfaces. Fix integers $m, k \geqq 1$ and choose points P_1, P_2, Q_1, Q_2 as in § 6. We shall denote by $\gamma_{m,k}$ the (four-sided) polygon $P_2Q_2P_1(-Q_2)Q_1$ as shown in Figure 5.

An apparent obstruction to applying Theorem 1 to this polygon is that condition (A) does not hold. This condition was assumed in order to use the theorems of [14, §§ 3 and 4]. However, by using [14, Prop. 1] and the fact that there is a closed hemisphere H containing $\gamma_{m,k}$ with $H \cap \gamma_{m,k} = \{Q_2, -Q_2\}$, all the results of these sections can easily be shown to hold for area minimiz-

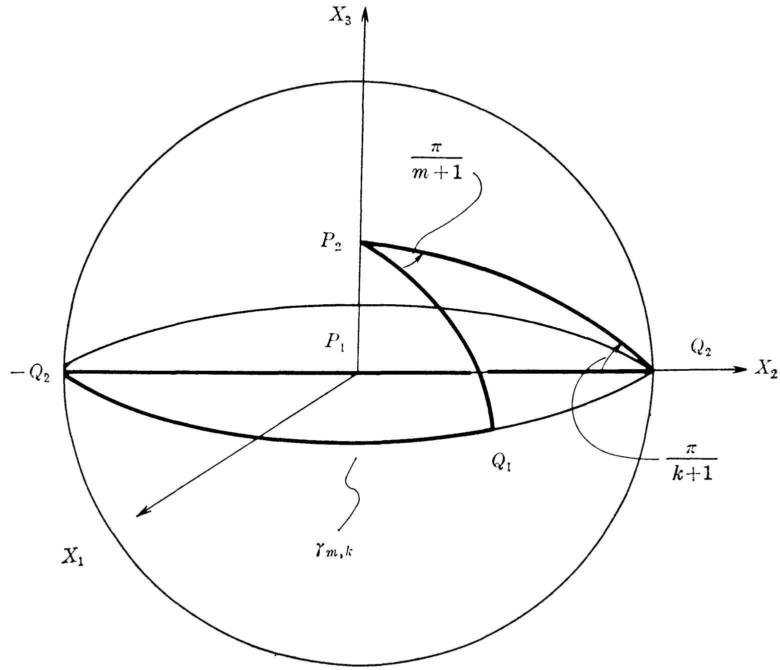

Figure 5.

ing surfaces with boundary $\gamma_{m,k}$. Hence we need only verify conditions (B), (C), and (D).

With reference to Figure 5 the convex hull of $\gamma_{m,k}$ can be described as the set bounded by the planes $X_1 = 0$ and $X_3 = 0$ and by the sphere which contains P_2 and meets $X_3 = 0$ in the circle $X_1^2 + X_2^2 \equiv 1$. Consideration of the family of planes through the X_2-axis quickly establishes condition (B).

Condition (D) can be verified as follows. Project S^3 stereographically from the point $-Q_2$ and choose coordinates on R^3 so that Q_2P_2 lies on the X_3-axis, $Q_1(-Q_2)$ lies on the X_2-axis, and $Q_2P_1(-Q_2)$ is mapped onto the plane $X_2 = 0$. (See Fig. 6.) Let Δ' be the closure in S^3 of the positive quadrant of the (X_1, X_2)-plane. We define a map $\pi\colon \mathcal{C}(\gamma_{m,k}) \to \Delta'$ by projecting $\mathcal{C}(\gamma_{m,k}) \sim \{-Q_2\}$ onto the (X_1, X_2)-plane along the X_3-axis and setting $\pi(-Q_2) = -Q_2$.

The map π is not continuous at $-Q_2$. However, it is sufficient for our purposes to know that $\pi \mid \mathfrak{M}_{\gamma_{m,k}}$ is continuous at $-Q_2$. This latter fact is true and is proved as follows. Using Hildebrandt [8] and the methods of § 4 we have that $\mathfrak{M}_{\gamma_{m,k}}$ is regular and analytic along its smooth boundary arcs and, since the angle at $-Q_2$ is $\pi/2$, it is also regular and analytic at $-Q_2$. Let $S_0 = S(Q_2P_1, Q_2Q_1)$. In Figure 6, S_0 is represented by a plane passing through the X_2-axis. Let S_1 be a geodesic hypersphere represented by another plane through the X_2-axis which lies above S_0 when X_1 is positive. The regularity

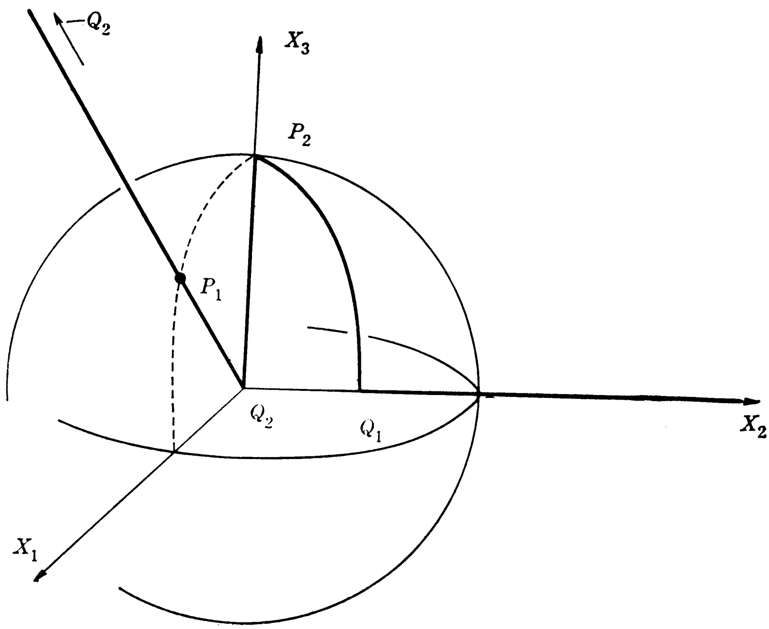

Figure 6.

of $\mathfrak{M}_{\gamma_{m,k}}$ at $-Q_2$ implies that small neighborhoods of $\mathfrak{M}_{\gamma_{m,k}}$ at $-Q_2$ will lie between S_0 and S_1 in $\mathcal{C}(\gamma_{m,k})$. The continuity of $\pi \mid \mathfrak{M}_{\gamma_{m,k}}$ at $-Q_2$ follows easily.

To verify the rest of condition D for the map π we make the following observations.

(a) If S is a great 2-sphere of S^3 such that $S \cap \gamma_{m,k}$ has four or more components, then S meets the interiors of each of the four geodesic subarcs of $\gamma_{m,k}$. This follows from the fact that for any geodesic γ of length $\leqq \pi$, either $\gamma \subset S$, or $\gamma \cap S = \{p\}$, or length $(\gamma) = \pi$ and $\gamma \cap S = \{p, -p\}$.

(b) Let S′ be the image of a great 2-sphere in the stereographic coordinate system above, and set $S^+ = S' \cap \{X \in R^3 \colon X_3 \geqq 0\}$. Then S^+ has a nonsingular perpendicular projection onto the (X_1, X_2)-plane if and only if S^+ lies in a hemisphere of S′ (when S′ is considered as a euclidean sphere in $\mathbf{R}^3$).

These facts make straightforward the proof that π satisfies condition D and therefore that $\mathfrak{M}_{\gamma_{m,k}}$ is an embedded disk.

For $m = 1$ and k arbitrary condition (D) is satisfied, but when $m > 1$, it fails at the vertex P_2. Nevertheless, for any m the methods of § 3 show that the extended surface at P_2 is analytically parameterized with a possible isolated drop in rank of the jacobian at P_2. Furthermore, if for a neighborhood N of P_2 on the surface $\mathfrak{M}_{\gamma_{m,k}}$ the set $N \sim \{P_2\}$ can be shown to lie in the region of $\mathcal{C}(\gamma_{m,k})$ where $X_1 \cos\big(2\pi/(m+1)\big) - X_2 \sin\big(2\pi/(m+1)\big) < 0$ (again with reference to Figure 5), then the arguments of § 4 will show that no branching can occur at P_2. These necessary local bounds for $\mathfrak{M}_{\gamma_{m,k}}$ can be obtained inductively by using $\mathfrak{M}_{\gamma_{m-1,k}}$ as a comparison surface for $\mathfrak{M}_{\gamma_{m,k}}$. In particular we shall show that when the boundaries $\gamma_{m-1,k}$ and $\gamma_{m,k}$ are fit together so that their convex hulls coincide, the open surfaces $\mathfrak{M}^{\circ}_{\gamma_{m-1,k}}$ and $\mathfrak{M}^{\circ}_{\gamma_{m,k}}$ are disjoint. The fact that $\mathfrak{M}_{\gamma_{m-1,k}}$ (or equivalently $\eta_{m-1,k}$) has a well-defined tangent plane at P_2 then provides the necessary local bound for $\mathfrak{M}_{\gamma_{m,k}}$ at P_2.

The fact claimed above can be proved either by methods developed in [13] or as follows. Let $\mathfrak{M}$ and $\mathfrak{M}'$ respectively denote the surfaces $\mathfrak{M}_{\gamma_{m-1,k}}$ and $\mathfrak{M}_{\gamma_{m,k}}$ and suppose $\mathfrak{M}$ and $\mathfrak{M}'$ are situated so that $\mathcal{C}(\partial\mathfrak{M}) = \mathcal{C}(\partial\mathfrak{M}')$. Let $\gamma = \mathfrak{M}^{\circ} \cap \mathfrak{M}'^{\circ}$. Then γ is the union of open analytic arcs which intersect exactly at points of tangency of the surface. (If $\mathfrak{M}$ and $\mathfrak{M}'$ are tangent at a point p, then in a small neighborhood of p on $\mathfrak{M}$ the distance from $\mathfrak{M}$ to $\mathfrak{M}'$ along the normal direction looks like a harmonic polynomial of degree $\geqq 2$.)

It is now possible to find components $\mathfrak{D}$ and $\mathfrak{D}'$ of $\mathfrak{M}^{\circ} \sim \gamma$ and $\mathfrak{M}^{\circ\prime} \sim \gamma$ respectively such that $\partial\mathfrak{D} = \partial\mathfrak{D}'$. This follows from the fact that the projection π discussed above simultaneously maps $\mathfrak{M}$ and $\mathfrak{M}'$, each in a one-to-one way, onto the disk Δ. If $\mathfrak{D}_0$ is any component of $\Delta^0 \sim \pi(\gamma)$, then $\mathfrak{D}$ and $\mathfrak{D}'$ can be chosen as the intersection of $\mathfrak{M}$ and $\mathfrak{M}'$ with $\pi^{-1}(\mathfrak{D}_0)$.

Assume $\gamma \neq \emptyset$. Then $\partial\mathfrak{D} \cap \gamma \neq \emptyset$. Let $\mathfrak{M}'' = (\mathfrak{M} \sim \mathfrak{D}) \cup \mathfrak{D}'$ and observe that $\mathfrak{M}''$ again minimizes area for the boundary $\gamma_{m-1,k}$. However, due to the isolated nature of the points of tangency of $\mathfrak{M}$ and $\mathfrak{M}'$ the surface $\mathfrak{D}'$ must meet the surface $\mathfrak{M} \sim \mathfrak{D}$ at an angle $<\pi$ almost everywhere along the "seam" $\partial\mathfrak{D}$. By deforming the surface slightly in a neighborhood of one of these points we can easily construct a parametric surface having boundary $\gamma_{m-1,k}$ and area strictly less than the area of $\mathfrak{M}$. This contradicts the minimality of $\mathfrak{M}$. Thus $\gamma = \emptyset$, which is what we were to prove.

Of course the methods of § 4 already apply to the other vertices of $\gamma_{m,k}$, and it follows that indefinite reflection of the surface $\mathfrak{M}_{\gamma_{m,k}}$ produces a complete, non-singular minimal submanifold which we denote $\eta_{m,k}$. Since the generators of $G_{\gamma_{m,k}}$ leave the graph $Sk_{m,k}$ invariant, we have that $\eta_{m,k}$ is compact.

THEOREM 4. *To each ordered pair of positive integers (m, k), where k is odd, there corresponds a compact, non-orientable minimal surface $\eta_{m,k}$ containing $\gamma_{m,k}$ and having Euler characteristic $1 - mk$.*

We recall that by Corollary 1.6 the real projective plane cannot be minimally immersed into S^3. However, by Theorems 2 and 4 we have the following

THEOREM 5. *Every compact surface but the real projective plane can be minimally immersed into S^3. For orientable surfaces the immersions can* *be chosen without self-intersections.*

PROOF OF THEOREM 4. Let $\mathfrak{M}_0$ be the surface obtained by reflecting $\mathfrak{M}_{\gamma_{m,k}}$ across the arc P_2Q_2. We observe as before that two successive reflections at P_2 (resp. Q_2) constitute a rotation of $2\pi/(m+1)$ (resp. $2\pi/(k+1)$) about C_1 (resp. C_2). These rotations are precisely the maps (a) and (b) given by (6.1) and (6.2). They generate the subgroup $G_0 = \mathbf{Z}_{m+1} \times \mathbf{Z}_{k+1}$ of $G_{\gamma_{m,k}}$.

We now consider the subset η' of $\eta_{m,k}$ defined by

$$\eta' = \bigcup_{g \in G_0} g(\mathfrak{M}_0) .$$

This set can also be constructed by reflecting $\mathfrak{M}_{\gamma_{m,k}}$ $(2m+2)$ times at P_2 and then reflecting the resulting configuration $k+1$ times at Q_2.

Let $r_0, \cdots, r_3$ denote the reflections across the geodesics containing P_2Q_2, P_2Q_1, $Q_2P_1(-Q_2)$, and $(-Q_2)Q_1$, respectively. It is straightfoward to check that

$$r_k a r_k = a^{-1} \tag{8.1}$$

$$r_k b r_k = b^{-1} \tag{8.2}$$

for $k = 0, 1, 2$.

We now show that η' is invariant under r_0, r_1 and r_2. Since $\mathfrak{M}_0$ is invariant under r_0, we have $\eta' = \bigcup_{g \in G_0} g \cdot r_0(\mathfrak{M}_0)$, and thus by (8.1) and (8.2) η' is invariant under r_0. For r_1 and r_2 we observe that

$$\mathfrak{M}_0 \cup r_1(\mathfrak{M}_0) = \mathfrak{M}_0 \cup a(\mathfrak{M}_0) \overset{\text{def.}}{=} \mathfrak{M}_1$$
$$\mathfrak{M}_0 \cup r_2(\mathfrak{M}_0) = \mathfrak{M}_0 \cup b(\mathfrak{M}_0) \overset{\text{def.}}{=} \mathfrak{M}_2 .$$

Hence $\eta' = \bigcup \{g(\mathfrak{M}_k); g \in G_0\}$ for $k = 1$ or 2, and since $\mathfrak{M}_k$ is r_k-invariant, it follows from (8.1) and (8.2) that η' is also r_k-invariant. Hence η' is invariant under the group G_1 generated by G_0 and the elements r_0, r_1, r_2.

Since k is odd we have that r_3 ($=$ rotation of π about C_2) is an element of G_0. Thus G_1 coincides $G_{\gamma_{m,k}}$, and therefore $\eta' = \eta_{m,k}$. Further, it is evident that the subgroup H_0 of G_0 given by $H_0 = \{g \in G_0 \colon g(\mathfrak{M}_0) = \mathfrak{M}_0\}$ is just the identity subgroup. Hence by Proposition 4.4 and Remark 4.5 the Euler characteristic $\chi(\eta_{m,k})$ of $\eta_{m,k}$ can be computed as

$$\begin{aligned} 2\pi\chi(\eta_{m,k}) &= (m+1)(k+1) \int_{\mathfrak{M}_0} K dH_2 \\ &= 2(m+1)(k+1) \int_{\mathfrak{M}_{m,k}} K dH_2 \\ &= 2\pi(1 - mk) \end{aligned}$$

where K is the Gauss curvature function.

It remains to show that when $\chi(\eta_{m,k})$ is even, $\eta_{m,k}$ is still non-orientable. This can be verified by considering a simple, non-contractible, closed curve on the piece of surface $\mathfrak{M}_{\gamma_{m,k}} \cup r_3(\mathfrak{M}_{\gamma_{m,k}}) \cup N$ where N is a small neighborhood of Q_2 on the surface $\eta_{m,k}$. Any such circuit is orientation reversing. This completes the proof.

Remark 8.1. It is possible to show that when k is even, the surface $\eta_{m,k}$ is orientable and has Euler characteristic $2(1 - mk)$. To see this it is helpful to note that r_3 lies in the center of $G_{\gamma_{m,k}}$.

9. Imbeddings into spherical space forms

The large groups of self-congruences of the surfaces M_Γ make it possible to construct minimal imbeddings of surfaces into many of the three-dimensional spherical space forms. It has been shown by T. Frankel [6] that if a compact surface M can be minimally imbedded into a space of the type S^3/G, then there exists a covering $p: M' \to M$ by a compact orientable surface M' such that the sequence

$$1 \longrightarrow \pi_1(M') \longrightarrow \pi_1(M) \longrightarrow G \longrightarrow 1$$

is exact. There are several immediate consequences of this.

(a) If $G \neq \{e\}$, then M cannot be homeomorphic to S^2.

(b) If $G \neq \mathbf{Z}_2$, then M cannot be the real projective plane.

(c) If G is not abelian, then M is also not a torus. If, furthermore, G does not have a subgroup $\approx \mathbf{Z}_n \times \mathbf{Z}_m$ of index 2, then the Euler characteristic of M is negative.

(d) If M is non-orientable, then G has even order. (The group of orientation preserving paths at $*$ has index 2 in $\pi_1(M, *)$ and contains $p_*(\pi_1(M, *))$.) Subject to these restrictions it is interesting to see how many imbeddings can be achieved.

The basic observation is the following. Suppose M_Γ is imbedded, and let $\mathcal{G}_\Gamma$ be the group of congruences of M_Γ in S^3. If $\mathcal{G}_\Gamma$ contains a subgroup H which acts freely on S^3 then the minimal surface M_Γ/H is imbedded in the space form S^3/H.

Recall that the group of congruences of the surface $\xi_{m-1,k-1}$ in S^3 corresponds to the group of symmetries in O(4) of the equation

$$\operatorname{Im}(z^m + |w|^{m-k}w^k) = 0 . \tag{9.1}$$

Hence, for each integer n such that $n/(m, k)$ and for each r with $(r, n) = 1$ we have that the surface $\xi_{m-1,k-1}$ is invariant under the group $\mathbf{Z}_n$ generated by the map

$$A_{n,r}(z, w) = (e^{\frac{2\pi i}{n}}z, e^{\frac{2\pi r i}{n}}w) . \tag{9.2}$$

Under projection $\xi_{m-1,k-1}$ covers a compact orientable minimal surface of genus

$(m-1)(k-1)+(n-1)mk$ imbedded in the lens space $L_{n,r}$.

If we express the Clifford torus as

$$\tau_{1,1} = \left\{\frac{1}{\sqrt{2}}(e^{i\theta}, e^{i\varphi}) \in \mathbf{C}^2 : (\theta, \varphi) \in \mathbf{R}^2\right\} \tag{9.3}$$

we see that it is invariant under $A_{n,r}$ for all n, r and covers a flat minimal torus in $L_{n,r}$. Moreover, from (9.3) it is clear that $\tau_{1,1}$ is also invariant under the group $\mathbf{Z}_{4n}$ generated by $g\colon \mathbf{C}^2 \to \mathbf{C}^2$ where $g(z, w) = (\alpha w, z)$ and $\alpha = e^{\pi i/n}$. This gives minimal imbeddings of flat Klein bottles into the spaces $L_{4n,1}$ for $n \geqq 1$.

We now observe that for each odd $l \geqq 1$ the surface $\xi_{n-1,n-1}$ is invariant under $A_{2n,r}$ whenever $(2n, r) = 1$. Under projection $\xi_{n-1,n-1}$ covers a surface which is non-orientable. (The geodesic joining Q_1 to Q_2 on $\mathfrak{M}_{\gamma_{n-1,n-1}}$ becomes an orientation reversing loop.)

Combining the above we have

THEOREM 6. *For each set of integers $n, r, m, k \geqq 1$ where $(n, r) = 1$ there exists a compact orientable minimal surface of genus $(m-1)(k-1)+(n-1)mk$ imbedded in the lens space $L_{n,r}$. For each set of integers $n, r, l \geqq 1$ where $(2n, r) = 1$ and l is odd there exists a compact, non-orientable minimal surface of Euler characteristic $1 - l(n-1)$ imbedded in $L_{2n,r}$.*

Furthermore there are minimal tori imbedded in each space $L_{n,r}$ and minimal Klein bottles in each $L_{4n,1}$.

By setting $n = 2$ and $k = 1$ and by recalling Theorem 4 we get a result for real projective space $\mathbf{R}P^3$ analogous to those above for the sphere.

COROLLARY 9.1. *Every compact, orientable surface but the sphere (which is prohibited by Frankel's theorem) can be minimally imbedded into $\mathbf{R}P^3$. The immersion of $\mathbf{R}P^2$ is unique and one-to-one.*

We now consider space forms arising from the binary dihedral groups, $\mathbf{D}_n^*$. For each integer $n > 1$, $\mathbf{D}_n^*$ is a group of order $4n$ with presentation

$$A^n = B^4 = 1, \quad BAB^{-1} = A^{-1} \qquad \text{if } n \text{ is odd},$$
$$A^{2n} = 1, \quad A^n = B^2, \quad BAB^{-1} = A^{-1} \qquad \text{if } n \text{ is even}.$$

This group can be represented freely on S^3 by defining

$$A(z, w) = \begin{cases} (e^{\frac{2\pi i}{n}} z, e^{\frac{2\pi i}{n}} w) & \text{if } n \text{ is odd},\\ (e^{\frac{\pi i}{n}} z, e^{\frac{\pi i}{n}} w) & \text{if } n \text{ is even},\end{cases}$$
$$B(z, w) = (-\bar{w}, \bar{z}).$$

Considering the equation $\operatorname{Im}\{z^{mn} + w^{mn}\} = 0$ shows that $\xi_{mn-1,mn-1}$ is invariant under $\mathbf{D}_n^*$ for each positive integer m such that $2/mn$. Furthermore, the flat torus (9.3) is also D_n^* invariant. This gives

THEOREM 7. *For each integer $m \geqq 0$ such that $2/mn$ there is a compact minimal surface of Euler characteristic $m(1 - mn/2)$ imbedded in $S^3/\mathbf{D}_n^*$.*

Note that whenever $n = 4n'$, each of the surfaces $\xi_{mn-1,mn-1}/\mathbf{D}_n^*$ for odd m is non-orientable.

Remark 9.2. For each non-homogeneous space form of the type S^3/H where H is a free representation of some $\mathbf{Z}_u \times D_v^*$ the same methods will again produce imbeddings. One simply uses the representations found in [20, p. 224].

10. Polar varieties

Let $\psi: \mathcal{R} \rightarrow R^3 \subseteq R^4$ represent a surface in the sense of § 1. The associated *Gauss map* $\psi^*: \mathcal{R} \rightarrow S^3$ is defined pointwise as the image of the unit normal in S^3 translated to the origin of $\mathbf{R}^4$. In local coordinates

$$\psi^* = \frac{1}{iF}\psi \wedge \partial\psi \wedge \bar{\partial}\psi \;.$$

(We identify $\bigwedge^3 \mathbf{R}^4$ with $\mathbf{R}^4$ by the $*$-isomorphism.) This definition is a natural generalization of the Gauss map for surfaces in $\mathbf{R}^3$. For example, we still have the "Weingarten equations" (1.6)(where $\eta = \psi^*$), and we have that the relative curvature $|1 - K|$ can be interpreted as the ratio of the induced volume elements dV_{ψ^*}/dV_{ψ}. Moreover, it is possible to show that *ψ^* is conformal if and only if ψ is minimal or parameterizes a constant curvature hypersphere* [13].

Assume that ψ is minimal. Since

$$\partial\psi^* = \frac{1}{iF}\psi \wedge \Phi \wedge \bar{\partial}\psi = \frac{1}{F}\varphi\partial\psi \tag{10.1}$$

where Φ and φ are given by (1.10) and (1.8) we have that $\langle\partial\psi^*, \partial\psi^*\rangle = \langle\bar{\partial}\psi^*, \bar{\partial}\psi^*\rangle = 0$ and $\langle\partial\psi^*, \bar{\partial}\psi^*\rangle = (1 - K)F$. Hence the metrics induced by ψ^* and ψ satisfy

$$\frac{ds^2_{\psi^*}}{ds^2_{\psi}} = 1 - K \;.$$

Furthermore, equation (1.11) and the fact that Φ and $\bar{\Phi}$ are complex linearly dependent show that

$$\partial\bar{\partial}\psi^* = -(1 - K)F\psi^* \tag{10.2}$$

and we quickly obtain

PROPOSITION 10.1. *The Gauss map of a minimal surface in S^3 again describes a minimal surface in S^3 with singularities occurring precisely at the points where $K = 1$. The curvature of the gaussian image surface is $K^* = -K/(1 - K)$.*

Hence, to each minimal surface in S^3 the Gauss map associates a second generalized minimal surface called the *polar variety*. It is easy to see that taking the Gauss map a second time produces the original surface, i.e. $\psi^{**} = \psi$. Hence the Gauss map acts as a pairing of generalized minimal surfaces in S^3 which is of particular interest in the compact case.

Observe that each point $p \in \mathcal{R}$ where $K = 1$ corresponds to a branch point on the polar variety of degree $= 1 +$ (order of the zero of $1 - K$) $= 1 + d_p$.

Hence, by Proposition 1.5 *the polar variety is non-singular if and only if $\mathcal{R}$ covers a torus or a Klein bottle.*

For any of the surfaces constructed by the methods of § 4, the polar variety can be explicitly exhibited. The general rule is as follows. Let Γ be the geodesic polygon used to construct M_Γ. Using the surface $\mathfrak{M}_\Gamma$ choose a unit normal ν_i to Γ at the vertex v_i for $1 \leqq i \leqq n$. Consider $\nu_1, \cdots, \nu_n$ as unit vectors in $\mathbf{R}^4$. Then $\nu_1, \cdots, \nu_n$ naturally describe a geodesic polygon Γ^* on S^3. The vertex angle of Γ^* at ν_i will be $\pi k_i/(k_i + 1)$ where the angle of Γ at ν_i is $\pi/(k_i + 1)$. Construct the surface M_Γ^*. From Lemma 4.2 we see that M_Γ^* will have analytically parameterized branch points at vertices where $k_i \neq 1$. The surface M_Γ^* is the polar variety of M_Γ.

With regard to the special surfaces constructed above we note that the polar variety of $\tau_{k,m}$ is $\tau_{m,k}$ $(\cong \tau_{k,m})$.

Remark 10.2. $G_\Gamma = G_\Gamma^*$.

11. Bipolar surfaces and Jacobi fields

Let $\psi: \mathcal{R} \to S^3 \subset \mathbf{R}^4$ be a minimal immersion and let ψ^* be its associated Gauss map. We view each map as $\mathbf{R}^4$-valued and define $\tilde{\psi}: \mathcal{R} \to S^5 \subset \mathbf{R}^6$ by $\tilde{\psi} = \psi \wedge \psi^*$. This mapping is again conformal and induces a metric on $\mathcal{R}$ of the form

$$d\tilde{s}^2 = (2 - K)F\,|\,dz\,|^2 = (2 - K)ds^2\ .$$

Even if we allow the original immersion ψ to have branch points, this immersion can be shown to be non-singular. Moreover, a straightforward computation using (10.1) and (10.2) shows that

$$\partial\bar{\partial}\tilde{\psi} = -(2 - K)F\tilde{\psi} \tag{11.1}$$

and thus *$\tilde{\psi}$ is a minimal immersion into S^5. We shall call this surface the bipolar surface.*

Evidently $\tilde{\psi} = -\tilde{\psi}^*$. Hence the bipolar surface represents a non-singular minimal immersion associated uniquely to each polar pair of generalized minimal surfaces in S^3.

Let $\mathfrak{M}$ be any minimal surface in S^3 and let η be a unit normal vector field on $\mathfrak{M}$. A *Jacobi field* on $\mathfrak{M}$ is a normal vector field $J = \varphi\eta$ where the function φ satisfies the equation $\Delta\varphi = -2(2 - \mathcal{K})\varphi$. Here Δ represents the laplacian of $\mathfrak{M}$, and $\mathcal{K}$ denotes its Gauss curvature. The *nullity* of $\mathfrak{M}$ is simply the dimension of the space of Jacobi fields which vanish on $\partial\mathfrak{M}$. (For a discussion of Jacobi fields on minimal surfaces see [19].)

Note that in a fixed local coordinate z on $\mathcal{R}$ the equations for Jacobi fields on $\psi(\mathcal{R})$ and $\psi^*(\mathcal{R})$ have the same form, namely $\partial\bar{\partial}\varphi = -(2 - K)F\varphi$. From (11.1) we see that each component of $\tilde{\psi}$ satisfies this equation. In fact we will show that the components of $\tilde{\psi}$ are just the Jacobi fields produced by infinitesimal rotations of S^3.

A function φ on a minimal surface in S^3 is called a *Killing-Jacobi field* if it represents the normal component of the restriction of a Killing field of S^3 to the surface. Such fields form a vector space K_ψ whose dimension ν $(3 \leqq \nu \leqq 6)$ is called the *Killing nullity* of the surface. Compact surfaces with $\nu = 3, 4$ and 5 are classified in [10].

Killing fields on S^3 come naturally from the Lie algebra $\mathfrak{so}(4)$. If we represent $\mathfrak{so}(4)$ as the skew-symmetric endomorphisms of $\mathbf{R}^4$, then we get a surjective linear map $L: \mathfrak{so}(4) \to K_\psi$ by setting

$$L(S) = \langle S(\psi), \psi^* \rangle \tag{11.2}$$

for a minimal immersion ψ. This is because the Killing fields restricted to the surface all have the form $S(\psi)$ for some $S \in \mathfrak{so}(4)$.

PROPOSITION 11.1. *The coordinate functions of the bipolar minimal immersion with respect to any orthonormal basis of* $\mathbf{R}^6$ *are the images under* (11.2) *of an orthonormal basis of* $\mathfrak{so}(4)$.

PROOF. Let $e_1, \cdots, e_4$ be an orthonormal basis for $\mathbf{R}^4$ and let S_{ij} be the skew-symmetric endomorphism of R^4 given by

$$S_{ij}(v) = \langle e_i, v \rangle e_j - \langle e_j, v \rangle e_i \tag{11.3}$$

for $1 \leqq i < j \leqq 4$. Observe that

$$L(S_{ij}) = \langle S_{ij}(\psi), \psi^* \rangle = \langle e_i \wedge e_j, \psi \wedge \psi^* \rangle .$$

The proposition follows immediately.

It follows that the bipolar image of a minimal surface in S^3 lies non-degenerately in $S^{\nu-1}$. The value of ν can be determined from [10]. In particular $\nu(S^2) = 3$, $\nu(\tau_{1,1}) = 4$, $\nu(\tau_{m,k}) = 5$ for $m > k \geqq 1$ and $\nu(\xi_{m,k}) = \nu(\eta_{m,k}) = 6$

for $m \neq 1$ and $k \neq 1$. Hence we have

PROPOSITION 11.2. *There exist non-singular minimal immersions of every surface of negative Euler characteristic into S^5 such that none of the images lies in a geodesic S^4. There is, moreover, a countable family of minimal immersions of the torus into S^4 where none of the images lies in a geodesic S^3.*

Dual to the immersion ψ is the minimal immersion $*\tilde{\psi} \overset{\text{def.}}{=} (1/iF)\partial\psi \wedge \bar{\partial}\psi$ which is isometric to $\tilde{\psi}$. $*\tilde{\psi}$ is the image of $\tilde{\psi}$ under the $*$-isomorphism on $\mathbf{R}^6 = \Lambda^2 \mathbf{R}^4$. Since $*\tilde{\psi}$ also satisfies (11.1) we have that for any 2-vector a

$$\partial\bar{\partial}\langle a, *\tilde{\psi}\rangle = -(2-K)F\langle a, *\tilde{\psi}\rangle . \tag{11.4}$$

This gives

PROPOSITION 11.3. *Let $\hat{T}$ denote the field of oriented unit tangent planes in $\mathbf{R}^4$ over a compact minimal surface $\mathfrak{M}$ in S^3. Then for any 2-vector a we have that*

$$\int_{\mathfrak{M}} \langle \hat{T}, a\rangle \omega = 0$$

where ω is the volume from the bipolar metric. If in particular we have $\langle \hat{T}, a\rangle \geqq 0$ for some $a \neq 0$, then $\nu(\mathfrak{M}) \leqq 5$ and $\mathfrak{M}$ is totally geodesic or an immersed torus.

PROOF. The first part follows from (11.4) and the divergence theorem. For the second part note that $\langle \hat{T}, a\rangle \geqq 0 \Rightarrow \langle \hat{T}, a\rangle \equiv 0$. This means that $\langle \tilde{\psi}, *a\rangle \equiv 0$. Let $e_1, \cdots, e_4$ be an orthonormal basis for $\mathbf{R}^4$ and write $*a = \sum_{i<j} a_{ij} e_i \wedge e_j$. Then $0 = \langle \tilde{\psi}, *a\rangle = \sum_{i<j} a_{ij}\langle \psi \wedge \psi^*, e_i \wedge e_j\rangle = \sum_{i<j} a_{ij}\langle S_{ij}(\psi), \psi^*\rangle = \langle S(\psi), \psi^*\rangle$ where $S = \sum a_{ij} S_{ij}$ and S_{ij} is given by (11.3). Hence the Killing nullity $\nu(\mathfrak{M})$ is $\leqq 5$, and the rest follows from the classification in [10].

Note. The kernel of the endomorphism L in $\mathfrak{so}(4)$ is a subalgebra corresponding to the group of self-congruences of $\mathfrak{M}$ in S^3. The dimension of this group is $6 - \nu$.

12. Intrinsic characterizations of minimal surfaces and associated constant mean curvature surfaces

It was proved by Ricci [2] that sufficient conditions for a riemannian metric $ds^2 = Edx^2 + 2Fdxdy + Gdy^2$ to be realized locally on a surface of constant mean curvature H in R^3 are that the Gauss curvature K satisfy $K < H^2$ and that the associated metric $d\hat{s}^2 = \sqrt{H^2 - K}ds^2$ be flat. Ricci's observation generalizes to a relevant and interesting statement in this context.

Let $\mathfrak{M}^3(c)$ denote the simply-connected, 3-dimensional space form of curvature c. This space can be viewed naturally as a submanifold of R^4 as follows.

$$\mathfrak{M}^3(c) = \{(x_1, \cdots, x_4) \in R^4 : \sqrt{|c|}\, q_c(x_1, \cdots, x_4) - 2x_4 = 0\} \tag{12.1}$$

where

$$q_c(x_1, \cdots, x_4) = \begin{cases} x_1^2 + \cdots + x_4^2 & \text{if } c \geqq 0 \\ x_1^2 + \cdots + x_3^2 - x_4^2 & \text{if } c < 0 \end{cases}$$

and where $\mathbf{R}^4$ is assumed to have the metric $d\sigma^2 = q_c(dx_1, \cdots, dx_4)$.

THEOREM 8. *Let ds^2 be a C^3 riemannian metric defined over a simply-connected surface S and let H^2 be any non-negative real number. Suppose that the Gauss curvature K of this metric satisfies*

$$K < H^2 \tag{12.2}$$

and furthermore suppose that the metric

$$d\hat{s}^2 = \sqrt{H^2 - K}\, ds^2 \tag{12.3}$$

is flat. Then for each constant $c \leqq H^2$ there exists a differentiable, 2π-periodic family of isometric immersions

$$\psi_{c,\theta} : S \longrightarrow \mathfrak{M}^3(c) \qquad \theta \in R$$

of constant mean curvature $\sqrt{H^2 - c}$. Moreover, up to congruences the maps $\psi_{c,\theta}$; $0 \leqq \theta \leqq \pi$ represent (extensions of) all local, isometric, constant mean curvature immersions of S into $\mathfrak{M}^3(c)$.

If, furthermore, the metric ds^2 was originally induced by an immersion of constant mean curvature H' into $\mathfrak{M}^3(c)$, then setting $H = \sqrt{(H')^2 + c}$ we have that

$$K \leqq H^2\,, \tag{12.2'}$$

the metric (12.3) is flat, and all the above conclusions hold.

Remark 12.1. If the hypothesis (12.2) is weakened to (12.2′) for general metrics, the above imbedding theorem does not hold. To see this consider the metric $ds^2 = (1 + |z|^{2\alpha})^2 |dz|^2$, defined over the complex plane, where α is a non-integer >3. The curvature K of this metric is given by

$$K = -\frac{2\alpha^2 |z^2|^{\alpha-1}}{(1 + |z^2|^{\alpha})^4}$$

and together with $H = 0$ satisfies (12.2′). Moreover, away from the point $z = 0$ the metric $\sqrt{-K}ds^2 = \sqrt{2}\,\alpha\, |z|^{\alpha-1} |dz|^2$ is flat. However, the plane with this metric cannot be (isometrically) minimally immersed into $\mathbf{R}^3$. This is seen as follows. Let β_{ij} be the second fundamental form of any such immersion. Then the function $f(z) = \beta_{11} - i\beta_{12}$ would be a well defined holomorphic function in the plane satisfying the equation

$$|f|^2 = -K(1 + |z^2|^\alpha)^4 = 2\alpha^2 |z^2|^{\alpha-1} .$$

This is impossible.

PROOF OF THEOREM 8. Assume that S is not the 2-sphere. By uniformization we can assume that S is either the unit disk or the plane and that the metric has the form $ds^2 = F(dx_1^2 + dx_2^2)$. The fact that the Gauss curvature of $d\hat{s}^2$ is identically zero means that

$$\Delta \log [(H^2 - K)F^2] \equiv 0 . \tag{12.4}$$

Hence there exists a holomorphic function $f(x_1 + ix_2)$, determined up to a multiplicative constant $e^{i\theta}$ and defined everywhere on S, such that

$$|f|^2 = F^2(H^2 - K) . \tag{12.5}$$

We now define a two-parameter family of second fundamental forms $\beta_{ij}(\theta, c)$ on S by

$$\begin{aligned} \beta_{11}(\theta, c) &= \mathrm{Re}\,\{e^{i\theta}f\} + \sqrt{H^2 - c}F \\ \beta_{22}(\theta, c) &= -\mathrm{Re}\,\{e^{i\theta}f\} + \sqrt{H^2 - c}F \\ \beta_{12}(\theta, c) &= \mathrm{Im}\,\{e^{i\theta}f\} = \beta_{21}(\theta, c) . \end{aligned}$$

Observe that for each pair of numbers (θ, c) $(c \leqq H^2)$ the forms ds^2 and $\beta(\theta, c)$ together satisfy the equations

$$(c - K)F^2 = \beta_{12}^2 - \beta_{11}\beta_{22} \tag{12.6}$$

$$\beta_{ij;k}(\theta, c) = \beta_{ik;j}(\theta, c); \qquad 1 \leqq i, j, k \leqq 2 \tag{12.7}$$

where the semi-colon denotes covariant differentiation with respect to the designated coordinate vector field. (The first equation is obvious; the second follows from a straightforward computation.)

Equations (12.6) and (12.7) are respectively the Gauss curvature and Mainardi-Codazzi equations for the first and second fundamental forms on a surface in $\mathfrak{M}^3(c)$. These equations are well known to be the integrability conditions necessary and sufficient for finding these forms on a surface in $\mathfrak{M}^3(c)$. Moreover by using the model (12.1) of $\mathfrak{M}^3(c)$ it is possible to write down, as in [12, p. 192], a first order, linear system of ordinary differential equations

$$X'(t) = A(\theta, \varphi, t)X(t) \tag{12.8}$$

(where the matrix A is class C^∞) which govern the imbedding of the curve $\{te^{i\varphi} \in S : t \in \mathbf{R}\}$ of S into $\mathfrak{M}^3(c) \subset \mathbf{R}^4$. From this picture the smooth dependence on θ is clear.

The uniqueness of the immersions is clear. Moreover, from (12.4) and (12.6–7) we can see that for fixed c there is only one number H' such that ds^2 can be found on a surface of constant mean curvature H' in $\mathfrak{M}^3(c)$.

The case where S is homeomorphic to the sphere S^2 is easy. Equation (12.4) gives rise to a differential form $f(z)dz^2$ (f as above) which is holomorphic in the conformal structure of the surface. This form must vanish, and thus by (12.5) we have $K \equiv H^2$ contrary to assumption.

For each $c \leqq H^2$ there is a standard immersion of S^2 into $\mathfrak{M}^3(c)$ with constant mean curvature $\sqrt{H^2 - c}$ [12]. This immersion is, moreover, unique. In fact, *any* conformal immersion of S^2, considered as a Riemann surface, into $\mathfrak{M}^3(c)$ with constant mean curvature $H_c = \sqrt{H^2 - c}$ induces a second fundamental form β such that the associated form $\omega = (\beta_{11} - H_c F - i\beta_{12})dz^2$ (where $d\sigma^2 = Fdz^2$ is the induced metric) is holomorphic on S^2. Hence $\omega = 0, \beta = H_c ds^2$ and $K \equiv H^2$; and the immersion is standard.

For the second part of the theorem we assume that ds^2 was inherited from an immersion ψ of S into $\mathfrak{M}^3(c_0)$ with constant mean curvature $\sqrt{H^2 - c_0}$. By the previous remarks we only need to worry when $S \not\approx S^2$. Choose the disk or the plane as global isothermal parameters for S and define a function f in these coordinates by $f = \beta_{11} - \sqrt{H^2 - c_0}F - i\beta_{12}$ where $F\delta_{ij}$ and β_{ij} are the first and second fundamental forms of the immersion ψ. We can now proceed exactly as above to construct the family of immersions $\psi_{c,\theta}$. This completes the proof.

Let M_Γ be a minimal surface in S^3 constructed by the methods of § 4. Lift the metric of M_Γ to the universal covering surface U_Γ of M_Γ. By Theorem 8 there exists for each $c \leqq 1$ a one-parameter family $\psi_{c,\theta}$ of complete, isometric constant mean curvature immersions of U_Γ into $\mathfrak{M}^3(c)$.

It should be clear that the symmetry of M_Γ will force a high degree of symmetry into the immersions of U_Γ. In particular let $\widetilde{\mathfrak{M}}_\Gamma \subset U_\Gamma$ be a domain mapped one-to-one onto $\mathfrak{M}_\Gamma \subset M_\Gamma$ by the covering map. Suppose that $\widetilde{\mathfrak{M}}_\Gamma^* \subset U_\Gamma$ is similarly mapped onto a domain $\mathfrak{M}_\Gamma^*$ which is the image of $\mathfrak{M}_\Gamma$ under an orientation preserving self-congruence of M_Γ. Then there is an isometry of $\mathfrak{M}^3(c)$ which takes $\psi_{c,\theta}(\widetilde{\mathfrak{M}}_\Gamma)$ onto $\psi_{c,\theta}(\widetilde{\mathfrak{M}}_\Gamma^*)$, and this isometry represents a congruence of the whole immersion $\psi_{c,\theta}$. This fact follows from the existence and uniqueness theorems discussed above. Hence, we have

PROPOSITION 12.2. *Let $\mathfrak{G}$ be the group of orientation-preserving isometries of M_Γ which extend to congruences in S^3. Let $\widetilde{\mathfrak{G}}$ be the extension of*

$\mathcal{G}$ to isometries of U_Γ by the deck transformations of the covering. Then each element of $\tilde{\mathcal{G}}$ extends to a congruence of each immersion $\psi_{c,\theta}$. That is, there exists for each (c, θ) a representation of $\tilde{\mathcal{G}}$ in Isom $(\mathfrak{M}^3(c))$ *which makes $\psi_{c,\theta}$ $\tilde{\mathcal{G}}$-equivariant.*

Remark 12.3. Two interesting directions of inquiry that now arise are:

(1) Which of the immersions $\psi_{c,\theta}: U_\Gamma \to \mathfrak{M}^3(c)$ factor to a compact surface?

(2) What are the properties of the complete surfaces $\psi_{0,\theta}(U_\Gamma)$ of constant mean curvature in $\mathbf{R}^3$ when $M_\Gamma = \xi_{m,k}, \tau_{m,k}$ or $\eta_{m,k}$?

The following easily proved facts should be useful for such considerations. Let $\pi: U_\Gamma \to M_\Gamma$ be the covering map.

(a) If γ is a great circle of S^3 which lies on M_Γ, then $\psi_{c,0}(\pi^{-1}(\gamma))$ is a curve of constant curvature $1 - c$ in $\mathfrak{M}^3(c)$.

(b) If M_Γ is invariant under reflection across a geodesic 2-sphere S in S^3 and if $\gamma = M_\Gamma \cap \mathrm{S}$, then for each component γ_α of $\pi^{-1}(\gamma)$, we have that $\psi_{c,0}(\gamma_\alpha)$ lies in a totally geodesic hypersurface S_α of $\mathfrak{M}^3(c)$ and the immersion $\psi_{c,0}$ is invariant under geodesic reflection across S_α.

Note. A simple example of the above phenomena is provided by the Clifford torus, $\tau_{1,1}$. For $0 < c \leqq 1$, $\psi_{c,0}(\tau_{1,1}) = \{(z, w) \in \mathbf{C}^2: |z|^2 = 1 \text{ and } |w|^2 = (1/c^2) - 1\} \subset S^3(1/c)$. When $c = 0$, $\psi_{0,0}(\tau_{1,1}) =$ the right circular cylinder of radius $1 = \{(z, w) \in \mathbf{C}^2: |z|^2 = 1 \text{ and } \mathrm{Im}(w) = 0\} \subset R^3$. When $c < 0$,

$$\begin{aligned}\psi_{c,0}(\tau_{1,1}) &= \{(z, w) \in \mathbf{C}^2: |z|^2 = 1 \text{ and } \mathrm{Re}(w^2) = -(1/c^2) - 1\} \\ &\subset \mathfrak{M}^3(c) = \{(z, w) \in \mathbf{C}^2 : |z|^2 + \mathrm{Re}(w^2) = -(1/c^2) \text{ and } \mathrm{Im}(w) > 0\}.\end{aligned}$$

Thus as c progresses down, the torus unfolds to a cylinder in $\mathbf{R}^3$ and then becomes a geodesic cylinder in hyperbolic space.

13. Conjugate surfaces and dual reflection principles

Using Theorem 8 we can generalize that concept of an associate surface which is defined for minimal surfaces in $\mathbf{R}^3$.

Let $\psi: \mathcal{R} \to \mathfrak{M}^3(c)$ be a surface of constant mean curvature in $\mathfrak{M}^3(c)$. Lift ψ to the universal covering surface $\tilde{\mathcal{R}}$ of $\mathcal{R}$ and denote by $\psi_\theta: \tilde{\mathcal{R}} \to \mathfrak{M}^3(c)$ the immersion $\psi_{c,\theta}$ given by Theorem 8. The surfaces ψ_θ, $0 \leqq \theta < \pi$, are defined to be the *associate surfaces* of ψ, and the surface $\psi_{\pi/2}$ is called the *conjugate surface* of ψ. Note that in general the surfaces $\psi_{c,\theta}$ are defined only to within isometries of $\mathfrak{M}^3(c)$. In this sense $\psi_{c,\theta}$ is π-periodic in θ, and the conjugate of a conjugate surface is just the original surface. For minimal surfaces in $\mathbf{R}^3$ the above definition agrees with the usual one. In the other cases the relationship among the associate surfaces is not as beautifully simple as in the classical case. However, when the mean curvature is zero there is still an interesting and useful reflection duality for conjugate surfaces.

Let $\psi\colon \mathfrak{M} \to \mathfrak{M}^3(c)$ be a surface in $\mathfrak{M}^3(c)$ and let $\gamma \subset \mathcal{R}$ be any curve. Then $\psi \mid \gamma$ is called an *arc of linear reflection* for the surface if $\psi \mid \gamma$ is a geodesic arc in $\mathfrak{M}^3(c)$ and if $\psi(\mathcal{R})$ is invariant under geodesic reflection across that arc. If, on the other hand, $\psi(\gamma)$ lies in a totally geodesic hypersurface S of $\mathfrak{M}^3(c)$ and if $\psi(\mathcal{R})$ is invariant under geodesic reflection across S, then $\psi \mid \gamma$ is called an *arc of planar reflection* for the surface.

PROPOSITION 13.1. *Let ψ and $\psi^\star$ be conjugate minimal surfaces in $\mathfrak{M}^3(c)$ and let γ be a curve on the parameter surface $\mathcal{R}$. Then $\psi \mid \gamma$ is an arc of linear reflection if and only if $\psi^\star \mid \gamma$ is an arc of planar reflection.*

PROOF. Suppose $c > 0$. By making a suitable normalization, we may assume that $c = 1$. Then by Proposition 3.1 it is sufficient to show the following: $\psi \mid \gamma$ is a great circle of S^3 if and only if $\psi^\star \mid \gamma$ is a curve of orthogonal intersection of the surface with a great 2-sphere.

Consider ψ and $\psi^\star$ as $\mathbf{R}^4$-valued functions with $|\psi| = |\psi^\star| = 1$. For a local coordinate $z = x_1 + ix_2$ on $\mathcal{R}$ the metric has the form $ds^2 = F\,|dz|^2$, and we set $a = \partial/\partial x_1 \log F$ and $b = \partial/\partial x_2 \log F$. Under these circumstances ψ will satisfy the system of equations

$$
\begin{aligned}
\psi_{,11} &= a\psi_{,1} - b\psi_{,2} + \alpha\eta - F\psi \\
\psi_{,22} &= -a\psi_{,1} + b\psi_{,2} - \alpha\eta - F\psi \\
\psi_{,12} &= b\psi_{,1} + a\psi_{,2} + \beta\eta \\
\eta_{,1} &= -\frac{1}{F}(\alpha\psi_{,1} + \beta\psi_{,2}) \\
\eta_{,2} &= -\frac{1}{F}(\beta\psi_{,1} - \alpha\psi_{,2})
\end{aligned}
\tag{13.1}
$$

where $\eta = (1/F)\psi \wedge \psi_{,1} \wedge \psi_{,2}$ and where $(\alpha - i\beta)dz^2$ is a holomorphic form on $\mathcal{R}$. Moreover, $\psi^\star$ will satisfy the same system of equations with α replaced by $-\beta$ and with β replaced by α. We denote this system by (13.1⋆).

By conformalizing the metric on a half-neighborhood and reflecting we can assume that γ is given locally by $x_2 = 0$. Let $\nu(x_1) = \psi(x_1, 0)$ and $\nu^\star(x_1) = \psi^\star(x_1, 0)$. From (13.1) and the fundamental theorem for space curves, we see that ν is a great circle of S^3 if and only if

$$
b(x_1, 0) = \alpha(x_1, 0) = 0\ . \tag{13.2}
$$

Similarly $\nu^\star$ is an arc of planar reflection if and only if $\psi_{,2}^\star(x_1, 0)$ has constant direction, i.e., the unit vector $(1/\sqrt{F})\psi_{,2}^\star(x_1, 0)$ is constant. From (13.1⋆) and the fundamental theorem for curves we see that this happens if and only if (13.2) holds, and the theorem for $c > 0$ follows.

Proofs for the cases where $c \leqq 0$ are entirely analogous.

Remark 13.2. For conjugate minimal surfaces in $\mathbf{R}^3$, normalized so that the corresponding component functions are conjugate harmonics, one can show that the line of (linear) reflection invariance on one surface is normal to the corresponding plane of invariance on the conjugate surface. Hence there is a tight relationship between the groups of reflection symmetries on conjugate surfaces in $\mathbf{R}^3$. The lattice groups of the classical Schwartz surfaces are a good example of this phenomenon.

Remark 13.3. By Proposition 13.1 the conjugate surface of a ruled surface (cf. § 7) is a "surface of rotation" i.e. it is invariant under the map $(z, w) \mapsto (e^{i\theta}z, w)$ for all θ. Furthermore, by Remark 12.3b each of the immersions $\psi_{c,\pi/2}$ associated to a ruled surface has a similar invariance. When $c = 0$ we obtain the classical constant mean curvature surfaces of rotation in $\mathbf{R}^3$.

14. Imbedded, periodic, constant mean curvature surfaces

Using the above observations it is possible to construct complete, constant mean curvature surfaces in a highly controlled way. Begin with a polygon Γ and the surface $\mathfrak{M}_\Gamma$ as in § 4. Let $\mathfrak{M}_\Gamma^\star$ be the conjugate surface. By Proposition 13.1, $\mathfrak{M}_\Gamma^\star$ is a minimal surface bounded by a geodesic polyhedron $\Gamma^\star$ with its boundary meeting the faces of $\Gamma^\star$ orthogonally. By Remark 12.3b each surface $\psi_{c,\pi/2}(\mathfrak{M}_\Gamma)$ for $c \leqq 1$ lies similarly in a geodesic polyhedron $\Gamma_c^\star \subset \mathfrak{M}^3(c)$. In each case a complete surface is generated by reflections across the faces of the polyhedron.

Hence to understand the structure of $\psi_{c,\pi/2}(U_\Gamma)$ it is useful to study Γ_c^*. We begin with some elementary observations.

Let Γ have vertices $v_1, \cdots, v_n$ and edges $\gamma_1, \cdots, \gamma_n$.

(a) $\Gamma_c^\star$ has exactly n faces $\pi_1, \cdots, \pi_n$ where each π_j contains $\psi_{c,\pi/2}(\gamma_j)$.

(b) The face π_{j-1} meets the face π_j at an angle equal to the angle of Γ at v_j, namely $\pi/(k_j + 1)$.

(c) Every orientation preserving self-congruence of M_Γ induces an orientation preserving self-congruence of $\psi_{c,\pi/2}(U_\Gamma)$.

These observations are not enough to determine $\Gamma_c^\star$ completely. For this we need to calculate the angle formed by the pair of "lines" $\pi_{j-1} \cap \pi_j$ and $\pi_j \cap \pi_{j+1}$ in each "plane" π_j.

For these calculations we restrict ourselves to the case $c = 0$ and prove some useful facts. Let $\gamma = \gamma_j$ for some j.

LEMMA 14.1. *Let s denote arc length along γ and let $\tilde{\beta} = \langle (d\eta/ds), e_2 \rangle$ where η is the unit normal to $\mathfrak{M}_\Gamma$ along γ and where e_2 is the unit tangent vector perpendicular to γ and interior to $\mathfrak{M}_\Gamma$. Let $\varphi(s)$ be the plane curve*

$\psi_{0,\pi/2} \mid \gamma$ lying on $\Gamma_0^\star \subset R^3$. Then

$$\varphi'(s) = \big(\cos B(s), \sin B(s)\big) \tag{14.1}$$

where

$$B(s) = s - \int^s \tilde{\beta}(t)dt\ . \tag{14.2}$$

PROOF. Choose local isothermal coordinates (x_1, x_2) for M_Γ so that γ corresponds to $x_2 = 0$. The immersion ψ then satisfies (13.1) where $b(x_1, 0) = \alpha(x_1, 0) = 0$. Hence the $\mathbf{R}^3$-valued map $\Psi = \psi_{0,\pi/2}$ satisfies the system

$$\begin{cases} \Psi_{,11} = a\Psi_{,1} - b\Psi_{,2} + (F - \beta)H \\ \Psi_{,22} = -a\Psi_{,1} + b\Psi_{,2} + (F + \beta)H \\ \Psi_{,12} = b\Psi_{,1} + a\Psi_{,2} + \alpha H \\ H_{,1} = -\dfrac{1}{F}\big((F - \beta)\Psi_{,1} + \alpha\Psi_{,2}\big) \\ H_{,2} = -\dfrac{1}{F}\big(\alpha\Psi_{,1} + (F + \beta)\Psi_{,2}\big) \end{cases} \tag{14.3}$$

where $H = (1/F)\Psi_{,1} \wedge \Psi_{,2}$. Restricting these equations to $x_2 = 0$ and reparameterizing by s we see that

(a) $(d/ds)\Psi_{,2}(s, 0) \equiv 0$.

(b) $\varphi(s)\big(=\Psi(s, 0)\big)$ lies in the plane $\Psi_{,2}(s, 0)^\perp$ and has normal vector field $\nu(s) = H(s, 0)$.

(c) φ' and ν satisfy the equations

$$\begin{aligned} \varphi'' &= (1 - \tilde{\beta})\nu \\ \nu' &= -(1 - \tilde{\beta})\varphi' \end{aligned} \tag{14.4}$$

where $\tilde{\beta} = (1/F)\beta$.

The result is now straightforward.

Consider η as an $\mathbf{R}^4$-valued unit vector. Then by (13.1) we see that

$$\frac{d\eta}{ds} = \tilde{\beta}e_2$$

and therefore

$$|\tilde{\beta}| = \left|\frac{d\eta}{ds}\right| . \tag{14.5}$$

Observe that as ψ traces a great circle on S^3, the Gauss map η also traces a great circle on S^3.

LEMMA 14.2. *For an interval (a, b) over which $\tilde{\beta}$ is never zero the integral*

$$\left|\int_a^b \tilde{\beta}(s)ds\right|$$

is just the distance on S^3 *from* $\eta(a, 0)$ *to* $\eta(b, 0)$, i.e., *the distance between the corresponding points on the polar surface. Furthermore,* $\tilde{\beta} = 0$ *only at the isolated zeros of the holomorphic form* ω, i.e., *where* $K = 1$.

PROOF. The first part follows from (14.5); the second from the observation that since $\alpha(s, 0) = 0$, $\beta = 0$ if and only if the second fundamental form vanishes.

Suppose that $K \neq 1$ in the interior of γ_j. We denote by γ_j^* the polar image of γ_j, and by $\gamma_j^\star$ the curve $\psi_{0,\pi/2}| \gamma_j$ which lies on π_j. For each j we set $e_j = \pi_j \cap \pi_{j+1}$ where, by convention, $\pi_{n+1} = \pi_1$. Then the normal to γ_j^* in π_j, which is the normal to the surface, is parallel to e_{j-1} at one end of γ_j^* and parallel to e_j at the other. Hence the angle between e_{j-1} and e_j is simply the change in direction of the normal to γ_j as it crosses the face π_j. This, combined with the lemmas above, gives

PROPOSITION 14.3. *The angle between the successive edges* e_{j-1} *and* e_j *on* $\Gamma_0^\star$ *is equal to*

$$\text{length}\,(\gamma_j) \pm \text{length}\,(\gamma_j^*)\ .$$

Consider the polygon $\Gamma_{m,k}$ defined in § 6. In this section it was shown

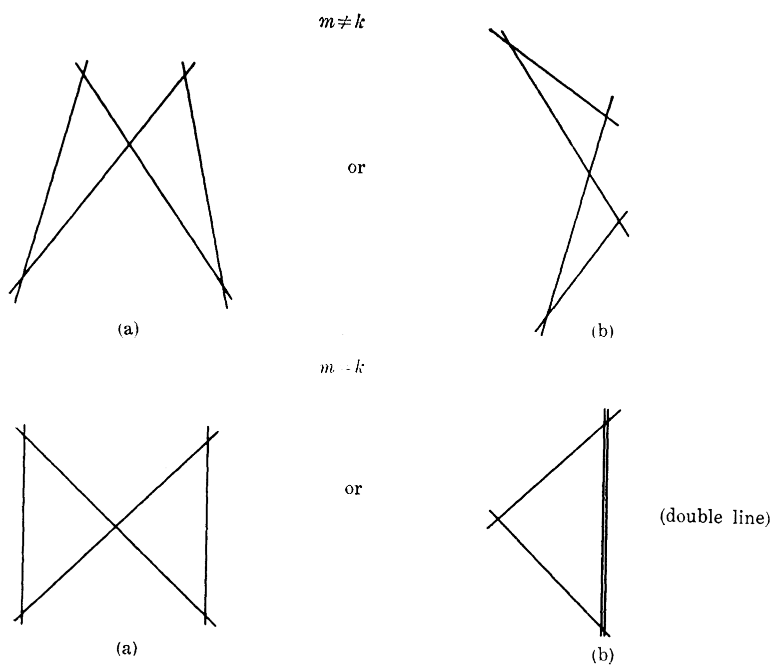

Figure 7.

that $K = 1$ only at the vertices of $\Gamma_{m,k}$. It is clear from inspection that for each edge γ of $\Gamma_{m,k}$ we have length (γ) = length $(\gamma^*) = \pi/2$.

Hence, by Proposition 14.3, the associated polyhedron $(\Gamma_{m,k})_0^\star \subset \mathbf{R}^3$ consists of four planes which intersect in mutually parallel lines at the successive angles $\pi/(m+1)$, $\pi/(k+1)$, $\pi/(m+1)$, $\pi/(k+1)$. Observe now that a rotation of $\mathfrak{M}_{\Gamma_{m,k}}$ by π about its center line of symmetry is an orientation preserving congruence. Hence, there is a rotation by π in euclidean space which leaves $\psi_{0,\pi/2}(\mathfrak{M}_{\Gamma_{m,k}})$ invariant and interchanges opposite vertices.

Hence a perpendicular cross-section of $(\Gamma_{m,k})_0^\star$ must be as shown in Figure 7.

Our final observation is that $\tilde{\beta}$ changes sign as we pass from one edge of

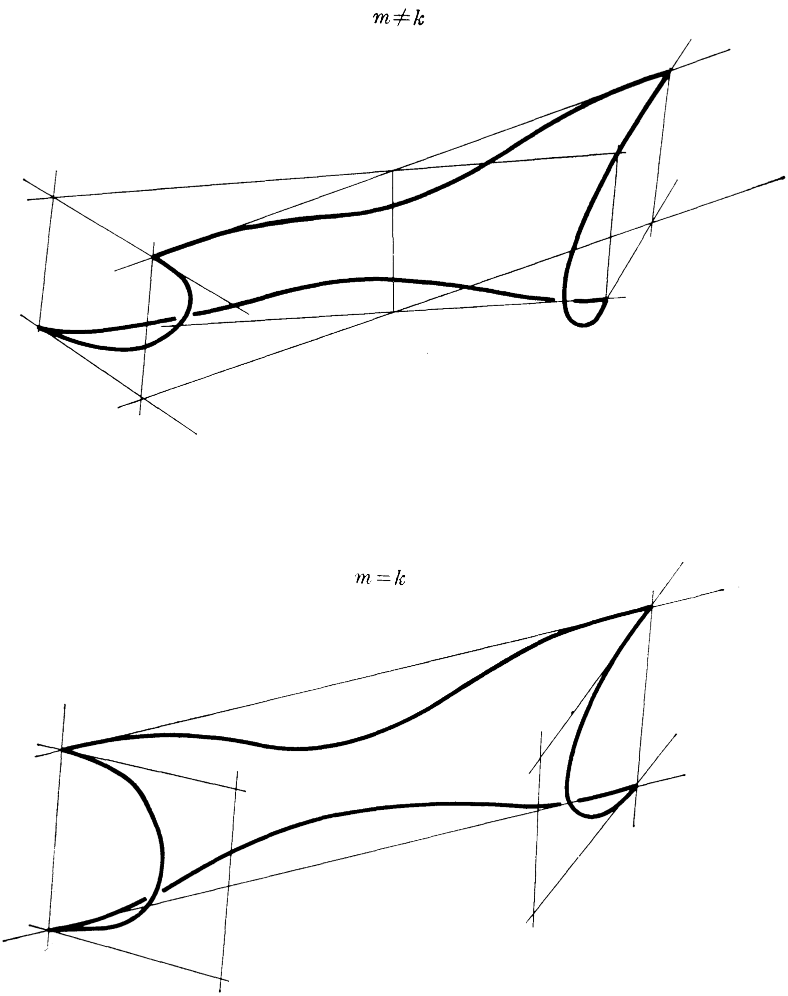

Figure 8.

$\Gamma_{m,k}$ to the next. Hence, *the change in the normal across the faces of* $\Gamma_{m,k}$ *is successively* π, 0, π, 0. It follows that only configurations of type (b) in Figure 7 are allowed.

A sketch of $\partial\psi_{0,\pi/2}(\mathfrak{M}_{\Gamma_{m,k}})$ is given in Figure 8.

Observe that when $m = k = 1$ we have a right circular cylinder, and in general when $m = k$ we get a complete surface which lies between two planes and has a fundamental domain which resembles the surface of an $(m+1)$-spoked wagon wheel without the rim. For $m = k = 2$ or 3 the surface is imbedded and appears from "above" as shown in Figure 9.

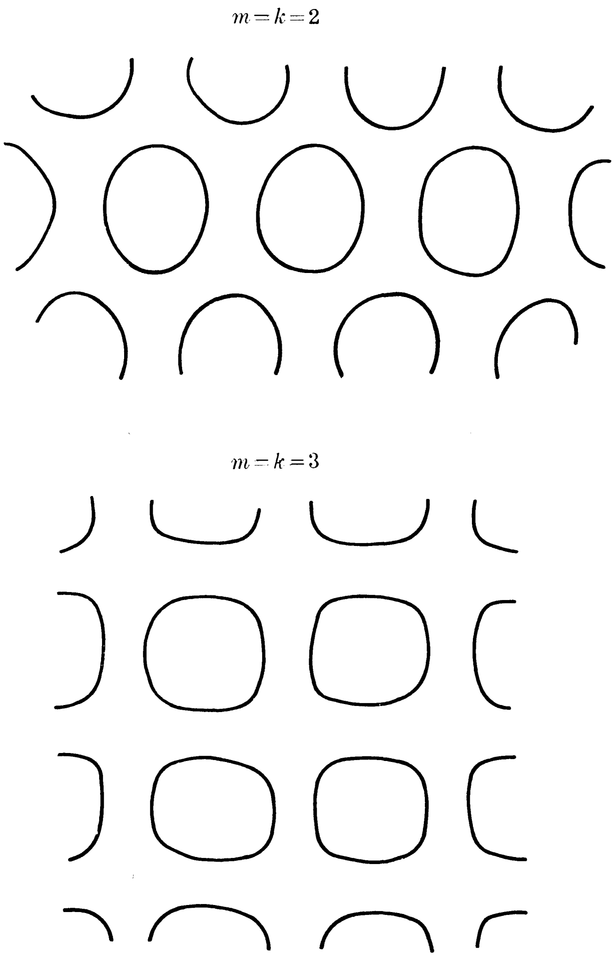

Figure 9.

Let $U_{m,m}$ be the universal riemannian covering of $\xi_{m,m}$, and recall that $\xi_{0,0} = S^2$. Then we have

THEOREM 9. *The surfaces* $\psi_{0,\pi/2}(U_{m,m})$ *for* $m = 0, 1, 2, 3$ *are complete, imbedded surfaces of constant mean curvature* 1 *in* $\mathbf{R}^3$ *which lie between two parallel planes.*

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