Asymptotic Symmetry and Local Behavior of Semilinear Elliptic Equations with Critical Sobolev Growth

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1. Introduction

In this paper we study non-negative smooth solutions of the conformally invariant equation

$$-\Delta u = u^{(n+2)/(n-2)}, \qquad u \geqq 0, \tag{1.1}$$

in a punctured ball, $B_1(0) \setminus \{0\} \subset \mathbb{R}^n$, $n \geqq 3$, with an isolated singularity at the origin. The model equation (1.1) arises in many physical contexts but its greatest interest in recent years lies in its relation to the Yamabe problem. From this geometric point of view, we think of u as defining the conformally flat metric $\bar{g}_{ij} = u^{4/(n-2)}\delta_{ij}$. Equation (1.1) then says that the metric $\bar{g}$ has constant scalar curvature. The recent work of Schoen and Yau [8], [9], [10] on conformally flat manifolds and the Yamabe problem has highlighted the importance of studying the distribution solutions of (1.1) and characterizing the singular set of u. A solution u of (1.1) with an isolated singularity is the simplest example of a singular distribution solution. This deceptively simple looking problem is analytically very difficult and requires the development of a new technique which we may call an asymptotic symmetry method. It is a "measure theoretic" variation of the Alexandrov reflection technique as developed by Gidas, Ni and Nirenberg [4], [5]. Loosely speaking, the heuristic idea of the asymptotic symmetry technique may be described as follows. After an inversion, the function u becomes defined in the complement of B_1, is strictly positive on ∂B_1, and in some sense "goes to zero" at infinity. If we could extend u to B_1 as a super solution of our problem,

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then we could start the reflection process at infinity and move all the way to ∂B_1. This would imply asymptotic radial symmetry at infinity.

Unfortunately, such an extension does not seem possible in general. On the other hand, in those directions in which $u(x)$ goes reasonably fast to zero for x going to infinity, we should be able to reflect u in planes that move reasonably close to the origin. The further away we stay from the origin, the larger the family of admissible reflections along which u remains monotone.

The justification of this heuristic idea requires a lot of machinery which we shall describe later. Using the asymptotic symmetry technique we may prove the following.

THEOREM 1.1. *Let $u \geqq 0$ be a $C^{2+\alpha}$ solution of*

$$-\Delta u = g(u) \quad in \quad B_1 \setminus \{0\}, \qquad n \geqq 3, \tag{1.2}$$

with an isolated singularity at the origin.

Assume that $g(t)$ is a locally Lipschitz function satisfying

$$\text{(i)} \quad g(t) \quad \textit{is nondecreasing}, \quad g(0) = 0,$$

and, for t sufficiently large,

$$\begin{aligned} &\text{(ii)} \quad t^{-(n+2)/(n-2)} g(t) \quad \textit{is nonincreasing}, \\ &\text{(iii)} \quad g(t) \geqq c t^p \quad \textit{for some} \quad p \geqq n/(n-2). \end{aligned} \tag{1.3}$$

Then

$$u(x) = (1 + O(|x|)) m(|x|) \quad as \quad x \to 0, \tag{1.4}$$

where $m(r) = \fint_{S^{n-1}} u(r, \omega)\, d\omega$ is the spherical average of u.

In order to get a more precise result we must understand the radial singular solutions of (1.2). For equation (1.1) this may be done (see [2]) by setting $t = -\log r$ and $\psi(t) = r^{(n-2)/2} \psi(r)$, where $\psi(r)$ is a non-negative radial solution of (1.1). Then $\psi(t) \geqq 0$ satisfies

$$\psi'' - \left(\frac{n-2}{2}\right)^2 \psi + \psi^{(n+2)/(n-2)} = 0, \qquad 0 < t < \infty. \tag{1.5}$$

The simplest singular solution of (1.1) corresponds to $\psi \equiv k = ((n-2)/2)^{(n-2)/2}$. Equation (1.5) can be integrated to give

$$\psi'^2 = \left(\frac{n-2}{2}\right)^2 \psi^2 - \frac{n-2}{n} \psi^{2n/(n-2)} + D. \tag{1.6}$$

It follows from (1.6) that the behavior of ψ is determined by the roots of

$$\left(\frac{n-2}{2}\right)^2\psi^2 - \frac{n-2}{n}\psi^{2n/(n-2)} + D = 0.$$

By the maximum principle, ψ cannot vanish for any finite t unless $\psi \equiv 0$, and this forces D to lie in the interval $0 \geqq D \geqq -(2/n)((n-2)/n)^n$. The case $D = 0$ corresponds to the regular family of solutions

$$u = \left(\frac{\lambda\sqrt{n(n-2)}}{\lambda^2 + r^2}\right)^{(n-2)/2}, \qquad \lambda > 0,$$

while for all other D there is a periodic translation invariant positive family of solutions $\psi_D(t)$ of (1.5). The other extreme case, $D = -(2/n)((n-2)/n)^n$ corresponds to the solution $\psi \equiv k$ or $u = k/r^{(n-2)/2}$.

Using Theorem (1.1) as a tool, we get the following characterization of the singular solutions of (1.1).

THEOREM 1.2. *Let u be a solution of* (1.1) *with a non-removable isolated singularity. Then there is a unique asymptotic constant D_∞ in the interval $0 > D_\infty \geqq -(2/n)((n-2)/n)^n$ and a radial singular solution $\psi(r) = \psi_D(\log r)/r^{(n-2)/2}$ so that*

$$u(x) = (1 + o(1))\psi(|x|) \quad \text{as} \quad x \to 0.$$

For the family of equations

$$(1.7) \qquad -\Delta u = u^\alpha, \qquad \frac{n}{n-2} \leqq \alpha < \frac{n+2}{n-2},$$

with subcritical Sobolev growth, a similar but much simpler analysis gives

THEOREM 1.3. *Let $u \geqq 0$ be a solution of* (1.8) *in $B_1\backslash\{0\}$ with a non-removable isolated singularity. Then*

(i) *for $n/(n-2) < \alpha < (n+2)/(n-2)$,*

$$u = (1 + o(1))c_0/|x|^{2/(\alpha-1)} \quad \text{as} \quad x \to 0,$$

where

$$c_0 = \left[2\frac{(n-2)}{(\alpha-1)^2}\left(\alpha - \frac{n}{n-2}\right)\right]^{1/(\alpha-1)};$$

(ii) *for $\alpha = n/(n-2)$,*

$$u = (1 + o(1))\left[\frac{(n-2)^2/2}{r^2\log(1/r)}\right]^{(n-2)/2}.$$

Part (i) of Theorem 1.3 was proved by Gidas and Spruck [6] using analytic techniques while part (ii) is an improvement of a result of Aviles [1].

We now discuss the organization of the paper. In order to make the heuristic ideas described earlier precise, we perform, in fact, an inversion

$$y = z + \frac{x}{|x|^2} \cdot v(x) = \frac{1}{|x|^{n-2}} u(y)$$

about a regular point z near zero, so that v has a good asymptotic expansion at infinity, but has a singularity in a cylinder of radius 1 far away from the origin. In Section 3 we prove a basic extension lemma that says that we can extend v to B_1 as a supersolution on a set $A \subset B_1$ of sufficiently small measure. This extension lemma is used in Section 4 to formulate our Reflection Theorem. The essential assumption of this theorem is that v decays to zero uniformly on rays parallel to the reflection direction except possibly for a certain set of exceptional rays which hit B_1. The set A is the intersection of the exceptional rays with B_1 and is required to have small enough measure. The proof of the Reflection Theorem utilizes some preliminary results of Section 2. In particular, Lemmas 2.3 and 2.4 show that the reflection process can always be started and can only terminate because of a difficulty on a compact set. In Section 5 we use potential theory and capacity estimates to obtain an estimate for the measure of the set of directions for which the hypotheses of the Reflection Theorem hold. In particular, Corollary 5.2 shows that the set of admissible directions on the unit sphere Σ_1 has measure $|\Sigma_1| - M^{-\delta}$ for some $\delta > 0$ if we are willing to reflect up to distance M from the origin. Section 6 makes precise the manner in which asymptotic symmetry follows from the results of Sections 4 and 5. Section 7 contains the proofs of the main results that we have stated. Finally, Section 8 contains additional global results about solutions with one or two singularities that are slight extensions of the corresponding results in [5]. This section requires only the preliminary results of Section 2 and may be read independently of the bulk of the paper.

2. Preliminary Results

In this section we collect some preliminary results which will be needed for our later analysis.

The following lemma is well known. The proof we give is a slight modification of Proposition 3.1 of [5]

LEMMA 2.1. *Let $u \geq 0$ be a C^2 solution of*

$$-\Delta u = g(u) \quad in \quad B_2 \setminus \{0\} \subset \mathbb{R}^n, \qquad n \geq 3, \tag{2.1}$$

where

(i) $g(t) \geq 0$ *for* $t \geq 0$,

(ii) $\underline{\lim}_{t \to \infty} g(t)/t^p > 0$ *for some* $p \geq n/(n-2)$.

Then, $g(u) \in L_1(B_1)$, $u \in L_p(B_1)$ *and u is a distribution solution in* B_1.

Proof: For $k > \max_{|x|=1} u(x)$, let $\psi(t) \geqq 0$ be a smooth nonincreasing function satisfying

$$(t) = \begin{cases} 1, & t < k, \\ 0, & t \geqq 2k. \end{cases}$$

Set $\Phi(t) = \int_0^t \varphi(s)\,ds$. Let $\eta = \eta(|x|)$ be a smooth nondecreasing function satisfying

$$\eta = \begin{cases} 0, & |x| < \varepsilon, \\ 1, & |x| \geqq 2\varepsilon. \end{cases}$$

Then (2.1) implies

$$\int_{B_1} \nabla u \cdot \nabla \eta \varphi(u) = \int_{B_1} \eta \varphi(u) g(u) + \int_{\partial B_1} u_\nu \, d\sigma.$$

On the other hand,

$$\int_{B_1} \nabla u \cdot \nabla \eta \varphi(u) = \int_{B_1} \eta \varphi'(u) |\nabla u|^2 + \int_{B_1} \nabla \Phi(u) \cdot \nabla \eta$$

$$\leqq -\int_{B_1} \Phi(u) \Delta \eta = O(\varepsilon^{n-2}).$$

Letting ε tend to zero gives

$$\int_{B_1 \cap \{u < k\}} g(u) \leqq -\int_{\partial B_1} U_\nu \, d\sigma.$$

Now letting $k \to \infty$, we conclude that $g(u) \in L_1(B_1)$ and $u \in L_p(B_1)$ by condition (ii).

To show that u is a distribution solution we must establish that

$$\int_{B_1} u \Delta \zeta + \zeta g(u) = 0 \quad \text{for all} \quad \zeta \in C_0^\infty(B_1). \tag{2.2}$$

Using $\eta\zeta$ as a test function in (2.1) with η as before gives

$$\int_{B_1} \nabla u \cdot \nabla \eta \zeta + \eta \zeta g(u) = 0.$$

But,

$$\int_{B_1} \nabla u \cdot \nabla \eta \zeta = \int_{B_1} u \, \Delta \eta \zeta = \int_{B_1} u(\eta \, \Delta \zeta + \zeta \, \Delta \eta + 2 \nabla \eta \nabla \zeta)$$

and so,

$$\left|\int_{B_1} \eta(u\Delta\zeta + \zeta g(u)\right| \leqq \int_{B_1} u|\zeta\Delta\eta + 2\nabla\eta\nabla\zeta| \leqq \frac{c}{\varepsilon^2}\int_{B_{2\varepsilon}\setminus B_\varepsilon} u$$

$$\leqq \frac{c}{\varepsilon^2}\cdot\varepsilon^{n(1-1/p)}\|u\|_{L_p(B_{2\varepsilon})}$$

$$= O(\varepsilon^{n-2-n/p}) \quad \text{as} \quad \varepsilon \to 0.$$

Since $u\Delta\zeta + \zeta g(u)$ is in $L_1(B_1)$, (2.2) follows from the dominated convergence theorem and the proof is complete.

One important consequence of Lemma 2.1 is the following

LEMMA 2.2. *Let u_1, u_2 be distribution solutions of $-\Delta u = g(u)$ in B_1, with u_2 as in Lemma 2.1. Assume u_1 is smooth near the origin and that $u_1 \leqq u_2$, $u_1 \not\equiv u_2$ a.e. in B_1. Then, $u_2 > u_1 + \varepsilon$ for some $\varepsilon > 0$ near the origin.*

In order to study a local solution $u \geqq 0$ of

$$-\Delta u = g(u) \quad \text{in} \quad B_2\setminus\{0\} \tag{2.3}$$

with a non-removable singularity at the origin, we perform an inversion about a regular point z of u with z near 0 and transform u to v by the Kelvin transform:

$$x = \frac{y-z}{|y-z|^2}, \qquad y = z + \frac{x}{|x|^2},$$
$$v(x) = \frac{1}{|x|^{n-2}}u(y). \tag{2.4}$$

Then v satisfies

$$-\Delta v(x) = f(x, v), \tag{2.5}$$

where

$$f = \frac{1}{|x|^{n+2}}g(|x|^{n-2}v).$$

Note that

$$f = O\left(\frac{1}{|x|^{n+2}}\right), \quad v = O\left(\frac{1}{|x|^{n-2}}\right) \quad \text{as} \quad |x| \to \infty$$

and that v has a harmonic asymptotic expansion at ∞:

$$v = \frac{1}{|x|^{n-2}}\left(a_0 + a_i \frac{x_i}{|x|^2}\right) + O\left(\frac{1}{|x|^n}\right), \qquad a_0 > 0,$$

$$(2.6) \qquad v_{x_i} = -(n-2)a_0 \frac{x_i}{|x|^n} + O\left(\frac{1}{|x|^n}\right),$$

$$v_{x_i x_j} = O\left(\frac{1}{|x|^n}\right),$$

where $a_0 = u(z)$, $a_i = u_{y_i}(z)$.

The following lemma, which is a simple consequence of the asymptotic expansion (2.6), will be used in the proof of the Reflection Theorem 4.1 to start the reflection process.

We use the notation $x_\lambda = (x', 2\lambda - x_n)$ to denote the reflection of the point $x = (x', x_n)$ in the plane $x_n = \lambda$.

LEMMA 2.3. *Let v be a function in a neighborhood of infinity satisfying the asymptotic expansion* (2.6). *Then there exist large positive constants $\bar{\lambda}$, R such that, if $\lambda \geqq \bar{\lambda}$,*

$$v(x) > v(x_\lambda) \quad \textit{for} \quad x_n < \lambda, \quad |x| > R.$$

Proof: Using the asymptotic expansion (2.6), we have

$$(2.7) \qquad v_{x_n} = -(n-2)a_0 \frac{x_n}{|x|^n} + O\left(\frac{1}{|x|^n}\right)$$

for $|x|$ large. In view of (2.6),

$$v(x) - v(x_\lambda) = a_0\left(\frac{1}{|x|^{n-2}} - \frac{1}{|x_\lambda|^{n-2}}\right) + \sum_{j<n} a_j x_j \left(\frac{1}{|x|^n} - \frac{1}{|x_\lambda|^n}\right)$$

$$(2.8) \qquad + a_n\left(\frac{(x - x_\lambda)_n}{|x|^n}\right) + a_n (x_\lambda)_n \left(\frac{1}{|x|^n} - \frac{1}{|x_\lambda|^n}\right)$$

$$+ O\left(\frac{1}{|x|^n}\right).$$

If $|x_\lambda| \geqq 2|x|$, and $|x|$ is sufficiently large we may conclude from (2.8) that

$$v(x) - v(x_\lambda) \geqq \frac{1}{2} a_0 \frac{1}{|x|^{n-2}} + O\left(\frac{1}{|x|^{n-1}}\right) \geqq \frac{a_0}{4|x|^{n-2}}.$$

So assume $|x_\lambda| < 2|x|$. We estimate the terms on the right-hand side of (2.8) as follows:

$$\frac{1}{|x|^{n-2}} - \frac{1}{|x_\lambda|^{n-2}} \geqq \frac{1}{|x|^{n-3}}\left(\frac{1}{|x|} - \frac{1}{|x_\lambda|}\right) \geqq \frac{1}{2}\frac{|x_\lambda| - |x|}{|x|^{n-1}},$$

$$\left|\Sigma a_j x_j\left(\frac{1}{|x|^n} - \frac{1}{|x_\lambda|^n}\right)\right| \leqq \frac{c|x|}{|x|^{n-1}}\left(\frac{1}{|x|} - \frac{1}{|x_\lambda|}\right)$$

$$= \frac{c}{|x|^{n-2}}\left(\frac{1}{|x|} - \frac{1}{|x_\lambda|}\right),$$

$$\frac{|x - x_\lambda|}{|x|^n} = \frac{2(\lambda - x_n)}{|x|^n} = \frac{1}{2\lambda}\frac{|x_\lambda|^2 - |x|^2}{|x|^n} \leqq \frac{3}{2\lambda}\frac{|x_\lambda| - |x|}{|x|^{n-1}},$$

$$|(x_\lambda)_n|\left(\frac{1}{|x|^n} - \frac{1}{|x_\lambda|^n}\right) \leqq |x_\lambda|\left(\frac{1}{|x|^n} - \frac{1}{|x_\lambda|^n}\right) \leqq \frac{c}{|x|^{n-2}}\left(\frac{1}{|x|} - \frac{1}{|x_\lambda|}\right).$$

Therefore, for λ large and $|x|$ large,

$$v(x) - v(x_\lambda) \geqq c_1\frac{|x_\lambda| - |x|}{|x|^{n-2}} - \frac{c_2}{|x|^n}, \tag{2.9}$$

Case (i): $|x_\lambda| - |x| > (c_2/c_1)(1/|x|)$.
Then (2.9) implies $v(x) - v(x_\lambda) > 0$.

Case (ii). $|x_\lambda| - |x| \leqq (c_2/c_1)(1/|x|)$.
Then $\lambda(\lambda - x_n) \leqq c$, or $x_n \geqq \lambda - c/\lambda$. For λ large we find from (2.7) that $v_{x_n} < 0$ along the interval (x', x_n) to $(x', (x_\lambda)_n)$ and so $v(x) \geqq v(x_\lambda)$. The proof is complete.

The next lemma will be used in the proof of the Reflection Theorem 4.1 to assert that the reflection property given in Lemma 2.3 must continue to hold until either the solution becomes symmetric or the property must fail on a compact subset of $\mathbb{R}^n$.

LEMMA 2.4. *Let v be a C^2 positive solution of*

$$-\Delta v = F(x) \quad in \quad |x| \geqq R, \tag{2.10}$$

where v has a harmonic asymptotic expansion (2.6) *at ∞. Suppose that, for $x_n > 0$,*

(a) $v(x', x_n) \leqq v(x', -x_n)$,
$v(x', x_n) \not\equiv v(x', -x_n)$,

(b) $F(x', x_n) \leqq F(x', -x_n)$.

Then there exist $\varepsilon > 0$, $S > R$ *such that*

(i) $v_{x_n} < 0$ *in* $|x_n| < \varepsilon$, $|x| > S$,

(ii) $v(x', x_n) < v(x', 2\lambda - x_n)$ *in* $x_n > \frac{1}{2}\varepsilon > \lambda$, $|x| > S$, *for* $\lambda \leqq \lambda_0$ *with* λ_0 *a suitably small multiple of* ε.

Proof: Let $w(x) = v(x', -x_n) - v(x', x_n)$ for $x_n > 0$, $|x| > R$. Then $\Delta w \leqq 0$. We choose $k > 0$ so small that

$$w(x) > \frac{kx_n}{|x|^n}$$

on $|x| = R + 1$, $x_n > 0$. This is possible since $w_{x_n} > 0$ on $x_n = 0$ by the Hopf boundary point lemma. Then the maximum principle ($x_n/|x|^n$ is harmonic) implies that

$$w(x) > k\frac{x_n}{|x|^n} \quad \text{on} \quad |x| > R + 1, \qquad x_n > 0.$$

In particular, $w_{x_n}(x', 0) = -2v_{x_n}(x', 0) > k/|x'|^n$. Using this and the asymptotic expansion (2.6), we have

$$v_{x_n}(x', h) \leqq v_{x_n}(x', 0) + \frac{C|h|}{|x|^n}$$

$$\leqq -\frac{1}{2}k\frac{1}{|x'|^n} + \frac{C|h|}{|x|^n} \leqq -\frac{1}{4}k\frac{1}{|x|^n}$$

for $|h| < k/4C$ and $|x|$ large. This proves part (i) of the lemma. To prove part (ii) we use the expansion (2.6) and the results of part (i). We first estimate

$$v(x', 2\lambda - x_n) - v(x', -x_n) \geqq \frac{-c\lambda}{|x|^n}(x_n + c)$$

for $x_n > 0$ and $|x|$ large. Then,

$$\begin{aligned} &v(x', 2\lambda - x_n) - v(x', x_n) \\ &\quad = (v(x', -x_n) - v(x', x_n)) + (v(x', 2\lambda - x_n) - v(x', -x_n)) \\ &\quad \geqq \frac{kx_n}{|x|^n} - \frac{c\lambda(x_n + c)}{|x|^n} = \frac{(k - c\lambda)x_n - c\lambda}{|x|^n} > 0 \end{aligned}$$

if $x_n \geqq \frac{1}{2}\varepsilon$ and λ is sufficiently small compared to ε. This completes the proof.

Remark 2.5. For our application to the study of isolated singularities, v is given by formulas (2.3)–(2.5) and

$$F(x) = \frac{1}{|x|^{n+2}} g\left(|x|^{n-2} v(x)\right).$$

Under the assumptions (1.3) of the introduction,

$$F(x_\lambda) \geqq \frac{1}{|x|^{n+2}} g\left(|x|^{n-2} v(x_\lambda)\right) \geqq F(x)$$

if $v(x_\lambda) \geqq v(x)$ for $x_n > \lambda$.

3. An Extension Lemma

In this section we shall prove an extension lemma that will provide the first step in adapting the method of moving planes to local situations.

LEMMA 3.1. *Let $v > 0$ be a $C^{1+\beta}$ function in $1 \leqq |x| \leqq 2$ satisfying*

$$(3.1) \qquad \begin{aligned} &-\Delta v \geqq f(x, v) \quad \text{in} \quad 1 \leqq |x| \leqq 2, \\ &0 < \delta_0 \leqq v \leqq 1/\delta_0, \qquad \|v\|_{C^{1+\beta}(B_2 - B_1)} \leqq M, \end{aligned}$$

where f is locally bounded in t, uniformly in $|x| \leqq 2$.

Then there exists $\sigma = \sigma(\delta_0, M, f)$ such that, for any open set $A \subset \overline{B}_1$ with $|A| < \sigma$, we can extend v to a Lipschitz function $\bar{v}$ in $\overline{B}_2$ satisfying

$$(3.2) \qquad \begin{aligned} &\text{(i)} \quad \bar{v} \geqq \tfrac{1}{2}\delta_0 \qquad \text{in} \quad \overline{B}_1, \\ &\text{(ii)} \quad -\Delta \bar{v} \geqq f(x, \bar{v}) \quad \text{in} \quad \left[\overline{\{1 < |x| < 2\} \cup A}\right]^0. \end{aligned}$$

Proof: We first extend v to $\overline{B}_1$ by choosing $\tilde{v} \in C^{1+\beta}(\overline{B}_1)$ satisfying

$$\begin{aligned} &\tfrac{1}{2}\delta_0 \leqq \tilde{v} \leqq 2/\delta_0 \quad \text{in} \quad \overline{B}_1, \\ &|\Delta \tilde{v}| \leqq C(\delta_0, M), \\ &\tilde{v} = v, \\ &\tilde{v}_\nu \geqq M + 2 \quad \text{on} \quad \partial B_1, \end{aligned}$$

(where ν is the exterior normal to ∂B_1). For example, we may take $\tilde{v}$ = harmonic extension of $v|\partial B$ + radial correction. For $|A|$ small enough, we can solve

$$-\Delta w = \left[C + \sup_{\substack{|x| \leqq 1 \\ 0 \leqq t \leqq 3/\delta_0}} |f(x,t)| \right] \chi_A \quad \text{in} \quad B_1,$$

$$w = 0 \qquad \text{on} \quad \partial B_1,$$

with $w \in C^{1+\beta}(B_1)$, $\|w\|_{C^{1+\beta}(B_1)} \leqq 1$ for any $\beta \in (0,1)$. Since $0 < \frac{1}{2}\delta_0 \leqq \tilde{v} + w \leqq 2/\delta_0 + 1 < 3/\delta_0$ it follows that

$$\text{(a)} \quad -\Delta(\tilde{v} + w) \geqq \sup|f| \geqq f(x, \tilde{v} + w) \quad \text{on} \quad A,$$

$$\text{(b)} \quad (\tilde{v} + w)_\nu \geqq M + 1 \qquad \text{on} \quad \partial B_1.$$

Then $\bar{v} = \tilde{v} + w$ is a Lipschitz extension of v to B_1 with the desired properties (3.2).

4. The Reflection Theorem

We are now in a position to prove a general "reflection theorem" that will eventually imply that solutions of (2.3) or (2.5) are asymptotically radial. Its statement is a bit complicated and its proof reasonably simple, given our preliminary results. The reader should keep in mind that the main hypothesis of the theorem, condition (e) below is formulated to insure that the good reflection property asserted in Lemmas 2.3 and 2.4 does not fail because of a difficulty on a compact set of small measure. The direction in which condition (e) holds is said to be an admissible direction. In Section 5 we shall show under very natural assumptions that there are many such directions.

Our formulation of the theorem makes essential use of the extension Lemma 3.1.

THEOREM 4.1. *Let $v > 0$ be a distribution solution of*

$$-\Delta v = f(x, v) \equiv F(x) \geqq 0 \quad \text{in} \quad |x| \geqq 1$$

satisfying

(a) *v is C^2 in $\{1 \leqq |x| \leqq 2\} \cup \{|x| > R\} \cup \{x_n > 1\}$,*
(b) *v has a harmonic asymptotic expansion* (2.6) *at ∞,*
(c) *$F(x) \leqq F(x_\lambda)$ in $x_n > \lambda > 0$ whenever $v(x) \leqq v(x_\lambda)$,*
(d) *v satisfies the assumptions* (3.1) *of Lemma* 3.1,
(e) *there exists a set $A' \subset \{(x', 0) : |x'| < 1\}$, $|A'| < \frac{1}{2}\sigma$, and a positive number $M > 1$ such that, if $x = (x', x_n)$ with $x' \notin A'$ and $x_n \geqq M$, then $v(x) \leqq \frac{1}{4}\delta_0$.*

Let $\bar{v}$ be the extension of v to $\mathbb{R}^n$ given by the extension Lemma 3.1 *corresponding to $A = \{x = (x', x_n) : |x| < 1,\ x' \in A'\}$. Then*

$$\bar{v}(x) \leqq \bar{v}(x_\lambda) \quad \text{for} \quad x_n > \lambda \geqq M. \tag{4.1}$$

Proof: Since $\bar{v}$ is strictly positive on compact sets and tends to zero uniformly at ∞, Lemma 2.3 implies that (4.1) holds for all sufficiently large λ. Consider the set of λ for which (4.1) holds. This set is closed because $w(x) = \bar{v}(x_\lambda) - \bar{v}(x)$ is lower semicontinuous. To see that it is also open, suppose (4.1) holds for some $\bar{\lambda} > M$. By the construction of $\bar{v}$, $\bar{v}(x) \not\equiv \bar{v}(x_\lambda)$. If (4.1) does not hold for all λ in some neighborhood of $\bar{\lambda}$, there is a sequence λ_j tending to $\bar{\lambda}$ and points x^j with $x_n^j > \lambda_j$ such that $\bar{v}(x^j) > \bar{v}(x^j_{\lambda_j})$. It follows from Lemma 2.4 (the plane $x_n = 0$ there corresponds to $x_n = \bar{\lambda}$ here) that a subsequence of the x_j converges to a point $\bar{x}$ with $\bar{x}_n \geqq \bar{\lambda}$. There are three cases to consider:

(i) $\bar{x}_n = \bar{\lambda}$ and $\bar{v}_{x_n}(\bar{x}) \geqq 0$,
(ii) $\bar{x}_n > \bar{\lambda}$ and $|\bar{x}_\lambda| > 1$,
(iii) $\bar{x}_n > \bar{\lambda}$ and $|\bar{x}_\lambda| \leqq 1$.

In all cases consider $w(x) = \bar{v}(x_{\bar{\lambda}}) - \bar{v}(x) \geqq 0$. Because of condition (c), w is superharmonic in $x_n \geqq \bar{\lambda}$. In case (i), w is a non-negative smooth superharmonic function in a small half-ball $B_\varepsilon^+(\bar{x}) = B_\varepsilon|\bar{x}| \cap \{x_n > \bar{\lambda}\}$ vanishing on $x_n = \bar{\lambda}$. By the Hopf boundary point lemma, $\bar{w}_{x_n}(\bar{x}) = -2\bar{v}_{x_n}(\bar{x}) > 0$, a contradiction. In case (ii), w is superharmonic and smooth in a neighborhood of $\bar{x}$ because of Lemma 2.2, and $w(\bar{x}) = 0$. This violates the maximum principle since $w(x) \not\equiv 0$.

In case (iii), $\bar{x}' \in A'$ for otherwise, one would have $w(\bar{x}) = v(\bar{x}_\lambda) - v(\bar{x}) \geqq \frac{1}{2}\delta_0 - \frac{1}{4}\delta_0 = \frac{1}{4}\delta_0$, a contradiction. Therefore, the extension Lemma 3.1 implies that w is again superharmonic in a neighborhood of $\bar{x}$. This is a contradiction as in case (ii) and completes the proof of the theorem.

5. The Family of Admissible Directions

In this section we shall obtain an estimate for the measure of the set of directions for which the hypothesis, condition (e) of the reflection Theorem 4.1, holds.

For this purpose the main properties of v at infinity that we shall use are the following:

(5.1)

(a) $v \geqq 0$ is superharmonic in $\mathbb{R}^n - B_1$,

(b) $\displaystyle\int_{\mathbb{R}^n - B_1} \frac{v^p}{|x|^\beta} < c_0$ for some $p \geqq 1,\ \beta < n$.

We want to use these two properties to estimate those cylindrical directions along which v has fast decay. Equivalently, we want to control the set of directions on which v does not decay.

For a given direction τ, we denote by $\Gamma(\tau)$ the half-infinite cylinder of radius 3 with axis τ, $\Gamma_k(\tau)$ being the portion of $\Gamma(\tau)$ in $B_{2^{k+1}} \setminus B_{2^k}$.

Given $\mu > 0$, define the exceptional set

$$A(k, \mu) = \left\{\tau: \left|P_\tau(\{v(x) > \mu\} \cap \Gamma_k(\tau))\right| > \mu\right\}.$$

Here, P_τ is orthogonal projection along the direction τ onto the plane $x \cdot \tau = 2^k$.

THEOREM 5.1. *$|A(k, \mu)| \leqq (C/\mu^2)2^{k/p(\beta-n)}$ with C independent of μ and k. In particular, if $\mu = 2^{-\delta k}$, $\delta = (n - \beta)/3p$, then*

$$|A(k)| = |A(k, \mu)| \leqq C2^{-\delta k}.$$

COROLLARY 5.2. *$|\bigcup_{k_0}^{\infty} A(k)| \leqq c2^{-\delta k_0}$ and if $\tau \notin \bigcup_{k_0}^{\infty} A(k)$, then τ is an admissible direction for $M \doteq 2^{k_0}$ with k_0 sufficiently large so that $\mu < \frac{1}{4}\delta_0$. In other words, the set of admissible directions for a given $M = 2^k$ and $\sigma = 2^{-\varepsilon k} = M^{-\varepsilon}$ has measure $|\Sigma_1| - M^{-\sigma}$ for suitable $\varepsilon, \delta > 0$.*

Proof of Theorem 5.1: We estimate $|A(k, \mu)|$ on the unit sphere Σ_1 by covering $A(k, \mu)$ by a finite union of spherical caps $\bigcup_{i=1}^{m} D(\tau_i)$ centered at τ_i of radius $C2^{-k}$. The constant C is an absolute constant independent of k chosen so that the radial projection of $P_\tau(\Gamma_k)$ is essentially $D(\tau)$. By a standard argument (see [3]), we may assume that the spherical caps centered at τ_i of radius $\frac{1}{2}C2^{-k}$ have "finite overlapping". Then,

$$|A(k, \mu)|_{\Sigma_1} \leqq \sum_{i=1}^{m} |D(\tau_1)|$$

$$\leqq C(2^{-k})^{n-1} \sum_{i=1}^{m} \left|P_{\tau_i}(\Gamma_k(\tau_i))\right| \tag{5.2}$$

$$\leqq C(2^{-k})^{n-1} \frac{1}{\mu} \sum_{i=1}^{m} \left|P_{\tau_i}(\{v(x) > \mu\} \cap \Gamma_k(\tau_i))\right|.$$

We shall estimate the right-hand side of (5.2) by an average of v. For that purpose let w be the capacitory potential of $2^{1-k}E$ in $B_8 - B_1$, $E = \bigcup_{i=1}^{m}\{v(x) >$

$\mu\} \cap \Gamma_k(\tau_i)$. That is, w is harmonic in $(B_8 - B_1) - E$, $w = 1$ on E, $w = 0$ on $\partial(B_8 - B_1)$. By the maximum principle, $v(2^{k-1}x) \geqq \mu w$ in $B_8 - B_1$.

Note that the capacity of $2^{1-k}E$ is given by

$$\operatorname{cap} 2^{1-k}E = \int_{B_8 - B_1} |\nabla w|^2\, dx = \int_{\partial(B_8 - B_1)} w_\nu\, d\sigma \tag{5.3}$$

(where ν is the interior normal) and that,

$$\begin{aligned} \frac{(n-2)}{R^{n-1}} \int_{\partial B_R} w &= \left(1 - \frac{1}{R^{n-2}}\right) \int_{\partial B_1} w_\nu, \qquad 1 < R < 2, \\ \frac{(n-2)}{R^{n-1}} \int_{\partial B_R} w &= \left(\frac{1}{R^{n-2}} - \frac{1}{8^{n-2}}\right) \int_{\partial B_8} w_\nu, \qquad 4 < R < 8. \end{aligned} \tag{5.4}$$

A simple argument using (5.3) and (5.4) gives us the important inequality

$$\int_{B_8 - B_1} |\nabla w|^2 \leqq c \fint_{B_8 - B_1} w \leqq \frac{c}{\mu} \fint_{B_{2^{k+2}} - B_{2^{k-1}}} v. \tag{5.5}$$

Denote by $\bar{P}_\tau$ the orthogonal projection P_τ followed by radial projection on Σ_1. By construction, $\bar{P}_\tau(\Gamma_k(\tau))$ is essentially $D(\tau)$ for large k. Given a point Q in $\bar{P}_\tau(E)$, let $\hat{Q}$ be the "first" point of $\bar{P}_\tau^{-1}(Q)$ on the section (curve) $\tilde{\gamma} = 2^{1-k}\gamma$ sitting over Q. Then

$$1 = \int_{\tilde{\gamma}} \frac{d}{ds} w\, ds.$$

Integration over $\bar{P}_\tau(E)$ on Σ_1 yields

$$|\bar{P}_\tau(E)| \leqq \int_{\bar{P}_\tau(E)} \int_{\tilde{\gamma}} |\nabla w|\, ds\, dr,$$

and so by Hölder's inequality

$$|\bar{P}_\tau(E)| \leqq c \int_{D(\tau)} \int_{\tilde{\gamma}} |\nabla w|^2\, ds\, dr. \tag{5.6}$$

Since the disks $D(\tau_i)$ have finite overlapping, we can sum the inequalities in (5.6) and obtain (recalling (5.2))

$$|A(k,\mu)| \leqq \frac{c}{\mu}\int_{B_8-B_1} |\nabla w|^2.$$

Using (5.5), this gives

$$|A(k,\mu)| \leqq \frac{c}{\mu^2}\fint_{R_k} v, \qquad R_k = B_{2^{k+2}} - B_{2^{k-1}}. \tag{5.7}$$

We now use assumption (5.16). We have

$$\begin{aligned} \fint_{R_k} v &\leqq \frac{c}{2^{nk}}\int_{R_k} v \\ &\leqq \frac{c}{2^{nk}}2^{nk(1-1/p)}2^{k\beta/p}\left(\int_{\mathbb{R}^n-B_1}\frac{v^p}{|x|^\beta}\right)^{1/p} \\ &\leqq c2^{k(\beta-n)/p}. \end{aligned} \tag{5.8}$$

Combining (5.7) and (5.8) we obtain the estimate

$$|A(k,\mu)| \leqq \frac{c}{\mu^2}2^{k(\beta-n)/p}.$$

The theorem is proven.

6. Asymptotic Symmetry

A final step that is needed, before we can discuss applications, is to clarify how asymptotic symmetry will follow from the Reflection Theorem.

For a given direction τ, denoted by $x_\lambda = x + 2(\lambda - x\cdot\tau)\tau$, the reflection of x in the plane $x\cdot\tau = \lambda$.

THEOREM 6.1. *Let v be a function on $\mathbb{R}^n - B_1$ with the property that, for some $M > 0$ and $\tau \in \mathscr{A} \subset \Sigma_1$,*

$$v(x) \leqq v(x_\lambda) \quad \text{if} \quad x\cdot\tau \geqq \lambda > M. \tag{6.1}$$

Then there are constants $\varepsilon_0 > 0$, $C > 0$ *independent of* M *such that, if* $|\Sigma_1| - |\mathscr{A}| < \varepsilon_0$, *then*

$$v(x) \geqq v(y) \quad \textit{whenever} \quad |x| > 1,\ |y| \geqq |x| + CM. \tag{6.2}$$

Proof: We want to show that we may compare $v(x)$ and $v(y)$ by a finite number of reflections taken from the admissible set $\mathscr{A}$. It suffices to show that (6.2) holds (for some C) if that angle $\alpha(x, y)$ between x and y is small enough. For we can then inductively find a sequence $z_1, z_2, \cdots, z_k$ of bounded length such that $z_1 = x$, $z_k = y$, $|z_{i+1}| > |z_i| + M$ and $\alpha(z_i, z_{i+1})$ small.

For concreteness, we shall suppose that $\alpha(x, y) \leqq \frac{1}{8}\pi$. Consider the cone Γ_x of directions of aperture $\frac{1}{4}\pi$ (with vertex at the origin) and axis direction $-x$. We fix $R \geqq |x| + 2M$ to be chosen later and look at those points z in Σ_R which can be obtained by reflection of x in a plane Π_τ with normal $\tau \in \Gamma_x$ such that Π_τ separates z from $B_M(0)$. If $\tau \in \mathscr{A}$ is admissible, then from (6.1), $v(x) \geqq v(z)$ and we say that z is admissible for x. That is,

$$\mathscr{A}_x = \{z \in \Sigma_R\colon x = z + 2(\lambda - x \cdot \tau)\tau \text{ with}$$

$$z \cdot \tau > \lambda \geqq M \text{ and } \tau \in \Gamma_x \cap \mathscr{A}\}.$$

Note that $z = x - 2(x \cdot \tau - \lambda)\tau$ and

$$|z|^2 = |x|^2 + 4\lambda(\lambda - x \cdot \tau) = R^2.$$

It follows that $R^2 - |x|^2 \leqq 4\lambda(\lambda + |x|)$ or $(2\lambda + |x|)^2 \geqq R^2$. Therefore, $2\lambda \geqq R - |x| \geqq 2M$ as long as $R \geqq |x| + 2M$.

Similarly, we define the admissible set for y, $\mathscr{A}_y$, as follows:

$$\mathscr{A}_y = \{z \in \Sigma_R\colon z = y + 2(\lambda - y \cdot \tau)\tau \text{ with}$$

$$y \cdot \tau > \lambda \geqq M \text{ and } -\tau \in \Gamma_Y,\ \tau \in \mathscr{A}\}.$$

Notice that if $z \in \mathscr{A}_y$, $v(z) \geqq v(y)$ and the direction $-\tau$ is in Γ_y (cone of directions of aperture $\frac{1}{8}\pi$ with axis $-y$). In order that $\lambda \geqq M$ be well defined, R must be chosen so that

$$\lambda^2 - \lambda y \cdot \tau + \frac{|y|^2 - R^2}{4} = 0.$$

For example, we may take

$$\sqrt{2}R \geqq |y| \geqq R \geqq |x| + 2\sqrt{2}M$$

and then both $\mathscr{A}_x$ and $\mathscr{A}_y$ are well defined.

By our construction $\mathscr{A}_x$ is just the set of admissible points for x in $\mathscr{C}_x \cap \Sigma_R$, where $\mathscr{C}_x$ is the cone with vertex x, axis $-x$ and aperture $\frac{1}{4}\pi$. For ε_0 small enough, $\mathscr{A}_x$ covers as big a proportion of $\mathscr{C}_x \cap \Sigma_R$ as we like. Similarly, $\mathscr{A}_y$ is the set of admissible points for y in $\mathscr{C}_y \cap \Sigma_R$ and covers as big a proportion of $\mathscr{C}_y \cap \Sigma_R$ as we want.

Therefore, we can arrange that

$$\left|\mathscr{A}_x \cap (\mathscr{C}_x \cap \Sigma_R) \cap (\mathscr{C}_y \cap \Sigma_R)\right| > \tfrac{1}{2}\left|(\mathscr{C}_x \cap \Sigma_R) \cap (\mathscr{C}_y \cap \Sigma_R)\right|,$$

$$\left|\mathscr{A}_y \cap (\mathscr{C}_x \cap \Sigma_R) \cap (\mathscr{C}_y \cap \Sigma_R)\right| > \tfrac{1}{2}\left|(\mathscr{C}_x \cap \Sigma_R) \cap (\mathscr{C}_y \cap \Sigma_R)\right|,$$

and thus there exist $z \in \mathscr{A}_x \cap \mathscr{A}_y$ with

$$v(y) \leqq v(z) \leqq v(x).$$

This completes the proof.

COROLLARY 6.2. *Let $v(x)$ satisfy the hypothesis of Theorem* 6.1. *Suppose in addition that $v \geqq 0$ is superharmonic. Then,*

$$v(x) = \inf_{\Sigma_R} v(1 + O(1/R)) \quad \textit{for} \quad |x| = R \quad \textit{as} \quad R \to \infty. \tag{6.3}$$

Proof: From Theorem 6.1, we have

$$\sup_{\partial B_{R+CM}} v \leqq \inf_{\partial B_R} v \leqq \sup_{\partial B_R} v \leqq \inf_{\partial B_{R-CM}} v. \tag{6.4}$$

But

$$v \geqq \left(\frac{R_1}{|x|}\right)^{n-2} \inf_{\partial B_{R_1}} v \quad \text{for} \quad |x| \geqq R_1.$$

In particular, for $R_1 = R - CM$, $|x| = R + CM$,

$$\inf_{\partial B_{R+CM}} v \geqq \left(\frac{R - CM}{R + CM}\right)^{n-2} \inf_{\partial B_{R-CM}} v,$$

or

$$\inf_{\partial B_{R-CM}} v \leqq (1 + O(1/R)) \inf_{\partial B_{R+CM}} v. \tag{6.5}$$

The combination of (6.4) and (6.5) gives (6.3).

7. Proofs of the Main Results

Proof of Theorem 1.1: As in Section 2, formulas (2.3)–(2.5), we perform an inversion about a regular point z near 0 and transform u to v by the Kelvin transform. Let us take $z = (1/\mu)e_n$ and locate the singularity of v at the point $-\mu e_n$. Then we can apply the Reflection Theorem 4.1 for any admissible direction τ in the cone $\alpha(\tau, e_n) \leqq \frac{1}{4}\pi$. Note that the estimates in measure, in Section 5, for the set of admissible directions hold uniformly for all approximations v. Since the reflection inequality is preserved under limits, if we let $\mu \to \infty$, the admissible directions for the limit are the lim sup of the admissible directions for the approximations. That is, $\chi(\limsup \mathscr{A}_\mu) = \limsup \chi(\mathscr{A}_\mu)$. Thus, by Fatou's lemma, the asymptotic symmetry results of Section 6 may be applied directly to our original solution u inverted through the singularity at 0. In particular, Corollary 6.2 shows that

$$v(x) = \inf_{\Sigma_R} v(1 + O(1/R)) \quad \text{for} \quad |x| = R \quad \text{as} \quad R \to \infty,$$

where

$$v(x) = \frac{1}{|x|^{n-2}} u\left(\frac{x}{|x|^2}\right).$$

It follows that, for $|y| = r$,

$$u(y) = \left(\inf_{|y|=r} u\right)(1 + O(r)) \quad \text{for} \quad |y| = r \quad \text{as} \quad r \to 0$$

and Theorem 1.1 follows. We now use Theorem 1.1 to prove Theorem 1.2.

Proof of Theorem 1.2: We first show that

$$\text{(7.1)} \qquad m(r) = O\left(\frac{1}{r^{(n-2)/2}}\right), \qquad m'(r) = O\left(\frac{1}{r^{n/2}}\right).$$

Since u is a distribution solution, we have

$$\int_{B_r} u\Delta\eta + \eta u^{(n+2)/(n-2)} = \int_{\partial B_r} u\eta_\nu - \eta u_\nu$$

for smooth η. Choosing $\eta = |x|^2$ we see, after some simple manipulations, that

$$\text{(7.2)} \qquad \int_{B_r} (r^2 - |x|^2) u^{(n+2)/(n-2)} + 2r\int_{\partial B_r} u = 2n\int_{B_r} u.$$

Since $u(x) = m(|x|)(1 + O(|x|))$ and

$$-m'(r) = -\int u_r \, d\omega = \frac{c}{r^{n-1}} \int_{B_r} u^{(n+2)/(n-2)} > 0, \tag{7.3}$$

we have, from (7.2),

$$\begin{aligned} r^{n+2} m^{(n+2)/(n-2)} &< c \int_0^r m(t) t^{n-1} \, dt \\ &\leqq c r^{4n/(n+2)} \left(\int_0^r m^{(n+2)/(n-2)} t^{n-1} \, dt \right)^{(n-2)/(n+2)} \\ &\leqq c r^{4n/(n+2)} \left(\int_0^R m^{(n+2)/(n-2)} t^{n-1} \, dt \right)^{(n-2)/(n+2)} \end{aligned} \tag{7.4}$$

for $r \leqq R$. From (7.4) it follows that

$$\int_0^R r^{n-1} m^{(n+2)/(n-2)} \, dr \leqq c R^{(n-2)/2} \tag{7.5}$$

and (7.5) gives $R^n m^{(n+2)/(n-2)} \leqq c R^{(n-2)/2}$ or $m(R) \leqq c / R^{(n-2)/2}$. Inserting this into (7.3) proves the preliminary estimates (7.1).

Now set $t = -\log r$ and

$$\psi(t, \theta) = r^{(n-2)/2} u(r, \theta), \qquad \theta \in S^{n-1}. \tag{7.6}$$

Then a simple calculation shows that ψ satisfies

$$\psi_{tt} + \Delta_\theta \psi - \left(\frac{n-2}{n} \right)^2 \psi + \psi^{(n+2)/(n-2)} = 0. \tag{7.7}$$

Let $\beta(t) = \int_{S^{n-1}} \psi \, d\theta = r^{(n-2)/2} m(r)$. Note that

$$\psi(t, \theta) = r^{(n-2)/2} m(r)(1 + O(r)) + \beta(t)(1 + O(e^{-t})),$$

$$\beta' + \frac{n-2}{n} \beta = -r^{n/2} m'(r) \geqq 0,$$

and that $\beta = O(1)$, $\beta' = O(1)$ from (7.1) and (7.8).

LEMMA 7.1.

$$\frac{\partial}{\partial t}(\psi - \beta) = \beta \, O(e^{-t}),$$

$$|\nabla_\theta (\psi - \beta)| = \beta \, O(e^{-t}).$$

Proof: From Theorem 1.1, we have

$$-\Delta(u-m) = m^{(n+2)/(n-2)}O(r) \quad \text{in} \quad \tfrac{1}{2}r < |x| < 2r,$$

say. Then by standard elliptic estimates (for example, Theorem 3.9 of [7]),

$$\begin{aligned}|\nabla(u-m)| &\leqq c\left(\frac{\sup|u-m|}{r} + r\sup m^{(n+2)/(n-2)}O(r)\right)\\ &\leqq c\left(\sup(m + r^2 m^{(n+2)/(n-2)})\right)\end{aligned}$$

on $|x| = r$.

But in $\frac{1}{2}r < |x| < 2r$, $\sup u$ (and thus $\sup m$) is comparable to $u(x)$. Since $u > 0$ satisfies $\Delta u + a(x)u = 0$ with $a(x) = u^{4/(n-2)} \leqq c/|x|^2$ and hence it satisfies the Harnack inequality (see [6] for more details) in the annulus $\frac{1}{4}r < |x| < 4r$. Therefore,

$$|\nabla(u-m)| \leqq c\left(m(r) + r^2 m^{4/(n-2)}m\right) \leqq cm(r) \quad \text{for} \quad |x| = r.$$

In particular,

$$\frac{\partial}{\partial r}(u-m) \leqq cm, \tag{7.9}$$
$$|\nabla_\theta(u-m)| \leqq crm.$$

Since $\psi(t,\theta) - \beta(t) = r^{(n-2)/2}(u(r,\theta) - m(r))$, $t = -\log r$, the lemma follows easily.

We now derive from (7.7) a standard energy estimate (equivalent to the well-known Pohazhaev identity) by multiplying (7.7) by ψ_t and integrating:

$$\int_{S^{n-1}}\left(\psi_t^2 - \left(\frac{n-2}{n}\right)^2\psi^2 + \frac{n-2}{n}\psi^{2n/(n-2)} - (\nabla_\theta\psi)^2\Big|_s^t = 0.\right. \tag{7.10}$$

Using Lemma 7.1, we convert (7.10) into our basic identity:

$$\left(\beta'^2 - \left(\frac{n-2}{n}\right)^2\beta^2 + \frac{n-2}{n}\beta^{2n/(n-2)}\right)\Bigg|_s^t = (\beta^2 + \beta'^2)O(e^{-t})\Big|_s^t. \tag{7.11}$$

Set

$$D(t) = \beta'^2 - \left(\frac{n-2}{n}\right)^2\beta^2 + \frac{n-2}{n}\beta^{2n/(n-2)}.$$

Then (7.11) can be rewritten as

$$D(t) = D(s) + (\beta^2 + \beta'^2)O(e^{-s}) + O(e^{-t}) \tag{7.12}$$

for $t \geqq s$. Equation (7.12) determines a unique asymptotic constant

$$D_\infty = \lim_{t\to\infty} D(t).$$

For, from (7.12),

$$D(k+1) - D(k) = O(e^{-k})$$

so that $|D(k+l+1) - D(k)| \leq ce^{-k}$. Therefore, the bounded Cauchy sequence $\{D(k)\}$ has a unique limit D_∞. Passing to the limit in (7.12) gives

$$D_\infty = D(s) + (\beta^2 + \beta'^2)O(e^{-s}).$$

Reversing, the roles of s and t, we arrive at the fundamental equation

$$D(t) = D_\infty + (\beta^2 + \beta'^2)O(e^{-t}), \tag{7.13}$$

or equivalently

$$\beta'^2 = \left(\frac{n-2}{n}\right)^2\beta^2 - \frac{n-2}{n}\beta^{2n/(n-2)} + D_\infty + (\beta^2 + \beta'^2)O(e^{-t}). \tag{7.14}$$

From (7.14) we see that the behavior of β is completely determined by the roots of the right-hand side of (7.14). In particular, D_∞ must lie in the interval

$$0 \geqq D_\infty \geqq \frac{-2}{n}\left(\frac{n-2}{n}\right)^n. \tag{7.15}$$

In Case $D_\infty < 0$ it follows easily from (7.14) that β is asymptotic to (a suitable translate of) the solution of the corresponding equation (1.6) of the introduction and Theorem 1.2 follows.

It remains to show that, in the case $D_\infty = 0$, u has a removable singularity (or equivalently, β decays to zero like $e^{-(n-2)t/2}$).

From (7.14) we see that β cannot have a local minimum and must ultimately decrease monotonically to zero (recall from the introduction that the asymptotic solution $\psi \equiv k$ corresponds to $D_\infty = -(2/n)((n-2)/n)^n$). Therefore,

$$\lim_{t\to\infty} \beta(t) = \lim_{t\to\infty} \beta'(t) = 0,$$

$$\beta' < 0 \quad \text{for} \quad t > t_0.$$

Returning to (7.14) we find that

$$\beta' \geqq \lambda^2 \beta^2 \quad \text{for} \quad t > t_0(\lambda)$$

for any $\lambda < (n-2)/2$. It follows that $\beta = O(e^{-\lambda t})$, $\beta' = O(e^{-\lambda t})$. Going back to (7.14) once more we see that

$$\frac{\beta'^2}{\beta^2} \geqq \left(\frac{n-2}{n}\right)^2 + O(e^{-t})$$

and so

$$\frac{-\beta'}{\beta} \geqq \frac{n-2}{n} - ce^{-t}, \qquad t \geqq t_0.$$

Integrating this we obtain

$$\beta(t) \leqq \beta(t_0) \exp\{ce^{t_0}\} e^{-(n-2)(t-t_0)/2}$$

and thus u has a removable singularity, and Theorem 1.2 is complete.

Proof of Theorem 1.3: We sketch the proof as there are no technical difficulties. As in the proof of Theorem 1.2, it is a simple matter to establish

$$(7.16) \qquad m(r) = O\left(\frac{1}{r^{2/(\alpha-1)}}\right), \qquad m'(r) = O\left(\frac{1}{r^{(\alpha+1)/(\alpha-1)}}\right).$$

Again, set $t = -\log r$ and

$$\psi(t,\theta) = r^{2/(\alpha-1)} u(r,\theta), \qquad \theta \in S^{n-1}.$$

Then

$$(7.17) \qquad \psi_{tt} + \Delta_\theta \psi + a\psi_t - c_0^{\alpha-1}\psi + \psi^\alpha = 0,$$

where

$$a = \frac{n-2}{\alpha-1}\left(\frac{n+2}{n-2} - \alpha\right) \quad \text{and} \quad c_0^{\alpha-1} = \frac{2(n-2)}{(\alpha-1)^2}\left(\alpha - \frac{n}{n-2}\right).$$

Let

$$\beta(t) = \int_{S^{n-1}} \psi \, d\theta = r^{2/(\alpha-1)} m(r).$$

As before, we have that

$$
\begin{aligned}
&\psi = \beta(1 + O(e^{-t})),\\
&\beta = O(1), \qquad \beta' = O(1),\\
&\frac{\partial}{\partial t}(\psi - \beta) = \beta O(e^{-t}),\\
&|\nabla_\theta(\psi - \beta)| = \beta O(e^{-t}).
\end{aligned} \tag{7.18}
$$

Averaging (7.17) over S^{n-1} we obtain

$$\beta'' + a\beta' - c_0^{\alpha-1}\beta + \beta^\alpha + \fint_{S^{n-1}}(\psi^\alpha - \beta^\alpha) = 0. \tag{7.19}$$

Multiplying (7.19) by β' and integrating yields

$$\tfrac{1}{2}\beta'^2\Big|_s^t + a\int_s^t \beta'^2\,dt - c_0^{\alpha-1}\tfrac{1}{2}\beta^2\Big|_s^t + \frac{\beta^{\alpha+1}}{\alpha+1}\Big|_s^t + O(e^{-s}) = 0. \tag{7.20}$$

It follows from (7.19) that $\int_s^\infty \beta'^2 < c$ and so $\beta' \to 0$ as $t \to \infty$. Multiplying (7.19) by β'' and integrating by parts, we find also that $\int_s^\infty \beta''^2 < c$ and so $\beta'' \to 0$ as $t \to \infty$. Passing to the limit as $t \to \infty$ in (7.19), we conclude that either

$$\lim_{t\to\infty} \beta(t) = c_0$$

or

$$\lim_{t\to\infty} \beta(T) = 0, \qquad \alpha > n/(n-2).$$

To complete the proof of part (i) of Theorem 1.3 we must show that the singularity is removable if $\beta \to 0$ as $t \to \infty$.

It is easy to see that the decay of β is controlled by the negative root of

$$r^2 + ar - c_0^{\alpha-1} = 0.$$

This root is

$$\lambda = -\tfrac{1}{2}\left(a + \sqrt{a^2 + 4c_0^{\alpha-1}}\right).$$

But $a^2 + 4c_0^{\alpha-1} = (n-2)^2$. Therefore,

$$-\lambda = \frac{a + (n-2)}{2} = \frac{2}{\alpha - 1}.$$

Using (7.18) to control the errors one can show that $\beta = O(e^{-2t/(\alpha-1)})$ and that the singularity of u is removable.

To complete the proof of part (ii) of Theorem 1.3 ($\alpha = n/(n-2)$, $c_0 = 0$, $a = n-2$), we must find the decay rate of β. Without too much trouble one sees that either $\beta = O(e^{-(n-2)t})$ and the singularity of u is removable or that

$$\lim_{t\to 0} t^{(n-2)/2}\beta(t) = \left(\frac{(n-2)^2}{n}\right)^{(n-2)/2}$$

This completes the proof.

8. Global Results

In this final section we include a global result for solutions in $\mathbb{R}^n$ with one or two singularities (one of which is located at infinity). Its proof follows easily from the results of Section 2.

THEOREM 8.1. *Let u be a C^2 solution of*

$$-\Delta u = g(u) \quad \text{in} \quad \mathbb{R}^n - \{0\}, \qquad n \geqq 3,$$

with an isolated singularity at the origin. Assume that $g(t)$ is a locally Lipschitz function satisfying

(i) *$g(t)$ is non-decreasing, $g(0) = 0$,*
(ii) *$t^{-(n+2)/(n-2)}g(t)$ is non-increasing*
(iii) *$g(t) \geqq ct^p$ for some $p \geqq n/(n-2)$ for t large.*

Then,

(a) *if the origin is a non-removable singularity, then u is radially symmetric about the origin;*
(b) *if the origin is a removable singularity, u is radially symmetric about some point of $\mathbb{R}^n$.*

COROLLARY 8.2. *Let $u \geq 0$ be a C^2 solution of*

$$\Delta u + u^\alpha = 0 \quad \text{in} \quad \mathbb{R}^n.$$

Then,

(a) *if $n/(n-2) \leqq \alpha < (n+2)/(n-2)$,*

$$u \equiv 0.$$

(b) *if $\alpha = (n+2)/(n-2)$,*

$$u = \left(\frac{\lambda\sqrt{n(n-2)}}{\lambda^2 + r^2}\right)^{(n-2)/2}, \qquad \lambda > 0,$$

for some origin.

Remark 8.3. For $\alpha = (n+2)/(n-2)$, the corollary implies by conformal invariance that there are no one-point singularities.

Proof of Corollary 8.2: Part (b) follows from the discussion of radial solutions given in the introduction. To prove part (a) observe that u satisfies

$$(r^{n-1}u')' = -r^{n-1}u^{\alpha}, \qquad u' < 0,$$

so that

$$r^{n-1}u'(r) = -\int_0^r s^{n-1}u^{\alpha}(s)\,ds \leqq \frac{-r^n}{n}u^{\alpha}(r).$$

Therefore, $u^{-\alpha}u'(r) \leqq -r/n$. Integrating this form 0 to r we obtain

$$\frac{1}{u^{\alpha-1}} \geqq \frac{1}{u^{\alpha-1}(0)} + \frac{\alpha-1}{2n}r^2.$$

This implies that

$$\begin{aligned} &u \leqq cr^{-2/(\alpha-1)}, \\ &|u'| \leqq cr^{-(\alpha+1)/(\alpha-1)}. \end{aligned} \tag{8.1}$$

To complete the proof we need the well-known Pohazhaev identity

$$\begin{aligned} &\tfrac{1}{2}R^n u'^2(R) + R^n\frac{u^{\alpha+1}}{\alpha+1}(R) + \frac{n-2}{n}R^{n-1}uu'(R) \\ &= \left(\frac{n}{\alpha+1} - \frac{n-2}{n}\right)\int_0^R r^{n-1}u^{\alpha+1}\,dr. \end{aligned} \tag{8.2}$$

Using (8.1) we see that the terms on the left-hand side of (8.2) tend to zero as $R \to \infty$. Since $n/(\alpha+1) - (n-2)/n > 0$, this implies that $\int_0^\infty r^{n-1}u^{\alpha+1} = 0$ or $u \equiv 0$.

Proof of Theorem 8.1: Suppose the origin is a non-removable singularity. Choose an arbitrary point $z \neq 0$ and, as in formulas (2.4), (2.5), let v be the Kelvin transform of u. Observe that $v(x)$ has a harmonic asymptotic development at infinity and is singular at the origin and at $x = -z/|z|^2$. This defines an axis (going through 0 and z) about which, as we shall show, v is axisymmetric. Consider any reflection direction τ orthogonal to this axis. For simplicity, suppose τ is the positive x_n direction. Then, by Lemma 2.3,

$$v(x) \leqq v(x_\lambda) \quad \text{for} \quad x_n \geqq \lambda \tag{8.3}$$

for all λ sufficiently large. Consider the set of $\lambda \geqq 0$ for which (8.3) holds. This set is closed by lower-semicontinuity of $w(x) = \bar{v}(x_\lambda) - \bar{v}(x)$. To see that it is also open, suppose (8.3) holds for some $\bar{\lambda} > 0$. By our assumption that the origin is non-removable, $v(x) \not\equiv v(x_{\bar{\lambda}})$. If (8.3) does not hold for all λ in some neighborhood of $\bar{\lambda}$, there is a sequence λ_j tending to $\bar{\lambda}$ and points x^j with $x_n^j > \lambda_j$ such that $\bar{v}(x^j) > \bar{v}(x^j_{\lambda_j})$. It follows from Lemma 2.4 (the plane $x_n = 0$ there corresponds to $x_n = \bar{\lambda}$ here) that a subsequence of the x_j converges to a point $\bar{x}$ with $\bar{x}_n \geqq \bar{\lambda}$. Necessarily,

$$\bar{v}(\bar{x}_{\bar{\lambda}}) = \bar{v}(\bar{x}).$$

Case (i): $\bar{x}_n = \lambda$ and $\bar{v}_{x_n}(\bar{x}) \geqq 0$.
Case (ii): $\bar{x}_n > \lambda$.

In case (ii), $w = \bar{v}(x_{\bar{\lambda}}) - \bar{v}(x)$ is superharmonic and non-negative in a neighborhood of $\bar{x}$, and $w(\bar{x}) = 0$. This violates the maximum principle since $w(x) \not\equiv 0$. In case (i), w is a non-negative smooth superharmonic in a small half-ball $B_\varepsilon^+(\bar{x}) = B_\varepsilon(\bar{x}) \cap \{x_n > \bar{\lambda}\}$ vanishing on $x_n = \lambda$.

By the Hopf boundary point lemma,

$$\bar{w}_{x_n}(\bar{x}) = -2\bar{v}_{x_n}(\bar{x}) > 0,$$

a contradiction. Therefore, we have shown that (8.3) holds for all $\lambda \geqq 0$ and so v is axisymmetric. This implies that $u(x)$ is axisymmetric about any axis through the origin, and so is radially symmetric about the origin. This proves part (a) of the theorem. To prove part (b) observe that the previous argument shows that, for any direction τ through the origin, it is symmetric in some plane $x \cdot \tau = \lambda \geqq 0$. Choosing a new point in this plane as the center of inversion, we find that u is symmetric about a second plane orthogonal to the first. After n steps we find the center of symmetry. Alternatively, one can invert about some point z and translate along a direction parallel to $\nabla u(z)$ to obtain the optimal asymptotic development at infinity as in [4]. Our previous reasoning then says that u is axisymmetric about an axis through z parallel to the vector $\nabla u(z)$. Choose another point $\bar{z}$ as the center of inversion. Then if $\nabla u(\bar{z})$ is not parallel (in the same sense), u is axisymmetric about a second axis. The intersection of these axes determines the center of symmetry. The other possibility namely that ∇u always points in the same direction is easily eliminated as there are no global non-negative solutions of

$$u'' + f(u) = 0.$$

This completes the proof.

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