Girsanov Change of Measure: SDE Inference

Math653: SDEs and applications

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$$dX_t = b(X_t)dt + \sigma(X_t)dB_t; \quad X_0 = x_0 \in \mathbb{R}.$$

Motivating Questions

- How to estimate the drift b from path data $\{X^m_{[0,T]}\}$?
- How to classify if a path is from SDE with b_1 or SDE with b_2? (Hypothesis test)

Motivation: Likelihood on Path Space

Likelihood on Path Space

Warm-up: i.i.d. case

To estimate θ_0 from data $X^{(1)}, \dots, X^{(M)} \overset{\text{i.i.d.}}{\sim} X$ with pdf u_{θ_0}, we maximize the likelihood

$$\ell_M(\theta) = \frac{1}{M} \sum_{m=1}^{M} \log u_\theta(X^{(m)}).$$

Note: u_θ is defined relative to a Lebesgue measure.

Inference for SDEs

How to estimate b from path data $\{X^m_{[0,T]}\}$ of SDE?

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t, \qquad t \in [0,T],$$

What is the analogue of the likelihood $\ell_M(\theta)$ for SDEs?
What should the *density of the path* $X_{[0,T]}$ mean?

Pseudo-Likelihood from Euler–Maruyama

On a grid $t_k = k\Delta$,

$$X_{t_{k+1}} \mid X_{t_k} \approx \mathcal{N}\big(X_{t_k} + b(X_{t_k})\Delta,\ \sigma^2(X_{t_k})\Delta\big).$$

This suggests the approximate log-likelihood of $X_{t_0:t_L}$:

$$\log u(x_{0:L}) \approx -\frac{1}{2}\sum_k \left(\log \sigma^2(x_{t_k}) + \frac{(\Delta x_k - b(x_{t_k})\Delta)^2}{\sigma^2(x_{t_k})\Delta}\right).$$

Temptation: Let $\Delta \to 0$, $\Delta x_k/\Delta \to \dot{X}_t$. $\downarrow$

$$\mathcal{E}(b; X_{[0,T]}) = \int_0^T \frac{|\dot{X}_t - b(X_t)|^2}{2\sigma^2(X_t)}\,\mathrm{d}t.$$

But $\dot{X}_t$ does not exist a.s. (nowhere differentiable).
What is wrong with this argument?

Reference Law: Choose a reference path law, typically the zero-drift:

$$X_{t_{k+1}} \mid X_{t_k} \approx \mathcal{N}\big(X_{t_k},\ \sigma^2(X_{t_k})\Delta\big).$$

A reference pdf of $X_{t_0:t_L}$: $u_0(x_{0:L})$

$$\log u_0(x_{0:L}) \approx -\frac{1}{2}\sum_k \Big(\log \sigma^2(x_{t_k}) + \frac{(\Delta x_k)^2}{\sigma^2(x_{t_k})\Delta}\Big).$$

The likelihood ratio of the drifted law versus the reference law:

$$\begin{aligned}\log \frac{u(x_{0:L})}{u_0(x_{0:L})} &\approx -\frac{1}{2}\sum_k \Big(\frac{(\Delta x_k - b(x_{t_k})\Delta)^2}{\sigma^2(x_{t_k})\Delta} - \frac{(\Delta x_k)^2}{\sigma^2(x_{t_k})\Delta}\Big) \\ &\approx -\frac{1}{2}\sum_k \Big(\frac{b^2(x_{t_k})\Delta^2 - 2b(x_{t_k})\Delta x_k \Delta}{\sigma^2(x_{t_k})\Delta}\Big) \\ &\xrightarrow{\Delta \to 0} -\int_0^T \frac{b^2(X_t)dt - 2b(X_t)dX_t}{2\sigma^2(X_t)} =: \mathcal{E}(b)\end{aligned}$$

Q: Ratio has limit. What is the limit of $u_0(x_{0:L})$ as $\Delta \to 0$?

Reference Law: The Right Likelihood Question

We cannot write the path pdf as the finite-dimensional case.
Denote it by $\mathbb{P}^b$ for SDE with drift b.
Choose a reference path law $\mathbb{P}^0$, and define the "ratio" between the two laws.

$$\frac{\mathrm{d}\mathbb{P}^b}{\mathrm{d}\mathbb{P}^0}\bigg|_{\mathcal{F}_T},$$

where $\mathcal{F}_T$ is the sigma-algebra generated by $X_{[0,T]}$.

Main message

Girsanov's theorem gives this Radon–Nikodym derivative when the diffusion is unchanged and only the drift is modified.

Change of Measure Basics

Change of Measure: What Replaces a Path Density?

Absolute continuity

$$\mathbb{Q} \ll \mathbb{P} \text{on} \mathcal{F}_T \iff \mathbb{P}(A) = 0 \implies \mathbb{Q}(A) = 0 \qquad \text{for all } A \in \mathcal{F}_T.$$

Radon–Nikodym derivative

Then there exists $Z_T \geq 0$, $\mathcal{F}_T$-measurable, such that

$$\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}[\mathbb{1}_A Z_T], \qquad Z_T = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_T}.$$

- Interpretation: Z_T is the weight that converts $\mathbb{P}$-expectations into $\mathbb{Q}$-expectations.
- **Density process:** (Z_t) is a nonnegative $\mathbb{P}$-martingale:

$$Z_t := \mathbb{E}_{\mathbb{P}}[Z_T \mid \mathcal{F}_t], \quad Z_t = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_t}.$$

- On path space, measure change is dynamic: the likelihood ratio itself evolves in time.

For SDE, what is the density process Z_t?

Return to the change of measure for r.v., using the Radon–Nikodym derivative.
Z: a nonnegative random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}_{\mathbb{P}}[Z] = 1$. Define a new measure $\mathbb{Q}$ by

$$\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}[\mathbb{1}_A Z], \quad A \in \mathcal{F}, \quad i.e., \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = Z.$$

Then for any random variable Y,

$$\mathbb{E}_{\mathbb{Q}}[Y] = \mathbb{E}_{\mathbb{P}}[YZ].$$

Example: Let $X \sim \mathcal{N}(\lambda, 1)$ under $\mathbb{P}$. Define $Z = \exp(-\lambda X + \frac{1}{2}\lambda^2)$, then $\mathbb{E}_{\mathbb{P}}[Z] = 1$ and under $\mathbb{Q}$ defined by $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} = Z$, the law of X is $\mathcal{N}(0, 1)$:

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}[e^{isX}] = \mathbb{E}_{\mathbb{P}}[e^{isX} Z] &= \mathbb{E}_{\mathbb{P}}\left[\exp\left((is - \lambda)X + \frac{1}{2}\lambda^2\right)\right] \\ &= \exp\left(\lambda(is - \lambda) + \frac{1}{2}(is - \lambda)^2 + \frac{1}{2}\lambda^2\right) = \exp\left(-\frac{1}{2}s^2\right).\end{aligned}$$

Brownian motion: Let B_t be $N(0, t)$ under $\mathbb{P}$. Define $Z_t = \exp(-\lambda B_t - \frac{1}{2}\lambda^2 t)$, then $\mathbb{E}_{\mathbb{P}}[Z_t] = 1$, and $W_t = B_t + \lambda t$ is $N(0, t)$ under $\mathbb{Q}$ defined by $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} = Z_t$.

Theorem (Girsanov's theorem for Brownian shift)

Let $(B_t)_{0\le t\le T}$ *be a Brownian motion under* $\mathbb{P}$, *adapted to the filtration* $(\mathcal{F}_t)_{0\le t\le T}$. *Fix* $\lambda \in \mathbb{R}$, *and define*

$$Z_t = \exp\left(-\lambda B_t - \frac{1}{2}\lambda^2 t\right), \qquad 0 \le t \le T.$$

Then $(Z_t)_{0\le t\le T}$ *is a martingale under* $\mathbb{P}$ *with* $\mathbb{E}_{\mathbb{P}}[Z_t] = 1$, $0 \le t \le T$. *Define a new probability measure* $\mathbb{Q}$ *on* $\mathcal{F}_T$ *by*

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{F}_T} = Z_T.$$

Then, under $\mathbb{Q}$, *the process* $W_t := B_t + \lambda t$, $0 \le t \le T$, *is a Brownian motion.*

- Equivalently, under $\mathbb{Q}$, the process $(B_t)_{0\le t\le T}$ has drift $-\lambda$, so that $B_t = W_t - \lambda t$.
- Z_t is a martingale by Itô's formula: $\mathrm{d}Z_t = -\lambda Z_t \,\mathrm{d}B_t$.