PDEs and Applications, take-home Midterm Exam

Your name: ________________________

This is an open-book take-home Exam, and you are supposed to complete the exam without getting help from others. Please show your work or explain how you reach your answers.

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1. (32 points). Recall that the Fourier sine series of π is

$$\pi \sim \sum_{n\geq 1, n \text{ odd}} \frac{4}{n}\sin(nx), \text{ for } x \in [0,\pi].$$

Let $f(x) = x$ for $x \in [0,\pi]$ and denote its Fourier cosine series and sine series by

$$F(x) = \sum_{n=0}^{\infty} A_n \cos(nx); \quad G(x) = \sum_{n=1}^{\infty} B_n \sin(nx).$$

(a) Sketch the even & odd periodic extensions of f on $[-2\pi, 2\pi]$ and evaluate $F(\pi)$ and $G(\pi)$.
(b) Determine the coefficients A_n for all $n = 0, 1, 2, \ldots$.
(c) Evaluate the infinite series $\frac{4}{\pi}(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots)$.

2. (34 points) Solve the initial boundary value problem:

$$\begin{cases} \partial_t u = \partial_{xx} u + \dfrac{x}{\pi} + e^t \sin x, & \text{for } 0 < x < \pi, t > 0; \\ u(0,t) = 1, u(\pi,t) = t; \\ u(x,0) = 1 - \dfrac{x}{\pi}, & \text{for } 0 \le x \le \pi. \end{cases}$$

3. (34 points) Find a solution, if exists, to the initial boundary value problem:

$$\begin{cases} \partial_{tt}u = 4\partial_{xx}u, \quad \text{for } 0 < x < \pi, t > 0; \\ \partial_x u(0,t) = 0,\ \partial_x u(\pi,t) = 0; \\ u(x,0) = \cos x,\ \partial_t u(x,0) = 0. \end{cases}$$

If a solution does not exist, explain why.

1. The table presents solutions to $\phi''(x) = -\lambda\phi$ with boundary values, and Fourier series formulas.

BOUNDARY VALUE PROBLEMS

Boundary conditions	$\phi(0)=0$ $\phi(L)=0$	$\frac{d\phi}{dx}(0)=0$ $\frac{d\phi}{dx}(L)=0$	$\phi(-L)=\phi(L)$ $\frac{d\phi}{dx}(-L)=\frac{d\phi}{dx}(L)$
Eigenvalues λ_n	$\left(\frac{n\pi}{L}\right)^2$ $n=1,\ 2,\ 3,\dots$	$\left(\frac{n\pi}{L}\right)^2$ $n=0,\ 1,\ 2,\ 3,\dots$	$\left(\frac{n\pi}{L}\right)^2$ $n=0,\ 1,\ 2,\ 3,\dots$
Eigenfunctions	$\sin\frac{n\pi x}{L}$	$\cos\frac{n\pi x}{L}$	$\sin\frac{n\pi x}{L}$ and $\cos\frac{n\pi x}{L}$
Series	$f(x)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$	$f(x)=\sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L}$	$f(x)=\sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{L}$ $+\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$
Coefficients	$B_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$	$A_0=\frac{1}{L}\int_0^L f(x)\,dx$ $A_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$	$a_0=\frac{1}{2L}\int_{-L}^{L} f(x)\,dx$ $a_n=\frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$ $b_n=\frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$

2. Solutions to some ODE problems:

$y'(t)=ay(t)+f(t);\ y(0)=y_0$	$y(t)=e^{at}y_0+\int_0^t e^{a(t-s)}f(s)ds$
$y''(t)=0;\ y(0)=A; y(L)=B$	$y(t)=A+\frac{B-A}{L}t$