Chapter 3: Fourier series

Fei Lu

Department of Mathematics, Johns Hopkins

Convergence of Fourier series

Convergent, i.e., $f_N \to f$ as $N \to \infty$?

$$f_N(x) := a_0 + \sum_{n=1}^{N} (a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x), \quad x \in [-L, L]$$

- L^2: $\|f_N - f\|_2^2 \to 0$, where $\|f\|_2^2 = \int_{-L}^{L} f(x)^2 dx$
- Point-wise: $f_N(x) \to \widetilde{f}(x)$ for each x
- Uniform: $\max_{x\in[-L,L]} |f_N(x) - \widetilde{f}(x)| \to 0$
  - Weierstrass test: $\sum_{n=1}^{\infty} |a_n| + |b_n| < \infty \Rightarrow$ uniform conv.

Convergence of Fourier series

Convergent, i.e., $f_N \to f$ as $N \to \infty$?

$$f_N(x) := a_0 + \sum_{n=1}^{N} (a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x), \quad x \in [-L, L]$$

- L^2: $\|f_N - f\|_2^2 \to 0$, where $\|f\|_2^2 = \int_{-L}^{L} f(x)^2 dx$
- Point-wise: $f_N(x) \to \widetilde{f}(x)$ for each x
- Uniform: $\max_{x\in[-L,L]} |f_N(x) - \widetilde{f}(x)| \to 0$
  - Weierstrass test: $\sum_{n=1}^{\infty} |a_n| + |b_n| < \infty \Rightarrow$ uniform conv.

A more precise notation (assuming convergence):

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x) = \widetilde{f}(x)$$

The Fourier coefficient of f (by orthogonality)

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx, \quad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x dx, \quad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x dx$$

Outline

Piecewise smooth functions

Definition

A function $f : [a,b] \to \mathbb{R}$ is piecewise continuous(PC) if it is continuous on $[a,b]$ except *jump discontinuous* at finitely many points. If both f and f' are piecewise continuous, then f is called piecewise smooth (PS) or piecewise C^1.

- PC: may have finitely many jump discontinuity, but $f(x^-)$ and $f(x^+)$ exist for all $x \in [a,b]$.
- If it is not defined at a jump discontinuity x, set it to be either $f(x^-)$ or $f(x^+)$.
- f satisfies Dirichlet property it is continuous except finitely many jump discontinuities and $[a,b]$ can be partitioned into a finitely many intervals s.t. f is monotone in each of them. (SV p15).

Are these functions PC or PS? Suppose that $x \in [-\pi, \pi]$:

function	PC	PS
$f_1(x) = \sin(10x)$;	Yes	Yes
$f_2(x) = \lvert x \rvert$;	Yes	Yes
$f_3(x) = x^{1/3}$;	Yes	No
$f_4(x) = \mathbf{1}_{[0,1]}(x)$	Yes	No
$f_5(x) = \begin{cases} -\ln(1-x), & -\pi \le x < 1; \\ 1, & 1 \le x \le \pi \end{cases}$	No	No

Periodic extension. If f is defined on $[-L, L]$, then its periodic extension is

$$\widetilde{f}(x) = \begin{cases} \vdots & \\ f(x+2L), & -3L < x < -L; \\ f(x), & -L < xL; \\ f(x-2L), & L < x < 3L; \\ \vdots & \end{cases}$$

- The end points?
- Example (how to make the extension in a sketch?)

Outline

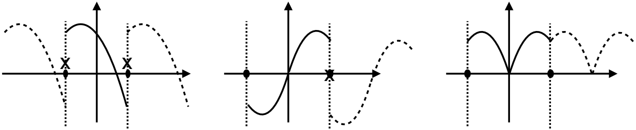
Theorem (Fourier Convergence Theorem)

If f is piecewise smooth on $[-L, L]$, then the Fourier series of f converges to

1. *the periodic extension $\bar{f}$, at where $\bar{f}$ is continuous;*
2. *the average $\frac{1}{2}\left[f(x^-) + f(x^+)\right]$ at where $\bar{f}$ has a jump discontinuity.*

- Set $f(L_+) = f((-L)_+)$ and $f((-L)_{)}) = f(L_-)$ — periodic extension.
- Note: 2 includes 1. Together:

$$\frac{1}{2}\left[f(x^-) + f(x^+)\right] = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$$



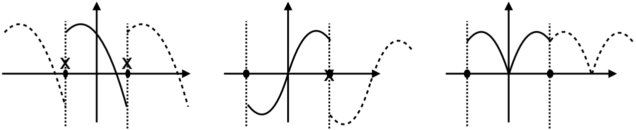
Theorem (Fourier Convergence Theorem)

If f is piecewise smooth on $[-L, L]$, then the Fourier series of f converges to

1. *the periodic extension $\bar{f}$, at where $\bar{f}$ is continuous;*
2. *the average $\frac{1}{2}[f(x^-) + f(x^+)]$ at where $\bar{f}$ has a jump discontinuity.*

- Set $f(L_+) = f((-L)_+)$ and $f((-L)_{)}) = f(L_-)$ — periodic extension.
- Note: 2 includes 1. Together:

$$\frac{1}{2}\left[f(x^-) + f(x^+)\right] = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$$



- Proof: use Dirichlet kernel: $D_N(x) = \frac{1}{2} + \sum_{n=1}^{N}\cos(nx) = \frac{\sin(N+\frac{1}{2})x}{2\sin\frac{x}{2}}$

Notation: f, periodic extension $\bar{f}$, Fourier series (limit) $\widetilde{f}(x)$

Sketch Fourier series Given f. Can we sketch the Fourier series $\widetilde{f} = a_0 + \sum_{n=1}^{\infty}(a_n \cos \frac{n\pi}{L}x + b_n \sin \frac{n\pi}{L}x)$ without knowing a_n, b_n?

Sketch Fourier series Given f. Can we sketch the Fourier series $\widetilde{f} = a_0 + \sum_{n=1}^{\infty}(a_n \cos \frac{n\pi}{L}x + b_n \sin \frac{n\pi}{L}x)$ without knowing a_n, b_n?

Yes! A simple application of the (powerful!) Fourier theorem: 3 steps
1. sketch f on $[-L, L]$
2. Period extension of f to $[-3L, 3L]$
3. skecch $\widetilde{f}$: same as $\bar{f}$ except average at jumps

Example: $f(x) = \begin{cases} 0, & -L \leq x < L/2; \\ 1, & L/2 \leq x \leq L \end{cases}$

Sketch Fourier series Given f. Can we sketch the Fourier series $\widetilde{f} = a_0 + \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x)$ without knowing a_n, b_n?

Yes! A simple application of the (powerful!) Fourier theorem: 3 steps
1. sketch f on $[-L, L]$
2. Period extension of f to $[-3L, 3L]$
3. skecch $\widetilde{f}$: same as $\bar{f}$ except average at jumps

Example: $f(x) = \begin{cases} 0, & -L \le x < L/2; \\ 1, & L/2 \le x \le L \end{cases}$

Q1: what if unbounded domain? $f(x) = \begin{cases} 0, & x < 0; \\ 1, & x \ge 0 \end{cases}$

Sketch Fourier series Given f. Can we sketch the Fourier series $\widetilde{f} = a_0 + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x)$ without knowing a_n, b_n?

Yes! A simple application of the (powerful!) Fourier theorem: 3 steps
1. sketch f on $[-L, L]$
2. Period extension of f to $[-3L, 3L]$
3. skecch $\widetilde{f}$: same as $\bar{f}$ except average at jumps

Example: $f(x) = \begin{cases} 0, & -L \le x < L/2; \\ 1, & L/2 \le x \le L \end{cases}$

Q1: what if unbounded domain? $f(x) = \begin{cases} 0, & x < 0; \\ 1, & x \ge 0 \end{cases}$

Q2: half domain: $f(x)$ defined only for $x \in [0, L]$?

(Recall in HE+BC(Dirichlet/Neumann) + IC: $x \in [0, L]$)

Outline

Fourier sine series

Fourier series of odd functions

When $f(x)$ on $[-L, L]$ is odd: $a_n = ? b_n = ?$

Fourier sine series

Fourier series of odd functions

When $f(x)$ on $[-L,L]$ is odd: $a_n = ? b_n = ?$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x dx = 0, b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x dx = B_n$$

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x$$

Fourier sine series: for $f(x)$ on $[0,L]$

$$f(x) \sim \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}x$$

Sketch Fourier sine series Given f, sketch the Fourier sine series $\widetilde{f} = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x$ without knowing B_n?

1. sketch f on $[0, L]$
2. Odd periodic extension of f to $[-3L, 3L]$: $\bar{f}_{odd}$
3. skecch $\widetilde{f}$: same as $\bar{f}_{odd}$ except average at jumps

Sketch Fourier sine series Given f, sketch the Fourier sine series $\widetilde{f} = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x$ without knowing B_n?

1. sketch f on $[0, L]$
2. Odd periodic extension of f to $[-3L, 3L]$: $\bar{f}_{odd}$
3. skecch $\widetilde{f}$: same as $\bar{f}_{odd}$ except average at jumps

Example: $f(x) = 100,\ x \in [0, L]$?
sketch:

Compute B_n: $B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x dx = \frac{200}{L} \int_0^L \sin \frac{n\pi}{L} x dx = \frac{400}{n\pi} \mathbf{1}_{\{n \text{ odd}\}}$

$$100 = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x = \frac{400}{\pi} \left[\sin \frac{\pi}{L} x + \frac{1}{3} \sin \frac{3\pi}{L} x + \cdots \right], \quad x \in (0, L)$$

- A series representation for π: $\frac{\pi}{4} = \sin \frac{\pi}{L} x + \frac{1}{3} \sin \frac{3\pi}{L} x + \cdots$ for $x \in (0, L)$ at $x = \frac{L}{2} \Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} + \cdots = \sum_{n \text{ odd}} (-1)^n \frac{1}{n}$
- Equality holds on $x \in (0, L)$, but not at $x = 0, x = L$.
- Discontinuity: $\widetilde{f}(0) = 0, \widetilde{f}(L) = 0$, but $f(x) = 100$

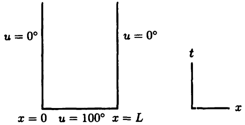
Physical example: HE+BC(Dirichlet) + IC: $x \in [0, L]$

$$\partial_t u = \kappa \partial_{xx} u, \quad \text{with } x \in (0, L), t > 0$$
$$u(0, t) = 0, u(L, t) = 0$$
$$u(x, 0) = f(x) \equiv 100, \quad x \in (0, L)$$

Solution: IF

$f(x)$ "=" $\sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x$,

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi}{L} x) e^{-\lambda_n \kappa t},$$



- "=" does not hold! The series $\widetilde{f} \neq f$ at $x = 0$, $x = L$.
- Physical meaning?
- numerical approximation ↓

Fourier series computation and the Gibbs Phenomenon
In numerical computation, we can only have finitely many terms.

$$f(x) \approx f_N(x) = \sum_{n=1}^{N} B_n \sin \frac{n\pi}{L} x$$

For $f(x) = 100, x \in [0, L]$, what will happen as $N \to \infty$?

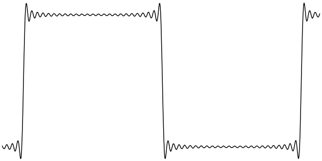
Fourier series computation and the Gibbs Phenomenon

In numerical computation, we can only have finitely many terms.

$$f(x) \approx f_N(x) = \sum_{n=1}^{N} B_n \sin \frac{n\pi}{L} x$$

For $f(x) = 100, x \in [0, L]$, what will happen as $N \to \infty$?

- for $x \in (0, L)$, $f_N(x) \to f(x)$
- $f_N(0) \to \widetilde{f}(0) = 0, f_N(L) \to \widetilde{f}(L) = 0$
- Gibbs phenomenon:
  overshoot(undershoot) at the jump
  discontinuity



$$\lim_{N\to\infty} f_N(0 + \frac{L}{2N}) \approx f(0^+) + [f(0^+) - f(0^-)] * 0.0895$$

Fourier cosine series

Similar to sine series:

- When $f(x)$ on $[-L, L]$ is EVEN: $b_n = 0 \rightarrow$ Fourier cosine series
- For $f(x)$ on $[0, L]$, even extension $\rightarrow$ Fourier cosine series

$$f(x) \sim \sum_{n=0}^{\infty} A_n \cos \frac{n\pi}{L} x$$

- Odd periodic extension to sketch $\widetilde{f}$.

$f(x)$ on $(0, L)$ by both sine and cosine series

Example: $f(x) = \cos \frac{2\pi}{L} x$ on $x \in (0, L)$

Sine series: $f(x) \sim \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x$ with $B_n = \frac{2}{L} \int_0^L \cos \frac{2\pi}{L} x \sin \frac{n\pi}{L} x dx$

Cosine series: $f(x) \sim \sum_{n=0}^{\infty} A_n \cos \frac{n\pi}{L} x$ with $A_n = 0$ if $n \neq 2, A_2 = 1$

$f(x)$ on $(0,L)$ by both sine and cosine series

Example: $f(x) = \cos\frac{2\pi}{L}x$ on $x \in (0,L)$

Sine series: $f(x) \sim \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}x$ with $B_n = \frac{2}{L}\int_0^L \cos\frac{2\pi}{L}x \sin\frac{n\pi}{L}x dx$

Cosine series: $f(x) \sim \sum_{n=0}^{\infty} A_n \cos\frac{n\pi}{L}x$ with $A_n = 0$ if $n \neq 2, A_2 = 1$

Even and odd parts

$$f(x) = f_{even}(x) + f_{odd}(x) = \frac{1}{2}[f(x)+f(-x)] + \frac{1}{2}[f(x)-f(-x)]$$

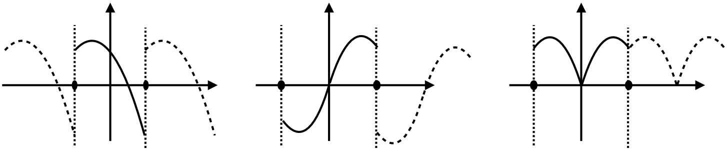
$$\widetilde{f}(x) = \widetilde{f}_{even}(x) + \widetilde{f}_{odd}(x) = \text{Cosine series + Sine Series}$$

Continuous Fourier Series

What condition on f makes its Fourier series continuous ($\in C$)?

Let f be piecewise smooth, and denote its Fourier (sine/cosine) series by $\widetilde{f}$.

- Fourier series $\widetilde{f} \in C$ and $\widetilde{f} = f$ on $[-L, L]$ iff $f(-L) = f(L)$ and $f \in C$;
- Fourier sine series $\widetilde{f} \in C$ and $\widetilde{f} = f$ on $[0, L]$ iff $f(0) = f(L) = 0$ and $f \in C$;
- Fourier cosine series $\widetilde{f} \in C$ and $\widetilde{f} = f$ on $[-L, L]$ iff $f \in C$.



Outline

Section 3.5 Term-by-term Integration

Question: can we exchange the order of the two operations:

$$\frac{d}{dx}\sum_{n=1}^{\infty} \text{“}=\text{”} \sum_{n=1}^{\infty}\frac{d}{dx}$$

Section 3.5 Term-by-term Integration

Question: can we exchange the order of the two operations:

$$\frac{d}{dx}\sum_{n=1}^{\infty} \text{``}=\text{''} \sum_{n=1}^{\infty}\frac{d}{dx}$$

Motivation: when solving PDE by separation of variables

$$\partial_t u = \kappa\partial_{xx}u, \text{ with } x\in(0,L), t>0$$
$$u(0,t)=0, u(L,t)=0$$
$$u(x,0)=f(x), \quad x\in[0,L]$$

We get

$$u(x,t)=\sum_{n=1}^{\infty} B_n \sin(\frac{n\pi}{L}x)e^{-\lambda_n\kappa t},$$

with B_n determined by

$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}x.$

To be addressed:

- Does the series converge?
-
$$\partial_t \sum_{n=1}^{\infty} \overset{?}{=} \kappa\partial_{xx}\sum_{n=1}^{\infty}$$

? $\partial_t \sum_{n=1}^{\infty} = \sum_{n=1}^{\infty}\partial_t$
? $\partial_{xx} \sum_{n=1}^{\infty} = \sum_{n=1}^{\infty}\partial_{xx}$

Example: Consider Fourier Sine series of $f(x) = x$, $x \in [0, L]$:

- Find the Fourier sine series of f
- Try term by term Diff. (TBTD)

Example: Consider Fourier Sine series of $f(x) = x$, $x \in [0, L]$:

- Find the Fourier sine series of f
- Try term by term Diff. (TBTD)

$$x = \sum_{n=1}^{\infty} \frac{2L}{n\pi}(-1)^{n+1} \sin\frac{n\pi x}{L} =: \widetilde{f}, \quad x \in [0, L)$$

TBTD:

$$1 \text{``} = \text{''} \sum_{n=1}^{\infty} 2(-1)^{n+1} \cos\frac{n\pi x}{L},$$

at $x = 0$: the RHS= $2\sum_{n=1}^{\infty}(-1)^{n+1}$ diverges!
$\Rightarrow$ **no TBTD**
Q: $f(x) = x$ is such a “good" function. What’s the problem?

TBTD of Fourier sine series f on $[0, L]$: f odd; f' even

f PC, f' PC $$f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} = \widetilde{f}$$

f' PC, f'' PC $$f'(x) \sim A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} = \widetilde{(f')}$$

TBTD of Fourier sine series f on $[0,L]$: f odd; f' even

f PC, f' PC $\qquad f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} = \widetilde{f}$

f' PC, f'' PC $\qquad f'(x) \sim A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} = \widetilde{(f')}$

If TBTD: $\qquad f'(x) \sim \sum_{n=1}^{\infty} b_n \frac{n\pi}{L} \cos \frac{n\pi x}{L}, \qquad$ (why?)

which requires $\qquad A_0 = 0; A_n = b_n \frac{n\pi}{L}.$

Thus (recall $b_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi}{L} x dx$)

$$0 = A_0 = \frac{1}{L}\int_0^L f'(x)dx = \frac{1}{L}[f(L) - f(0)] \quad \Rightarrow \quad f(L) = f(0)$$

$$A_n = \frac{2}{L}\int_0^L f'(x) \cos \frac{n\pi}{L} x dx =$$

Integration by parts: $\int_a^b u dv = uv\big|_a^b - \int_a^b v du$ if u, v continuous and PS.

TBTD of Fourier sine series f on $[0,L]$

- f PS $\Rightarrow$ its Fourier sine series converges:

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} = \frac{1}{2}\left[f(x^-) + f(x^+)\right]$$

- f' PS, $\Rightarrow$ Fourier series of f' converges
  if in addition, f continuous: $\Rightarrow$

$$f'(x) \sim \frac{1}{L}[f(L) - f(0)] + \sum_{n=1}^{\infty}\left[\frac{n\pi}{L}b_n + \frac{2}{L}[(-1)^n f(L) - f(0)]\right]\cos\frac{n\pi x}{L}$$

Theorem: TBTD if f, f' are PS, f continuous and $f(L) = f(0) = 0$.

TBTD of Fourier cosine series f on $[0, L]$

- f PS $\Rightarrow$ its Fourier sine series converges:

$$f(x) \sim \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{L} = \frac{1}{2}\left[f(x^-) + f(x^+)\right]$$

- f' PS, $\Rightarrow$ Fourier series of f' converges
  if in addition, f continuous: $\Rightarrow$ (check it)

$$f'(x) \sim \sum_{n=1}^{\infty} \frac{n\pi}{L} a_n (-1) \sin \frac{n\pi x}{L}$$

Theorem: TBTD if f, f' are PS, f continuous.

TBTD of Fourier series f on $[-L, L]$

- f PS $\Rightarrow$ its Fourier series converges:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} = \frac{1}{2}\left[f(x^-) + f(x^+)\right]$$

- f' PS, $\Rightarrow$ Fourier series of f' converges
  if in addition, f continuous: $\Rightarrow$

$$f'(x) \sim$$

Theorem: TBTD if f, f' are PS, f continuous and $f(L) = f(-L)$.

Back to PDE:

$\partial_t u = \kappa \partial_{xx} u,$ with $x \in (0, L), t > 0$

$u(0,t) = 0, u(L,t) = 0$

$u(x,0) = f(x), \quad x \in [0, L]$

We get

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi}{L} x) e^{-\lambda_n \kappa t},$$

with B_n determined by

$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x.$

To be addressed:

- Does the series converge?
-
$$\partial_t \sum_{n=1}^{\infty} \overset{?}{=} \kappa \partial_{xx} \sum_{n=1}^{\infty}$$

? $\partial_t \sum_{n=1}^{\infty} = \sum_{n=1}^{\infty} \partial_t$

? $\partial_{xx} \sum_{n=1}^{\infty} = \sum_{n=1}^{\infty} \partial_{xx}$

- for each t: $u(x,t)$ is conti.& $\partial_x u$ PS, BC $\Rightarrow$ TBTD sine series

  $\partial_x u$ is conti.& $\partial_{xx} u$ PS $\Rightarrow$ TBTD cosine series

  $\Rightarrow \quad \partial_{xx} \sum_{n=1}^{\infty} = \sum_{n=1}^{\infty} \partial_{xx}$
- $\partial_t u$ PS $\Rightarrow \quad \partial_t \sum_{n=1}^{\infty} = \sum_{n=1}^{\infty} \partial_t$

Method of eigenfunction expansion (a generalization separation of variables) Seek solution of the form

$$u(x,t) = \sum_{n=0}^{\infty} a_n(t) \cos \frac{n\pi}{L} x + b_n(t) \sin \frac{n\pi}{L} x,$$

- PDE+ BC determines the eigenfunctions to use
- works for equation with source $\partial_t u = \kappa \partial_{xx} u + Q(x,t)$
- solve $a_n(t), b_n(t)$ from the PDE + IC

*3.4.9 Consider the heat equation with a known source $q(x,t)$:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + q(x,t) \quad \text{with} \quad u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0.$$

Assume that $q(x,t)$ (for each $t > 0$) is a piecewise smooth function of x. Also assume that u and $\partial u/\partial x$ are continuous functions of x (for $t > 0$) and $\partial^2 u/\partial x^2$ and $\partial u/\partial t$ are piecewise smooth. Thus,

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin\frac{n\pi x}{L}.$$

What ordinary differential equation does $b_n(t)$ satisfy? Do not solve this differential equation.

Outline

Outline