Chapter 2: Method of Separation of Variables

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Solution to the IBVP in 1D:

$$\partial_t u = \kappa \partial_{xx} u + Q(x,t), \quad \text{with } x \in (0,L), t > 0$$
$$u(x,0) = f(x)$$
$$\text{BC: } u(0,t) = \phi(t), u(L,t) = \psi(t)$$

2D (parabolic equations)? Uniqueness of solution?

Outline

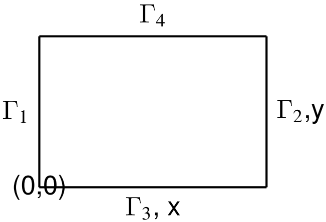
1. Laplace's equation inside a rectangular

Consider the Laplace's equation

$$\nabla^2 u = \partial_{xx} u + \partial_{yy} u = 0, \qquad 0 \le x \le L, 0 \le y \le H$$

$$u|_{\Gamma_1} = g_1(y); \qquad u|_{\Gamma_2} = g_2(y);$$

$$u|_{\Gamma_3} = f_1(x); \qquad u|_{\Gamma_4} = f_2(x);$$



- Equilibrium of the HE
- How to solve it? 1D: $\partial_{xx} u = 0 \Rightarrow u(x) = c_1 x + c_2$.
  Separation of variables?
  Linear and homogeneous: PDE, BC

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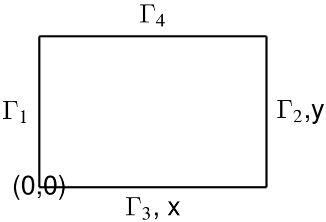

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$$\nabla^2 u_1 = 0, \quad u_1|_{\Gamma_1} = g_1; \quad u_1|_{\Gamma_2} = 0; \quad u_1|_{\Gamma_3} = 0; \quad u_1|_{\Gamma_4} = 0;$$

$$\nabla^2 u_2 = 0, \quad u_2|_{\Gamma_1} = 0; \quad u_2|_{\Gamma_2} = g_2; \quad u_2|_{\Gamma_3} = 0; \quad u_2|_{\Gamma_4} = 0;$$

$$\nabla^2 u_3 = 0, \quad u_3|_{\Gamma_1} = 0; \quad u_3|_{\Gamma_2} = 0; \quad u_3|_{\Gamma_3} = f_1; \quad u_3|_{\Gamma_4} = 0;$$

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Solve u_1 by Separation of Variables:

$$\nabla^2 u_1 = 0,$$

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$$u_1|_{\Gamma_4} = 0;$$

1. Seek solution $u_1(x,y) = h(x)\phi(y)$:

$$\frac{h''(x)}{h} = -\frac{\phi''(y)}{\phi} = \lambda$$

2. Eigenvalue problem:

$$\phi''(y) = -\lambda\phi(y), \quad \phi(0) = \phi(H) = 0$$

$$\phi_n(y) = \sin(\frac{n\pi}{H}y), \quad \lambda_n = (\frac{n\pi}{H})^2, \quad n = 1, 2, \cdots,$$

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3. Solve h:

$$h''(x) = \lambda h(x), \quad h(L) = 0$$

- $\lambda > 0$: $h(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$

$$h_n(x) = a_n \sinh(\sqrt{\lambda_n}(x - L))$$

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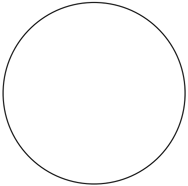
4. Determine a_n

$$u_1(x, y) = \sum_{n=1}^{\infty} a_n \sinh(\sqrt{\lambda_n}(x - L))\phi_n(y).$$

2.5.2 Laplace equation on a disk



$$\nabla^2 u = 0, \quad (x, y) \in Disk$$
$$u|_\Gamma = f$$

$$x = r\cos\theta; \quad y = r\sin\theta$$

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}, \quad 0 < r < a, -\pi < \theta < \pi$$

- BC: $u(a, \pi) = u(a, -\pi); \quad \partial_\theta u(a, \pi) = \partial_\theta u(a, -\pi)$
  $u(a, \theta) = f(\theta); \quad u(0, \theta) = ?$
- Separation of variables?
  linear homo: PDE, BC

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- 1. Seek solution $u(r,\theta) = G(r)\phi(\theta)$:
$$\frac{r(rG')'}{G}(r) = -\frac{\phi''(\theta)}{\phi} = \lambda$$
- 2. EigenvalueP:
$\phi''(\theta) = -\lambda\phi(\theta), \quad \phi(-\pi) = \phi(\pi); \phi'(-\pi) = \phi'(\pi)$
- 3. $G(r)$: $\frac{r(rG')'}{G} = \lambda_n$; BC?

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- 3. $G(r)$: $\frac{r(rG')'}{G} = \lambda_n$; BC? $\quad G(0)$ is bounded
- 4. Solution: $\lambda_n = n^2$, $\phi_n = \cos(n\theta), \sin(n\theta), \quad n = 0, 1, \ldots$
$r^2G'' + rG' - n^2G = 0 \quad \Rightarrow$ (Euler's method:) $G(r) = r^{\pm n}$ or $\ln r$

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty}[A_n r^n \cos(n\theta) + B_n r^n \sin(n\theta)]$$

*2.5.3. Solve Laplace's equation *outside* a circular disk ($r \geq a$) subject to the boundary condition

(a) $u(a, \theta) = \ln 2 + 4 \cos 3\theta$

(b) $u(a, \theta) = f(\theta)$

You may assume that $u(r, \theta)$ remains finite as $r \to \infty$.

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BC: $u(a, \pi) = u(a, -\pi); \quad \partial_\theta u(a, \pi) = \partial_\theta u(a, -\pi)$

$u(a, \theta) = g(\theta); \quad \lim_{r\to\infty} u(r, \theta) < \infty$

Outline

Section 2.5 Laplace's equation: solution examples

Energy method

Section 2.5 Laplace's equation: qualitative properties

Energy method for uniqueness of solution

Reference: SV: page 33.
Suppose that u_1, u_2 are two solutions:

$$u_t = k\,\Delta u \quad \text{in } \Omega; \quad u = 0 \quad \text{on } \partial\Omega; \quad u(x,0) = f(x) \quad \text{for } x \in \Omega.$$

Then, $w = u_1 - u_2$ satisfies

$$w_t = k\,\Delta w \quad \text{in } \Omega; \quad w = 0 \quad \text{on } \partial\Omega; \quad w(x,0) = 0, \quad \text{for } x \in \Omega.$$

Energy function does not grow over time:

$$E(t) = \frac{1}{2}\int_\Omega w(x,t)^2\,dx. \quad \frac{d}{dt}E(t) = \cdots \leq 0$$

Thus, $E(t) \leq E(0) = 0$ and $w(x,t) \equiv 0$ (since w is continuous).

Therefore, $u_1 \equiv u_2$. The solution is unique.

Outline

2.5.3 Qualitative properties

Mean value property $u(P)$ is the average of u in $\partial B_r(P) \subset \mathrm{D}$

- The temperature at any point is the average of the temperature along any circle (inside domain) centered at the point.
- Examples (1) 1D case. (2) on disk: $u(0,\theta) = a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)d\theta$

Theorem (Mean value property, SV Theorem 3.4)

Let $\Delta u = 0$ in $D \subseteq \mathbb{R}^n$. Then for any $B_R(x) \subset D$,

$$u(x) = \frac{n}{\omega_n R^n}\int_{B_R(x)} u(y)dy, \quad u(x) = \frac{1}{\omega_n R^{n-1}}\int_{\partial B_R(x)} u(\sigma)d\sigma,$$

where ω_n is the surface area of the unit sphere.

Proof hint: show that $g'(r) = 0$ for
$g(r) := \frac{1}{\omega_n r^{n-1}}\int_{\partial B_r(x)} u(\sigma)d\sigma = \frac{1}{\omega_n}\int_{\partial B_1(x)} u(x+r\sigma)d\sigma$.

Maximum principle

Theorem (Maximum principle, SV Theorem 3.4& 3.7)

Let $\Delta u = 0$ in a domain (an open connected set) $D \subseteq \mathbb{R}^n$.
If u attains its maximum or minimum at $p \in D$, then u is a constant.
If D is bounded, and u is not a constant, then

$$u(x) < \max_{x \in \partial D} u, \quad u(x) > \min_{x \in \partial D} u$$

- Proof by MVP.
- In non-constant steady state the temperature cannot attain its maximum in the interior:

$$u(P) = \max_{\overline{D}} u \Rightarrow P \in \partial D$$

- Is it true for the three types of boundary conditions?

Wellposedness and uniqueness

Definition: a DE problem is well-posed if there exists a unique solution that *depends continuously* on the nonhomogeneous data.

Theorem

$\nabla^2 u = 0$ on a smooth domain D with $u|_{\partial D} = f(x)$ is well-posed.

"Proof".

- Existence: physical intuition, for compatible f.
  solution on $\mathbb{R}^d$; then constraint on D (Reading: Craig Evans, Partial Differential Equations)
- Continuous dependence on BC
- Uniqueness

□

Solvability condition For $\nabla^2 u = 0$, we have (Divergence theorem)

$$\oint \nabla u \cdot \mathbf{n} dS = \int \nabla^2 u dV = 0$$

- If Neumann BC $-K_0 \nabla u \cdot \mathbf{n} = g$, then we must have $\oint g dS = 0$
- The net heat flow through the boundary must be zero for a steady state (with no source).

Summary of Chp 2: Separation of variables

- Heat equation + BC + IC; Laplace +BC
- Linear + homogeneous $\Rightarrow$ Principle of superposition
- Separation of variables

Solution of HE + BC+ IC: $\partial_t u = \kappa \partial_{xx} u$, $u(x,0) = f(x)$

Dirichlet $x \in (0,L)$	$f(x) \quad = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi}{L}x)$
$u(0,t) = u(L,t) = 0$	$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi}{L}x) e^{-\lambda_n \kappa t}$

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Mixed $x \in (-L,L)$ $\quad f(x) = A_0 + \sum_{n=1}^{\infty} (A_n \cos \frac{n\pi}{L}x + B_n \sin \frac{n\pi}{L}x)$

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$u(0,t) = u(L,t)$

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$u(0,t) = u(L,t)$

Question

- When $f(x)$ can be written as series? Convergence?
- If the series of f converge, will $u(x,t)$ series converge?
- If converge, will u continuous/differentiable/satisfy HE?