

Chapter 10: Fourier transform

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10.2 Heat equation on an infinite domain

10.3 Fourier transform pair

10.4 Fourier transform and heat equation

10.5: Fourier sine and cosine transforms

10.6 Examples using Fourier transform

Outline

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10.2 Heat equation on an infinite domain

Previous: PDE on bounded region (1D rod or 2D rectangular/circle).

What if **infinite regions**? What is the “boundary” condition?

- ▶ Influence at boundary is negligible
- ▶ Bounded total energy: $\int |u(x, t)| dx < \infty$
- ▶ Equilibrium?

1D infinite rod:

$$\partial_t u = \partial_{xx} u, \quad x \in (-\infty, \infty)$$

$$u(x, 0) = f(x), \quad x \in (-\infty, \infty)$$

$$\text{BC: } u(x, t) \rightarrow 0 \text{ as } |x| \rightarrow \infty$$

10.2 Heat equation on an infinite domain

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Eigenvalue problem

$$\phi'' = -\lambda \phi, \quad x \in (-\infty, \infty)$$

$$\phi(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty?$$

Linear + Homogeneous

→ separation of variables

▶ seek $u(x, t) = h(t)\phi(x) \rightarrow$

▶ superposition?

$$u(x, t) = \sum_{n=1}^{\infty} h_n(0) e^{-\lambda_n t} \phi_n(x)$$

$$\lambda > 0 :$$

$$\lambda = 0 :$$

$$\lambda < 0 :$$

10.2 Heat equation on an infinite domain

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$\lambda = 0 :$

$\lambda < 0 :$

ϕ bounded

10.2 Heat equation on an infinite domain

We have eigenfunction $\phi_\lambda(x) = c_1 \sin \sqrt{\lambda}x + c_2 \cos \sqrt{\lambda}x$ for every $\lambda \geq 0$. Uncountably many!

Fourier series:

$$u(x, t) = \sum_{n=1}^{\infty} h_n(0) e^{-\lambda_n t} \phi_n(x)$$

$h_n(0)$ from IC

Orthogonality
Completeness

Now: ?? $u(x, t) = \sum_{\lambda \geq 0} h_\lambda(0) e^{-\lambda t} \phi_\lambda(x)$??

$$\begin{aligned} &= \int_0^\infty [c_1(\lambda) \sin \sqrt{\lambda}x + c_2(\lambda) \cos \sqrt{\lambda}x] e^{-\lambda t} d\lambda \\ &= \int_0^\infty [A(\omega) e^{i\omega x} + B(\omega) e^{-i\omega x}] e^{-\omega^2 t} d\omega \end{aligned}$$

$A(\omega)$ and $B(\omega)$ from IC?

$\phi_\lambda(x)$: Orthogonality, Completeness?

Complex exponential

$$u(x, t) = \int_{-\infty}^{\infty} e^{-\omega^2 t} c(\omega) e^{-i\omega x} d\omega$$

$$f(x) = \int_{-\infty}^{\infty} c(\omega) e^{-i\omega x} d\omega$$

$c(\omega)$ from f ?

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10.3 Fourier transform pair

To determine $c(\omega)$: (from Fourier series to Fourier transform)

Fourier series on $[-L, L]$:

$$(\lambda_n, \phi_n) = \left(\left(\frac{n\pi}{L} \right)^2, e^{-i \frac{n\pi}{L} x} \right)$$

$$\frac{f(x_+) + f(x_-)}{2} = \sum_{n=-\infty}^{\infty} c_n e^{-i \frac{n\pi}{L} x}$$

$$c_n = \frac{1}{2L} \int_{-L}^L f(y) e^{i \frac{n\pi}{L} y} dy$$

10.3 Fourier transform pair

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Fourier Transform on $(-\infty, \infty)$:

$$(\lambda, \phi_\lambda) = \left(\lambda, e^{\mp i \sqrt{\lambda} x} \right)$$

$$\frac{f(x_+) + f(x_-)}{2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} c(\omega) e^{-i\omega x} d\omega$$

$$c(\omega) = \int_{-L}^L f(y) e^{i\omega y} dy$$

$$\frac{f(x_+) + f(x_-)}{2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(y) e^{i\omega y} dy \right) e^{-i\omega x} d\omega$$

Fourier Transform

$$\hat{f}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx = \mathcal{F}[f](\omega)$$

Inverse Fourier Transform

$$\frac{f(x_+) + f(x_-)}{2} = \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-i\omega x} d\omega$$

- ▶ the coefficient $1/(2\pi)$ is a conventional notion

Example: Fourier transforms of Gaussian

$$\mathcal{F}^{-1}[e^{-\alpha\omega^2}](x) = \sqrt{\frac{\pi}{\alpha}} e^{-\frac{x^2}{4\alpha}}$$

$$\mathcal{F}[e^{-\beta x^2}](\omega) = \sqrt{\frac{1}{4\pi\beta}} e^{-\frac{\omega^2}{4\beta}}$$

Proof:

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10.4 Fourier transform and heat equation

$$\partial_t u = \partial_{xx} u, \quad x \in (-\infty, \infty)$$

$$u(x, 0) = f(x), \quad x \in (-\infty, \infty)$$

$$\text{BC: } u(x, t) \rightarrow 0 \text{ as } |x| \rightarrow \infty$$

Outline: 1. Apply Fourier transform

$$\mathcal{F}[\partial_t u] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \partial_t u(x, t) e^{i\omega x} dx = \partial_t \hat{u}$$

$$\mathcal{F}[\partial_{xx} u] = -\omega^2 \hat{u}$$

$$\mathcal{F}[\partial_t u] = \mathcal{F}[\partial_{xx} u] \Leftrightarrow \partial_t \hat{u} = -\omega^2 \hat{u}$$

2. Solve for $\hat{u}$

3. Inverse FT for $u(x, t)$

Gaussian kernel:

$$u(x, t) = \mathcal{F}^{-1}[\widehat{u}(\omega, t)](x) = \dots = \int_{-\infty}^{\infty} f(y) \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x-y|^2}{4t}} dy$$

- ▶ Influence function: $u(x, t)$ depends on the entire IC
- ▶ $t \rightarrow 0$: $\delta(x - y)$

Fundamental solution of the heat equation: $u(x, 0) = \delta(x)$

$$u(x, t) =$$

Convolution Theorem:

$$\mathcal{F}(f * g) = 2\pi \widehat{f} \widehat{g}; \quad \frac{1}{2\pi} f * g = \mathcal{F}^{-1}[\widehat{f} \widehat{g}]$$

- ▶ Proof: by definition
- ▶ In the solution to (HE): (with $G(t, x) = \frac{1}{2\pi} G_t(x)$)

$$u(x, t) = \frac{1}{2\pi} [f * G_t](x)$$

$$\widehat{u}(\omega, t) = \widehat{f}(\omega) \widehat{G}_t(\omega)$$

Convolution Theorem:

$$\mathcal{F}(f * g) = 2\pi \widehat{f} \widehat{g}; \quad \frac{1}{2\pi} f * g = \mathcal{F}^{-1}[\widehat{f} \widehat{g}]$$

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$$u(x, t) = \frac{1}{2\pi} [f * G_t](x)$$

$$\widehat{u}(\omega, t) = \widehat{f}(\omega) \widehat{G}_t(\omega)$$

Parseval's Equality

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} g(x) \overline{g(x)} dx = \int_{-\infty}^{\infty} \widehat{g}(\omega) \overline{\widehat{g}(\omega)} d\omega$$

- ▶ The Fourier series version?
- ▶ Proof: $\frac{1}{2\pi} f * g(0) = \mathcal{F}^{-1}[\widehat{f} \widehat{g}](0)$ and use that $\widehat{\widehat{f}} = \widehat{f(-x)}$.

Summary: using Fourier transform to solve heat equation

$$\partial_t u = \partial_{xx} u, \quad x \in (-\infty, \infty)$$

$$u(x, 0) = f(x), \quad x \in (-\infty, \infty)$$

$$\text{BC: } u(x, t) \rightarrow 0, \partial_x u(x, t) \rightarrow 0 \text{ as } |x| \rightarrow \infty$$

Apply Fourier transform

$$\mathcal{F}[\partial_t u] = \partial_t \hat{u};$$

$$\mathcal{F}[\partial_{xx} u] = -\omega^2 \hat{u};$$

$$\mathcal{F}[\partial_t u] = \mathcal{F}[\partial_{xx} u] \Leftrightarrow \partial_t \hat{u} = -\omega^2 \hat{u}$$

$$\Rightarrow \hat{u}(\omega, t) = \hat{f}(\omega) e^{-\omega^2 t}$$

$$\mathcal{F}^{-1}[e^{-\alpha \omega^2}](x) = \sqrt{\frac{\pi}{\alpha}} e^{-\frac{x^2}{4\alpha}}$$

$$\text{Denote } G(x, t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x-y|^2}{4t}}$$

$$u(x, t) = \mathcal{F}^{-1}[\hat{u}(\cdot, t)](x) = 2\pi \mathcal{F}^{-1}[\hat{f} \hat{G}(\cdot, t)](x) = [f * G(\cdot, t)](x)$$

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Motivating example

$$\partial_t u = \partial_{xx} u, \quad x \in (0, \infty)$$

$$u(x, 0) = f(x), \quad x \in (0, \infty)$$

$$u(x, 0) = 0;$$

Separation of variables:

- ▶ $u(x, t) \sim h(t)\phi(x)$
- ▶ eigenvalue problem

$$\phi'' = -\lambda\phi, \quad x \in (0, \infty)$$

$$\phi(0) = 0;$$

$$S[f](\omega) = \hat{f}^s(\omega) = \frac{2}{\pi} \int_0^{\infty} f(x) \sin(\omega x) dx$$

Fourier Sine Transform

$$S^{-1}[\hat{f}^s](x) = f(x) = \int_0^{\infty} \hat{f}^s(\omega) \sin(\omega x) d\omega$$

Fourier Cosine Transform

$$C[f](\omega) = \hat{f}^c(\omega) = \frac{2}{\pi} \int_0^{\infty} f(x) \cos(\omega x) dx$$

$$C^{-1}[\hat{f}^c](x) = f(x) = \int_0^{\infty} \hat{f}^c(\omega) \cos(\omega x) d\omega$$

► Transform of derivatives

$$S[\partial_x f](\omega) = -\omega C[f]; \quad C[\partial_x f](\omega) = \frac{2}{\pi} f(0) - \omega S[f](\omega);$$

► applications: solve the (HE)

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10.6 Application examples

1. Wave equation

$$\partial_{tt}u = c^2 \partial_{xx}u, \quad x \in (-\infty, \infty)$$

$$u(x, 0) = f(x), \partial_t u(x, 0) = 0;$$

To solve it, apply Fourier transform

$$\partial_{tt}\hat{u} = -c^2 w^2 \hat{u}(t, w), \quad w \in (-\infty, \infty)$$

$$\hat{u}(0, w) = \hat{f}(w), \partial_t \hat{u}(0, w) = 0;$$

$$u(x, t) = \frac{1}{2}[f(x - ct) + f(x + ct)]$$

2. Laplace equation on a semi-infinite strip

$$\partial_{xx}u + \partial_{yy}u = 0, \quad x \in (0, L), y > 0$$

$$u(0, y) = g_1(y); \quad u(L, y) = g_2(y);$$

$$u(x, 0) = f(x); \textit{bounded}$$

- ▶ 1. homogenize BC (u_2)

$$u = u_1 + u_2$$

- ▶ 2. Solve them:

FT

Eigenfunction expansion

2. Laplace equation on a semi-infinite strip

$$\begin{aligned}\partial_{xx}u + \partial_{yy}u &= 0, \quad x \in (0, L), y > 0 \\ u(0, y) &= g_1(y); \quad u(L, y) = g_2(y); \\ u(x, 0) &= f(x); \textit{bounded}\end{aligned}$$

► 1. homogenize BC (u_2)

$$u = u_1 + u_2$$

► 2. Solve them:

FT

Eigenfunction expansion

$$\begin{aligned}\partial_{xx}u_1 + \partial_{yy}u_1 &= 0, \quad x \in (0, L) \\ u_1(0, y) &= g_1(y); \quad u_1(L, y) = g_2(y); \\ u(x, 0) &= 0; \textit{bounded}\end{aligned}$$

FT in y

$$\begin{aligned}\partial_{xx}u_2 + \partial_{yy}u_2 &= 0, \quad x \in (0, L) \\ u_2(0, y) &= 0; \quad u_2(L, y) = 0; \\ u_2(x, 0) &= f(x); \textit{bounded}\end{aligned}$$

Fourier series in x .

3-4. Laplace equation on half-plane or a quarter-plane
Reading.

5. HE on the plane

$$\partial_t u = \partial_{xx} u + \partial_{yy} u, \quad x, y \in (-\infty, \infty)$$

$$u(x, y, 0) = f(x, y)$$

- ▶ the BCs by default
- ▶ Double Fourier transform
- ▶ solution

$$u(x, y, t) = \int_{-\infty}^{\infty} A(w_1, w_2) e^{-i(w_1 x + w_2 y)} e^{-(w_1^2 + w_2^2)t} dw_1 dw_2$$

Homework 10:

Exe 10.6.11c: Eq. on the first quarter plane with BC

$$\partial_t u = \partial_{xx} u + \partial_{yy} u, \quad x > 0, y > 0$$

$$u(x, y, 0) = f(x, y)$$

$$u(0, y, t) = 0, \partial_y u(x, 0, t) = 0$$

Hint: make extensions of f to the entire plane. The extensions (even/odd/mixed) should satisfy the BC.