Chapter 9: Green's function

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Section: 9.2 1D heat equation
Section 9.3 Green's function and ODE-BVP

Exe 9.2.1: (a) Solve the IBVP using Green's formula instead of TBTD;
(b) consider BC $u(0,t) = A(t)$, $u(L,t) = B(t)$.

$$\partial_t u = \kappa \partial_{xx} u + Q(x,t),$$
$$u(0,t) = 0 = u(L,t)$$
$$u(x,0) = g(x)$$

Outline:
1. Operator $L = \kappa\partial_{xx}$ and BC. $\to$ Sturm-Liuville Theorem $\{(\lambda_n, \phi_n)\}$
2. MEE: $u(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x)$, where $\{\phi_n\}$ is a complete basis.
Solve $a_m(t)$ from DE:

$$\partial_t \langle u, \phi_m\rangle = \langle \kappa\partial_{xx} u, \phi_m\rangle + \langle Q, \phi_m\rangle,$$

where Green's formula: $\int_a^b (uLv - vLu) = p(uv' - u'v)\big|_a^b \Rightarrow$

$$\langle \partial_{xx} u, \phi_m\rangle = \langle u, \phi_m''\rangle + (u\phi_m' - \phi_m \partial_x u)\big|_a^b$$

Homogeneous HE:

$$\partial_t u = \kappa \partial_{xx} u, \quad x \in (0, L)$$
$$u(0,t) = 0 = u(L,t)$$
$$u(x,0) = g(x)$$

Solution: $a_n = \frac{2}{L}\int_0^L g(x_0)\sin\frac{n\pi x_0}{L}dx_0$

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n \kappa t}\sin\frac{n\pi x}{L}$$
$$= \int_0^L g(x_0) G(x,t;x_0,0) dx_0$$

Non-homogeneous HE:

$$\partial_t u = \kappa \partial_{xx} u + Q(x,t), \quad x \in (0, L)$$
$$u(0,t) = 0 = u(L,t)$$
$$u(x,0) = g(x)$$

Solution: $a_n(t) = \ldots$

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) e^{-\lambda_n \kappa t}\sin\frac{n\pi x}{L}$$
$$= \int_0^L g(x_0) G(x,t;x_0,0) dx_0$$
$$+ \int_0^L \int_0^t g(x_0) G(x,t-s;x_0,0) ds dx_0$$

Green's function for HE:

$$G(x,t;x_0,0) := \sum_{n=1}^{\infty} \frac{2}{L} e^{-\lambda_n \kappa t}\sin\frac{n\pi x}{L}\sin\frac{n\pi x_0}{L}$$

1. Causality principle: $u(x,t)$ depends on the past, not the future.
2. Depend on $t-s$

9.3 Green's function and BVP-ODE

A. Deviation using 1D steady-state of HE

Suppose $Q(x,t) = Q(x)$.
$u_E(x) = \lim_{t\to\infty} u(x,t)$.
Then, $0 = \kappa\partial_{xx}u_E + Q(x)$.

$$\partial_{xx}u_E = f(x) = -Q(x)/\kappa,$$
$$u_E(0) = 0 = u_E(L)$$

$$\begin{aligned} u_E(x) &= \lim_{t\to\infty} u(x,t) \\ &= \lim_{t\to\infty} \int_0^L g(y)G(x,t;y,0)dy \\ &+ \int_0^L \int_0^t [-\kappa f(y)]G(x,t;y,s)dyds \\ &= \int_0^L f(y)G(x,y)dy \end{aligned}$$

$$G(x,y) = \frac{2}{L}\sum_{n=1}^{\infty} \lambda_n^{-1}\phi_n(x)\phi_n(y)$$

with $\lambda_n = (\frac{n\pi}{L})^2$, $\phi_n(x) = \sin(\frac{n\pi}{L}x)$

Green's function for the DE
$\partial_{xx}u = f$; $u(0) = u(L) = 0$:

$$u(x) = \int_0^L f(y)G(x,y)dy$$

for all continuous f.

B: Derive Green's function by *variation of parameters* for the DE.

C. Green's function for general self-adjoint operator

Derivation by Method of Eigen-function Expansion:

$$Lu = f \text{ with homo-BC.}$$

Suppose that L has eigenpairs $\{(\lambda_n, \phi_n)\}_{n=1}^{\infty}$

$$\lambda_n > 0,\ \forall n \geq 1$$

Show that the Green's function is

$$G(x,y) = \sum_{n=1}^{\infty} \frac{1}{\langle \phi_n, \phi_n \rangle} \lambda_n^{-1} \phi_n(x) \phi_n(y)$$

or $G = \sum_{n=1}^{\infty} \frac{1}{\langle \phi_n, \phi_n \rangle} \lambda_n^{-1} \phi_n \otimes \phi_n$

- From $u(x) = \langle f, G(x,\cdot) \rangle = \int f(y) G(x,y) dy$
- Example: $\partial_{xx} u = f$; $u(0) = u(L) = 0$
- Q: Explicit function expression of G?

D. Dirac Delta function and Green's function

Dirac Delta function $\delta(x-x_0)$

Definition: a generalized function

$$\delta(x-x_0) = \begin{cases} 0 & x \neq x_0 \\ \infty & x = x_0 \end{cases}$$

in the sense that $\forall f$ continuous,

$$f(x_0) = \int f(x)\delta(x-x_0)dx.$$

It is NOT a "function", but an "action".

- $\int \delta(x-x_0)dx = 1$.
- $\delta(x-x_0) = \delta(x_0-x)$.
- derivative of Heaviside function $\delta(x-x_0) = H'(x-x_0)$

Green's function $G(x, x_0)$

$Lu = f$ with homo-BCs.

$$u(x) = \int f(y)G(x,y)dy.$$

Formally, if $f(x) = \delta(x-x_0)$, then

$$u(x) = \int \delta(x-x_0)G(x,y)dy = G(x,x_0)$$

$$LG(x,x_0) = \delta(x-x_0)$$

Q: δ is not a function,
what does the equation mean?

DE for Green's function

$$LG(x, x_0) = \delta(x - x_0), \quad x, x_0 \in (a, b)$$

- weak form: for all $v \in C_b^2$,

$$\langle LG(\cdot, x_0), v \rangle = \langle \delta(x - x_0), v \rangle$$

- Green's formula: $\langle LG(\cdot, x_0), v \rangle = [Gv' - \partial_x Gv]\big|_a^b - \langle G(\cdot, x_0), Lv \rangle \leftarrow$
G is continuous, $\partial_x G(x, x_0)$ exists and $\partial_x G(x, x_0)\big|_{x_0^-}^{x_0^+} = 1$.

Example: find the Green's function by solving

$$\frac{d^2}{dx^2} G(x, x_0) = \delta(x - x_0),$$
$$G(0, x_0) = 0 = G(L, x_0)$$

$$G(x, x_0) = \begin{cases} -\frac{L - x_0}{L} x, & x < x_0; \\ -\frac{x_0}{L}(L - x), & x > x_0. \end{cases}$$

E. Non-homogeneous BC

Exe: Solve $\partial_{xx}u = f$; $u(0) = \alpha, u(L) = \beta$ using the Green's function.

- By Green's formula

$$\int_0^L [u(x)\partial_{xx}G(x,x_0) - G(x,x_0)\partial_{xx}u]dx = u\partial_x G(x,x_0) - \partial_x u G(x,x_0)\Big|_0^L$$

- Applying the Eq. and BC:

$$\int_0^L [u(x)\delta(x-x_0) - G(x,x_0)f(x)]dx = \beta\partial_x G(L,x_0) - \alpha\partial_x G(0,x_0)$$

$$u(x_0) - \int_0^L f(x)G(x,x_0)dx = g(x_0)$$

- Recall $\partial_x G(L,x_0) = \frac{x_0}{L}$, $\partial_x G(0,x_0) = \frac{x_0-L}{L}$.

$$u(x_0) = \int_0^L f(x)G(x,x_0)dx + \beta\frac{x_0}{L} - \alpha\frac{x_0-L}{L}, \quad \forall x_0 \in (0,L)$$

9.4 Fredholm Alternative and Generalized Green's functions

$$Lu = f \text{ with homo-BC.}$$

Suppose that L has eigenpairs $\{(\lambda_n, \phi_n)\}_{n=1}^{\infty}$

$$\lambda_n > 0, \; \forall n \geq m, \quad \lambda_1 = \ldots = \lambda_m = 0$$

The generalized Green's function is a solution to

$$LG(x, x_0) = \delta(x - x_0) - \sum_{n=1}^{m} c_n \frac{1}{\langle \phi_n, \phi_n \rangle} \phi_n(x) \phi_n(x_0)$$

$$G(x, y) = \sum_{n=m+1}^{\infty} \frac{1}{\langle \phi_n, \phi_n \rangle} \lambda_n^{-1} \phi_n(x) \phi_n(y) + \sum_{n=1}^{m} c_n \frac{1}{\langle \phi_n, \phi_n \rangle} \phi_n(x) \phi_n(y)$$

or $G = \sum_{n=m+1}^{\infty} \frac{1}{\langle \phi_n, \phi_n \rangle} \lambda_n^{-1} \phi_n \otimes \phi_n + \sum_{n=1}^{m} c_n \frac{1}{\langle \phi_n, \phi_n \rangle} \phi_n \otimes \phi_n$