Chapter 7: High dimensional PDEs

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Outline

Section 7.2: Separation of time variable d-D

We have solved 1D equations. Will everything work for 2D& 3D?

1D: $x \in [0, L]$

HE $\partial_t u = \partial_{xx} u$

WE $\partial_{tt} u = \partial_{xx} u$

BC & IC

$u(t, x) = h(t)\phi(x) :\downarrow$

$$\phi''(x) = -\lambda\phi; \quad \text{BC}$$

$$h'(t) \text{ or } h''(t) = -\lambda_n h, \quad \text{IC}$$

$\Rightarrow (\lambda_n, \phi_n), h_n$

$u(t, x) = \sum_{n=1}^{\infty} h_n(t)\phi_n(x)$

Sturm-Liouville:

$(p\phi')' + q\phi = -\lambda\sigma\phi$

Method of eigenfunction expansion

$\partial_{xx} u \rightarrow ?$

2D: $\nabla^2 u = (\partial_{xx} + \partial_{yy})u$

3D: $\nabla^2 u = (\partial_{xx} + \partial_{yy} + \partial_{zz})u$

d-D: $\nabla^2 u = \sum_{i=1}^{d} \partial_{x_i x_i} u$

Second order PDEs

HE $\partial_t u = \nabla^2 u$

WE $\partial_{tt} u = \nabla^2 u$

BC: $[A(t)u + B(t)\nabla u \cdot \mathbf{n}]\big|_{\partial\Omega} = 0$

IC: $u(x, 0)$, $\partial_t u(x, 0)$

Separate time variable

Consider (WE)

$$\partial_{tt}u = c^2(\partial_{xx}u + \partial_{yy}u)$$
$$u(x,y,0) = \alpha(x,y); \partial_t u(x,y,0) = \beta(x,y);$$
$$\text{BC:} \quad [A(t)u + B(t)\nabla u \cdot \mathbf{n}]\big|_{\partial\Omega} = 0;$$

Seek $u(x,y,t) = h(t)\phi(x,y)$:
from the equation, we have

$$h''(t)\phi = c^2(\partial_{xx}\phi + \partial_{yy}\phi)h$$
$$\frac{h''}{c^2 h} = \frac{\partial_{xx}\phi + \partial_{yy}\phi}{\phi} = -\lambda$$

(HE): Similar

How do we solve
$\partial_{xx}\phi + \partial_{yy}\phi = -\lambda\phi$?
Recall Laplace equ. in Chp 2.

Outline

7.3 Vibrating Rectangular Membrane

BC: $x \in [0, L]; y \in [0, H]$,

$$u(0, y, t) = 0 = u(L, y, t);$$
$$u(x, 0, t) = 0 = u(x, H, t);$$

Simulation video: 5:50

Eigenvalue problem

$$\partial_{xx}\phi + \partial_{yy}\phi = -\lambda\phi;$$
$$\phi(0, y) = 0 = \phi(L, y);$$
$$\phi(x, 0) = 0 = \phi(x, H); \nearrow$$

Separation of variables (again)

$\phi(x, y) = f(x)g(y)$

$$\frac{f''}{f} = -\lambda - \frac{g''}{g} = -\mu$$

$\lambda_{n,m} = (\frac{n\pi}{L})^2 + (\frac{m\pi}{H})^2$,

$\phi_{n,m}(x, y) = \sin(\frac{n\pi}{L}x)\sin(\frac{m\pi}{H}y)$

Back to the equation,

$$\partial_{tt} u = c^2(\partial_{xx} u + \partial_{yy} u)$$
$$\text{IC: } u(x,y,0) = \alpha(x,y); \partial_t u(x,y,0) = \beta(x,y);$$
$$\text{BC: } \ldots$$

SoV: $u(x,t) = h_{n,m}(t)\phi_{n,m}(x,y)$
To determine $h_{n,m}(t)$:

$$h''(t) = -\lambda_{n,m} c^2 h,$$
$$\text{IC } h(0), h'(0) = ?$$

$h_{n,m}(t) = a_{n,m}\cos(c\sqrt{\lambda_{n,m}}t) + b_{n,m}\sin(c\sqrt{\lambda_{n,m}}t)$

Back to the equation,

$$\partial_{tt}u = c^2(\partial_{xx}u + \partial_{yy}u)$$
$$\text{IC: } u(x,y,0) = \alpha(x,y); \partial_t u(x,y,0) = \beta(x,y);$$
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To determine $h_{n,m}(t)$:

$$h''(t) = -\lambda_{n,m}c^2 h,$$
$$\text{IC } h(0), h'(0) = ?$$

$h_{n,m}(t) = a_{n,m}\cos(c\sqrt{\lambda_{n,m}}t) + b_{n,m}\sin(c\sqrt{\lambda_{n,m}}t)$ General solution

(MEE): $u(x,t) = \sum_{n,m=1}^{\infty} h_{n,m}(t)\phi_{n,m}(x,y)$
Orthogonal? Complete?

Outline

1D Sturm-Liouville Theorem

$$(p(x)\phi')' + q(x)\phi = -\lambda\sigma\phi$$
$$\beta_1\phi(a) + \beta_2\phi'(a) = 0;$$
$$\beta_3\phi(b) + \beta_4\phi'(b) = 0;$$

regular **SLEP** $\{(\lambda_n, \phi_n)\}$ s.t.

1-2 $\{\lambda_n \uparrow \infty\}_{n=1}^{\infty}$ strictly

3 ϕ_n is unique to λ_n;
ϕ_n has $n-1$ zeros

4 $\{\phi_n\}_{n=1}^{\infty}$ is complete.

5 $\{\phi_n\}_{n=1}^{\infty}$ are orthogonal

6 Rayleigh quotient $\lambda_n = -\frac{\langle L\phi_n, \phi_n\rangle}{\langle \phi_n, \phi_n\rangle_\sigma}$;

2D& 3D Helmholtz Eq.

$$\nabla^2\phi = -\lambda\sigma\phi$$
$$a\phi + b\nabla\phi\cdot\mathbf{n}\,|_{\partial\Omega} = 0;$$

$$\nabla\cdot(p\nabla\phi) + q\phi = -\lambda\sigma\phi$$

1-2 $\{\lambda_n \nearrow \infty\}_{n=1}^{\infty}$, may repeat

3 $\lambda_n = \lambda_{n+1}$: multiple eigen-fun

4 $\{\phi_n\}_{n=1}^{\infty}$ is complete.

5 $\{\phi_n\}_{n=1}^{\infty}$ are orthogonal
(Gram-schmidt orthogonalization)

6 Rayleigh quotient $\lambda_n = -\frac{\langle L\phi_n, \phi_n\rangle}{\langle \phi_n, \phi_n\rangle_\sigma}$;

Outline

7.5: Green's formula

Orthogonality of eigenfunctions:

$$L\phi = \nabla^2\phi = -\lambda\phi \qquad \text{in } \Omega$$
$$a\phi + b\nabla\phi\cdot\mathbf{n} = 0 \qquad \text{on } \partial\Omega$$

Green's formula ↗

$$\int_\Omega (uLv - vLu)dx = \oint_{\partial\Omega}(u\nabla v - v\nabla u)\cdot\mathbf{n}dS$$

Self-adjoint operator

$L = L^*$: $\langle u, Lv\rangle = \langle L^*u, v\rangle$

$\int_\Omega (uLv - vLu)dx = 0$

L is self-adjoint on the function spaces

- $C_0^2 = \{w : w \in C^2(\Omega), w|_{\partial\Omega=0}\}$
- $C_I^2 = \{w : w \in C^2(\Omega), \nabla w\cdot\mathbf{n}|_{\partial\Omega=0}\}$

Proof of the Green's formula:
Divergence theorem

$$\int_\Omega \nabla\mathbf{A}dx = \oint \mathbf{A}\cdot\mathbf{n}dS$$

Green's formula: application

Orthogonality of eigenfunctions L self-adjoint
Let $\{\lambda_i, \phi_i\}$ and $\{\lambda_j, \phi_j\}$ be two eigen-pairs, $\lambda_i \neq \lambda_j$. Then $\langle \phi_i, \phi_j \rangle = 0$.
Proof:

Question: How about ϕ_i and ϕ_j for $\lambda_i = \lambda_j$? Gram-Schmidt

Outline

7.6: Rayleigh Quotient and Laplace's Equation

Let L be self-adjoint. If $L\phi = -\lambda\phi$, then $\lambda = -\dfrac{\langle L\phi, \phi\rangle}{\langle \phi, \phi\rangle}$

Application 1. $L = \nabla^2$. Suppose:

$$\nabla^2\phi = -\lambda\phi \qquad \text{in } \Omega$$
$$\phi = 0 \qquad \text{on } \partial\Omega,$$

show that $\lambda > 0$.
($\Rightarrow$ all eigenvalues > 0)

If insulated BC: $\nabla u(t,\cdot)\cdot\mathbf{n} = 0$?

7.6: Rayleigh Quotient and Laplace's Equation

Let L be self-adjoint. If $L\phi = -\lambda\phi$, then $\lambda = -\dfrac{\langle L\phi, \phi\rangle}{\langle \phi, \phi\rangle}$

Application 2. Heat equation:

$$\int_\Omega u\nabla^2 u dx = -\int_\Omega |\nabla u|^2 dx + \oint u\nabla u \cdot \mathbf{n} dS$$

$$\begin{aligned} \partial_t u &= \nabla^2 u \text{ in } \Omega \\ u(t,\cdot) &= 0 \text{ on } \partial\Omega, \\ u(0,x) &= f(x) \end{aligned}$$

1. show that $\lim_{t\to\infty} u(t,x) = 0$, $\forall x$.
Hint: solution & $\lambda > 0$

2. If insulated BC: $\nabla u(t,\cdot)\cdot\mathbf{n} = 0$?

Outline

7.7: Vibrating Circular Membrane

Consider Ω = disk with radius a

$$\partial_{tt} u = \nabla^2 u \text{ in } \Omega$$
$$u(a, \theta, t) = 0, \theta \in [-\phi, \phi]$$
$$u(r, \theta, 0) = \alpha(r, \theta), \partial_t u(r, \theta, t) = \beta(r, \theta)$$

Simulation video

Outline:

1. Separation of variables

$$u(r, \theta, t) = \phi(r, \theta) h(t)$$

$$h''(t)\phi = c^2(\nabla^2 \phi) h$$
$$\frac{h''}{c^2 h} = \frac{\nabla^2 \phi}{\phi} = -\lambda$$

2. Solve the eigenvalue problem
$\nabla^2 \phi = -\lambda \phi$ & BC
   - complete and orthogonal
3. Eigenfunction expansion

$$u(r, \theta, t) = \sum_{n=1}^{\infty} h_n(t) \phi_n(r, \theta)$$

7.7: Vibrating Circular Membrane

2. Solve the eigenvalue problem

$$\nabla^2\phi = -\lambda\phi, \quad r \in (0,a), \theta \in (-\pi,\pi)$$
$$\phi(a,\theta) = 0, \quad \theta \in (-\pi,\pi)$$

Seek $\phi(r,\theta) = f(r)g(\theta)$:

- Eigenvalue Problem A:

$$g''(\theta) = -\mu g, \quad \theta \in (-\pi,\pi)$$
$$g(-\pi) = g(\pi), g'(-\pi) = g'(\pi)$$

- Eigenvalue Problem B:

$$r(rf')' + (\lambda r^2 - \mu)f = 0, \quad r \in (0,a)$$

$f(a) = 0, f(0)$ bounded

7.7: Vibrating Circular Membrane

2. Solve the eigenvalue problem

$$\nabla^2\phi = -\lambda\phi, \quad r \in (0,a), \theta \in (-\pi,\pi)$$
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- Eigenvalue Problem B:

$$r(rf')' + (\lambda r^2 - \mu)f = 0, \quad r \in (0,a)$$
$f(a) = 0, f(0)$ bounded

Eigenvalue Problem A:

$$\mu_m = m^2, m \geq 0;$$
$g_m(\theta) = \sin m\theta$, or $\cos m\theta$.

Eigenvalue Problem B: SLEP?

$$(rf')' - \frac{m^2}{r}f = -\lambda rf, \quad r \in (0,a)$$

Still valid: singular SLEP.

Bessel functions

Eigenvalue Problem B:

$r(rf')' + (\lambda r^2 - \mu)f = 0, \quad r \in (0, a)$
$f(a) = 0, f(0)$ bounded

$r^2 f'' + rf' + (\lambda r^2 - m^2)f = 0$
A change of variables (will $\lambda = 0$?)
$z = \sqrt{\lambda} r \rightarrow$

$$z^2 f'' + zf' + (z^2 - m^2)f = 0$$

(Bessel's DE of order m)

- no exact closed form solution
- can have good estimates

Bessel functions

Eigenvalue Problem B:

$$r(rf')' + (\lambda r^2 - \mu)f = 0, \quad r \in (0, a)$$

$f(a) = 0, f(0)$ bounded

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(Bessel's DE of order m)

- no exact closed form solution
- can have good estimates

$$f'' + z^{-1} f' + (1 - m^2 z^{-2})f = 0$$

non-singular / singular;

- non-singular $z \geq \epsilon > 0$: OK
- singular point $z = 0$

Near $z = 0$:

- $m \neq 0$: $z^2 << m^2$
  f bounded $\rightarrow z^2 f << m^2 f$
  f', f'' can be large: keep them

  $$z^2 f'' + zf' - m^2 f \approx 0$$

  Solution $f(z) \approx z^m$ or z^{-m}
- $m = 0$: $f(z) \approx 1$ or $\ln z$

Bessel functions

First kind of order m (well-behaved)

$$J_m(z) = \begin{cases} 1, & m = 0 \\ \frac{1}{2^m m!} z^m, & m > 0 \end{cases}$$

2nd kind of order m (singular)

$$Y_m(z) = \begin{cases} \frac{2}{\pi} \ln z, & m = 0 \\ -\frac{2^m (m-1)!}{\pi} z^{-m}, & m > 0 \end{cases}$$

Eigenvalue Problem B:

$$f(r) = c_1 J_m(\sqrt{\lambda} r) + c_2 Y_m(\sqrt{\lambda} r)$$

- $f(0)$ bounded $\to c_2 = 0$.
- $f(a) = 0 \to$ determine $\lambda_{n,m} = (z_{n,m}/a)^2$: zeros of $J_m(\sqrt{\lambda} r)$.

Eigenfunctions: $J_m(\sqrt{\lambda_{n,m}} r)$; complete and orthogonal $\sigma(r) = r$

back to the Vibrating Circular Membrane

Consider $\Omega =$ disk with radius a

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$$u(r, \theta, 0) = \alpha(r, \theta), \partial_t u(r, \theta, t) = \beta(r, \theta)$$

1. Separation of variables $u(r, \theta, t) = \phi(r, \theta) h(t)$
2. Solve the eigenvalue problem: $\nabla^2 \phi = -\lambda \phi$ + BC
   - complete and orthogonal
3. Eigenfunction expansion

$$u(r, \theta, t) = \sum_{n,m=1}^{\infty} h_{n,m}(t) \phi_{n,m}(r, \theta)$$

$$J_m(\sqrt{\lambda_{n,m}} r) \begin{pmatrix} \cos(m\theta) \\ \sin(m\theta) \end{pmatrix} \begin{pmatrix} \cos(c\sqrt{\lambda_{n,m}} t) \\ \sin(c\sqrt{\lambda_{n,m}} t) \end{pmatrix}$$