Chapter 5: Sturm-Liouville Eigenvalue Problem

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Solution to the IBVP?

$$c(x)\rho(x)\partial_t u = K_0 \partial_{xx} u + Q(x,t), \quad \text{with } x \in (0,L), t \geq 0$$

IC: $u(x,0) = f(x)$,

BC: $u(0,t) = \phi(t), u(L,t) = \psi(t)$

Outline

Section 5.5: Sturm-Liouville Eigenvalue Problem

Regular SLEP:

$$(p(x)\phi')' + q(x)\phi = -\lambda\sigma\phi$$
$$\beta_1\phi(a) + \beta_2\phi'(a) = 0;$$
$$\beta_3\phi(b) + \beta_4\phi'(b) = 0;$$

$$p', q, \sigma \in C[a,b],$$
$$p(x) > 0, \sigma(x) > 0, \forall x \in [a,b]$$
$$\beta_1^2 + \beta_2^2 > 0, \beta_3^2 + \beta_4^2 > 0,$$

Theorem (Sturm-Liouville Theorems)

A regular SLEP has eigenvalues and eigenfunctions $\{(\lambda_n, \phi_n)\}$ *s.t.*

1-2 $\{\lambda_n\}_{n=1}^{\infty}$ *are real and strictly increasing to* ∞

3 ϕ_n *is the unique (up to a *factor) solution to* λ_n*;* ϕ_n *has* $n-1$ *zeros*

4 $\{\phi_n\}_{n=1}^{\infty}$ *is complete. That is, any piecewise smooth* f *can be represented by a generalized Fourier series*
$f(x) \sim \sum_{n=1}^{\infty} a_n\phi_n(x) = \frac{1}{2}[f(x_-) + f(x_+)]$

5 $\{\phi_n\}_{n=1}^{\infty}$ *are orthogonal:* $\langle\phi_n, \phi_m\rangle_\sigma = 0$ *if* $n \neq m$*;* $\langle\phi_n, \phi_n\rangle_\sigma > 0$

6 *Rayleigh quotient* $\lambda_n = -\frac{\langle L\phi_n, \phi_n\rangle}{\langle\phi_n, \phi_n\rangle_\sigma}$*;*

$\langle f, g\rangle := \int_a^b f(x)g(x)dx; \quad \langle f, g\rangle_\sigma := \int_a^b f(x)g(x)\sigma(x)dx$. TO PROVE: 5136.

5. orthogonality: $\langle \phi_n, \phi_m \rangle_\sigma = 0$ if $n \neq m$; $\langle \phi_n, \phi_n \rangle_\sigma > 0$
Denote: $L\phi = (p\phi')' + q\phi = -\lambda\sigma\phi$. A linear operator.
Recall: $\langle \phi_n, \phi_m \rangle_\sigma = \int_a^b \sigma(x)\phi_n(x)\phi_m(x)dx$.
Proof: Let (λ_n, ϕ_n) and (λ_m, ϕ_m) solve the SLEP with $\lambda_n \neq \lambda_m$. Then

$$L\phi_n = -\lambda_n\sigma\phi_n \qquad \int_a^b L\phi_n(x)\phi_m(x)dx = -\lambda_n \langle \phi_n, \phi_m \rangle_\sigma$$

$$L\phi_m = -\lambda_m\sigma\phi_m \qquad \int_a^b L\phi_m(x)\phi_n(x)dx = -\lambda_m \langle \phi_n, \phi_m \rangle_\sigma$$

$$\int_a^b [\phi_m L\phi_n - \phi_n L\phi_m]dx = -(\lambda_n - \lambda_m)\langle \phi_n, \phi_m \rangle_\sigma.$$

Thus, if LHS =0, then we obtain $\langle \phi_n, \phi_m \rangle_\sigma = 0$ bc. $\lambda_n \neq \lambda_m$.

Green's formula: $\int_a^b [uLv - vLu]dx = p(uv' - vu') \,|_a^b$.

$+$ regular BC: $\beta_1\phi(a) + \beta_2\phi'(a) = 0; \beta_3\phi(b) + \beta_4\phi'(b) = 0;$

$$\beta_1^2 + \beta_2^2 > 0, \beta_3^2 + \beta_4^2 > 0,$$

$\Rightarrow \mathbf{LHS = 0}$

Green's formula: for any $u, v \in C^2$, $L\phi := (p\phi')' + q\phi$,

$$\int_a^b [uLv - vLu]dx = p(uv' - vu') \,\Big|_a^b \,.$$

Proof:
Lagrange identity:

$$uLv - vLu = [p(uv' - vu')]'$$

Green's formula: for any $u, v \in C^2$, $L\phi := (p\phi')' + q\phi$,

$$\int_a^b [uLv - vLu]dx = p(uv' - vu') \mid_a^b .$$

Self-adjoint operator: with regular BC: $\int_a^b [uLv - vLu]dx = 0$.
Equivalently, $\int_a^b uLvdx = \int_a^b vLudx$ for any u, v with BC.

$$\langle L^* u, v\rangle = \langle u, Lv\rangle = \langle v, Lu\rangle; \quad L^* = L$$

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Remark: matrix $A \in \mathbb{R}^{d\times d}$. Then, for any $u, v \in \mathbb{R}^d$,

$$\langle A^* u, v \rangle = \langle u, Av \rangle = \langle v, Au \rangle. \quad A^* = A^\top = A$$

1. All eigenvalues are real.

Proof: take adjoint to both sides of the equation.
(Either without using orthogonality or with it.)

3. Uniqueness: for each λ_n, there is a unique normal eigenfunction.
Proof: by Lagrange identity.
(The role of the regular BC.)

Section 5.5: Quiz

1. The eigenvalues for the SLEP must be non-negative.
2. The Green's formula $\int_a^b [uLv - vLu]dx = p(uv' - vu') \mid_a^b$ can be applied to the BC $\phi(-a) = \phi(a)$, $\phi'(-a) = \phi'(a)$ when the interval is $(-a, a)$.

Outline

Section 5.6 Rayleigh quotient

Regular SLEP:

$$L\phi = (p(x)\phi')' + q(x)\phi = -\lambda\sigma\phi \qquad p', q, \sigma \in C[a,b],$$
$$\beta_1\phi(a) + \beta_2\phi'(a) = 0; \qquad p(x) > 0, \sigma(x) > 0, \forall x \in [a,b]$$
$$\beta_3\phi(b) + \beta_4\phi'(b) = 0; \qquad \beta_1^2 + \beta_2^2 > 0, \beta_3^2 + \beta_4^2 > 0,$$

A regular SLEP has eigenvalues and eigenfunctions $\{(\lambda_n, \phi_n)\}$ s.t.

6 **Rayleigh quotient** $\lambda_n = -\frac{\langle L\phi_n, \phi_n\rangle}{\langle \phi_n, \phi_n\rangle_\sigma}$.

$\langle f, g\rangle := \int_a^b f(x)g(x)dx$; $\quad \langle f, g\rangle_\sigma := \int_a^b f(x)g(x)\sigma(x)dx$.

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$\langle f, g\rangle := \int_a^b f(x)g(x)dx; \quad \langle f, g\rangle_\sigma := \int_a^b f(x)g(x)\sigma(x)dx.$

Remark: Note that $\langle L\phi_n, \phi_n\rangle = p\phi_n\phi_n'|_a^b + \int_a^b [q\phi_n^2 - p(\phi_n')^2]dx$

- **Nonnegative eigenvalues** if $p\phi_n\phi_n'|_a^b \leq 0, q \leq 0$
- We use only p. NO need of its derivative
- We use only ϕ_n, ϕ_n'. No need of ϕ''

$\rightarrow$ weak/distribution solution in Sobolev spaces

Minimax principle

$$\lambda_1 = \min_{u \in H_0} -\frac{\langle Lu, u \rangle}{\langle u, u \rangle_\sigma}$$

$H_0 = \{u \in C^1 : u \text{ satisfies BC}, \langle u, u \rangle_\sigma > 0\}$

- ϕ_1 is a minimizer
- We can estimate λ_1 by trial functions u (Example next)
- Similarly, for larger eigenvalues, recursively,

$$\lambda_n = \min_{u \in H_{n-1}} -\frac{\langle Lu, u \rangle}{\langle u, u \rangle_\sigma} = \max_{S_n \subset H_0} \min_{u \in S_n^\perp} -\frac{\langle Lu, u \rangle}{\langle u, u \rangle_\sigma}$$

with $H_{n-1} = \text{span}\{\phi_1, \cdots, \phi_{n-1}\}^\perp$ and $S_n \subset H_0$ with dim=n .

Proof: $\{\phi_n\}$ complete and orthogonal. Let $u = \sum_{i=1}^{\infty} a_i \phi_i$.
Use TBTD. $\langle Lu, u \rangle = ...$

$$RQ[u] = -\frac{\langle Lu, u \rangle}{\langle u, u \rangle_\sigma} = ...$$

Estimate bounds for λ_1 by the Minimax principle:

$$\lambda_1 = \min_{u \in H_0} -\frac{\langle Lu, u\rangle}{\langle u, u\rangle_\sigma}$$

$H_0 = \{u \in C^1 : u$ satisfies BC, $\langle u, u\rangle_\sigma > 0\}$;
$\langle Lu, u\rangle = pu'u|_a^b + \int_a^b [qu^2 - p(u')^2]dx$.

Example: $\phi'' = -\lambda\phi, \quad \phi(0) = \phi(1) = 0$. Can we choose trial function to get a close estimate of $\lambda_1 = \pi^2$?

- satisfies BC
- close to ϕ_1 (unknown). It has no zero in $[0, 1]$

1. $u(x) = x\mathbf{1}_{[0,0.5]} + (1-x)\mathbf{1}_{[0.5,1]}$.
   $-\langle Lu, u\rangle = 1$, $\langle u, u\rangle_\sigma = 1/12$. $\Rightarrow \lambda_1 \le 12$.
2. $u(x) = x(1-x)$. $\cdots\cdots\cdots \Rightarrow \lambda_1 \le 10$.

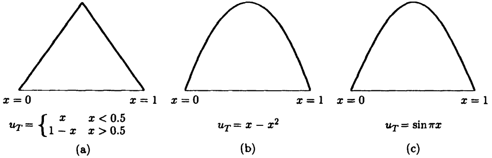

Figure 5.6.1 Trial functions: continuous, satisfy the boundary conditions, and are of one sign.

Outline

Section 5.7 Variation of non-uniform string

PDE: $\rho(x)\partial_{tt}u = T_0\partial_{xx}u,\quad$ with $x \in (0, L), t \geq 0$

IC: $u(x, 0) = f(x), \partial_t u(x, 0) = g(x)$

BC: $u(0, t) = 0, u(L, t) = 0$

Separation of variables (or Eigenfunction expansion):

$$u(x,t) = \sum_{n=0}^{\infty}[a_n \cos(\sqrt{\lambda_n}t) + b_n \sin(\sqrt{(}\lambda_n t))]\phi_n(x).$$

(Recall: when $\rho \equiv \rho_0$, we have $\lambda_n^2 = \frac{n\pi T_0}{\rho_0 L}$ and $\phi_n(x) = \sin\frac{n\pi x}{L}$)

Question: range of λ_1 for a non-uniform string?

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Question: range of λ_1 for a non-uniform string?

Assume: $0 < \rho_{min} \leq \rho(x) \leq \rho_{max}$.

The SLEP: $T_0\phi'' = -\lambda\rho(x)\phi$ with BC: $\phi(0) = 0 = \phi(L)$.

$$? \quad \leq \lambda_1 = \min_{\phi \in H_0} -\frac{\langle T_0\phi'', \phi\rangle}{\langle \phi, \phi\rangle_\rho} \leq \quad ?$$

Outline

Section 5.8: BC of the 3rd kind

Consider either heat equation or wave equation

	$\partial_t u = \kappa \partial_{xx} u$	$\partial_{tt} u = c^2 \partial_{xx} u$
IC	$u(x,0) = 0;$	$u(x,0) = f(x), \partial_t u(x,0) = g(x)$
BC	$u(0,t) = 0; \partial_x u(L,t) = -hu(L,t).$	
$h > 0$	cooling	a restoring force
$h = 0$	insulated	zero speed
$h < 0$	heating	a destabilizing force

Physical:
To solve them, use separation of variables

- 1 separate variable
- 2 solve the eigenvalue problem
  Q1: Sturm-Liouville theorem?
- 3 solve time ODEs with λ_n and ICs

Solve the eigenvalue problem: (estimate λ_n, find ϕ_n)

$$\phi''(x) = -\lambda\phi; \quad \phi(0) = 0; \phi'(L) + h\phi(L) = 0.$$

1. $\lambda > 0$:

   $$\phi(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x).$$

   Apply BC to estimate c_1, c_2 and find possible λ.
2. $\lambda = 0$ (exe)

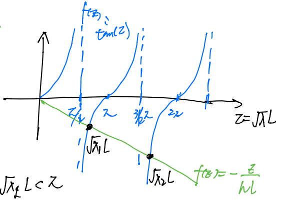
   $$\phi(x) = c_1 + c_2 x.$$

3. $\lambda < 0$ (exe)

   $$\phi(x) = c_1 e^{-\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}.$$

Case 1, $\lambda > 0$: Estimate λ_n from $-\frac{1}{hL}z = \tan z$ with $z = \sqrt{\lambda L}$:

$h>0$

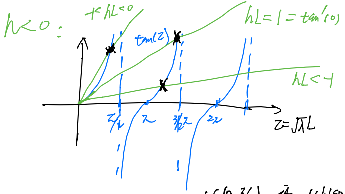


$\frac{\pi}{2} < \sqrt{\lambda_1} L < \pi$

$\frac{3}{2}\pi < \sqrt{\lambda_2} L < 2\pi$

$(n-\frac{1}{2})\pi < \sqrt{\lambda_n} L < n\pi \Rightarrow \lambda_n \sim \left((n-\frac{1}{2})\frac{\pi}{L}\right)^2$

$h<0$:



$\sqrt{\lambda_1}$ in the three cases $\begin{cases} \in (0, \frac{\pi}{2}), & \text{if } -1 < hL < 0, \\ \in (\pi, \frac{3}{2}\pi), & \text{if } hL \le -1. \end{cases}$

$\lambda_n \sim \left((n+\frac{1}{2})\frac{\pi}{L}\right)^2$ if $hL < -1$

Summary:

summary of cases ① ② ③

eigenfunction for $\begin{cases} \varphi''(x)+\lambda\varphi=0 \\ \varphi(0)=0,\ \varphi'(L)+h\varphi(L)=0 \end{cases}$

	$\lambda>0$	$\lambda=0$	$\lambda<0$	$\lambda_n \nearrow \infty$
$h>0$	$\sin\sqrt{\lambda}x$	No	No	$\lambda_n \sim \left((n-\frac{1}{2})\frac{\pi}{L}\right)^2$
$h=0$	$\sin\sqrt{\lambda}x$	No	No	
$h<0$, $-1<hL<0$	"	No	No	$\lambda_n \sim \left((n-\frac{1}{2})\frac{\pi}{L}\right)^2$
$hL=-1$	"	x	No	$\lambda_n \sim \left((n+\frac{1}{2})\frac{\pi}{L}\right)^2$
$hL<-1$	"	No	$\sinh(\sqrt{-\lambda_1}x)$	$\lambda_n \sim \left((n+\frac{1}{2})\frac{\pi}{L}\right)^2$

- The smallest eigenvalue (λ_1) depends on h.

Exe: complete the other cases.