Chapter 1: Heat equation, Diffusion

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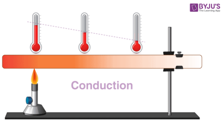
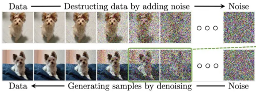


Diffusion



$$\partial_t u = \partial_{xx} u + Q(x,t)$$

Section 1.2: Conduction of heat
Section 1.3: Initial boundary conditions
Section 1.4: Equilibrium
Section 1.5 Heat equation in 2D and 3D

Outline

Section 1.2: Conduction of heat

How does heat “move”?

Section 1.2: Conduction of heat
How does heat "move"?

Consider the **thermal energy** in an ideal 1D rod:

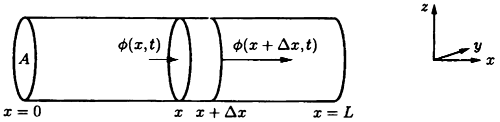


Figure 1.2.1 One-dimensional rod with heat energy flowing into and out of a thin slice.

- $\phi(x,t)$ = Heat flux (energy per unit time flowing per unit surface area)
- Total energy in a slice $(x, x+\Delta x)$: $\int_x^{x+\Delta x} e(z,t)dz$

$$\underbrace{e(x,t)}_{\text{Energy density}} = \underbrace{u(x,t)}_{\text{Temperature}} \; c(x)\rho(x) \tag{1}$$

  - $c(x)$ = heat capacity energy per unit mass to raise the temperature 1 unit
  - $\rho(x)$ = mass density
- Conservation law: total energy = flow in $-$ out + generated

$\Rightarrow$ study heat conduction via temperature evolution

Section 1.2: Conduction of heat

How does heat “move”?

Conservation of energy (rate of change in-time in $(x, x+\Delta x)$)

$$\text{total energy } = \text{flow in} - \text{out} + \text{generated} \tag{2}$$

$$\frac{d}{dt}\int_x^{x+\Delta x} e(z,t)dz = \phi(x,t) - \phi(x+\Delta x, t) + \int_x^{x+\Delta x} Q(z,t)dz \tag{3}$$

$\Delta x \to 0$, (Recall FTC: $\frac{1}{\Delta x}\int_x^{x+\Delta x} f(y)dy \to f(x)$ for $f \in C([x, x+b])$)

$$\partial_t e = -\partial_x \phi + Q(x,t) \tag{4}$$

Section 1.2: Conduction of heat

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Recall $e(x,t) = u(x,t)c(x)\rho(x)$, and
Fourier's law: $\phi = -K_0 \partial_x u$ [i.e., the heat flow depends linearly on $\partial_x u$]

$$\boxed{\partial_t u \, c(x)\rho(x) = K_0 \partial_{xx} u + Q(x,t)}$$

If uniform rod: $c(x) \equiv c_0$, $\rho(x) \equiv \rho_0 \to \kappa = \frac{K_0}{c_0 \rho_0}$; no source $Q = 0$; then

Heat Equation: $\boxed{\partial_t u = \kappa \partial_{xx} u}$

Heat/Diffusion Equation: $\boxed{\partial_t u = \kappa \partial_{xx} u}$

Diffusion: spread of heat/chemical/...

- diffusion of heat
  - $u(x,t)$ temperature; κ thermal diffusivity
  - Conservation of energy; Fourier's law

Heat/Diffusion Equation: $\boxed{\partial_t u = \kappa \partial_{xx} u}$

Diffusion: spread of heat/chemical/...

- diffusion of heat
  - $u(x,t)$ temperature; κ thermal diffusivity
  - Conservation of energy; Fourier's law
- diffusion of chemicals (perfumes or pullutants)
  - $u(x,t)$ concentration density; κ chemical diffusivity;
  - Conservation of mass; Fick's law

Diffusion

Reading: Diffusion (wiki); Brownian motion (Wiki)

Outline

Initial and boundary conditions

Heat Equation: $$\boxed{\partial_t u = \kappa \partial_{xx} u}$$

Any solution to it?

Initial and boundary conditions

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Any solution to it? Infinitely many

Constant	$u_0(x,t) \equiv 1$
Linear in x	$u_1(x,t) = x$
Gaussian density	$u_2(x,t) = \dfrac{1}{2\pi\sqrt{t}} e^{-\frac{x^2}{2t}}$

Initial and boundary conditions

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Linear in x	$u_1(x,t) = x$
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- Any linear combination of those (principle of superposition)

$$u(x,t) = c_0 u_0 + c_1 u_1 + c_2 u_2,$$

  for any constant $c_0, c_1, c_2 \in \mathbb{R}$

To determine a solution, need to specify initial boundary conditions

Initial and boundary conditions

Heat Equation: $\partial_t u = \kappa \partial_{xx} u$

How many initial boundary conditions do we need?

Recall ODE: for $t \geq t_0$

- $y'(t) = f(y,t)$, with $y(t_0) = y_0$;
- $\frac{d^k}{dt^k} y = f(y, y^{(1)}, \ldots, y^{(k-1)}, t)$, with $y(t_0), y'(t_0), \ldots, y^{(k)}(t_0)$;
  (Exe: what condition do we need on the k-ICs? How about IBVP?)

Initial and boundary conditions

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Domain of equation

$$t \geq t_0, x \in D, \text{ with } D = \mathbb{R}^d \text{ or } D = (0, L).$$

Initial condition for HE

$$u(x, t_0) = f(x), \text{ for all } x \in D$$

- when $D = \mathbb{R}^d$: IC determines the solution
- when $D = (0, L)$: need Boundary conditions

IVBP

Heat equation on a bounded interval

$$\partial_t u = \kappa \partial_{xx} u, \quad \text{with } x \in (0, L), t \geq 0$$

Initial condition $u(x,0) = f(x), x \in [0, L]$

Boundary conditions boundaries $x = 0, x = L$

Dirichlet	$u(0,t) = \phi(t), u(L,t) = \psi(t)$	prescribed tempt.
Neumann	$\partial_x u(0,t) = \phi(t), \partial_x u(L,t) = \psi(t)$	heat flux
	$\partial_x u(0,t) = \partial_x u(L,t) = 0$	insulated bd
Robin	$a_1 \partial_x u(0,t) + a_0 u(0,t) = \phi(t)$	Newton's law of cooling
mixed	$b_1 \partial_x u(L,t) + b_0 u(L,t) = \psi(t)$	

Exe: read Section 1.3.

Outline

Equilibrium Temperature Distribution

Q: What is Equilibrium and why?

The steady state; a state of rest or balance due to equal action of opposing forces.

Recall ODE: $y' = f(y)$, how to find its equilibrium? Stability?

Reading for fun: Equilibrium of dynamics systems

1. **Prescribed Temperature** Consider the IBVP

$$\begin{aligned}
\partial_t u = & \ \kappa \partial_{xx} u, \quad \text{with } x \in (0, L), t \geq 0 \\
u(x, 0) = & f(x) \\
u(0, t) = & \phi(t), u(L, t) = \psi(t)
\end{aligned}$$

At equilibrium: $\partial_t \widetilde{u} = 0$, $\widetilde{u}(0, t) = \phi(t) \equiv T_1, \widetilde{u}(L, t) = \psi(t) \equiv T_2$:

$$\begin{aligned}
\partial_{xx} \widetilde{u} &= 0, \\
\widetilde{u}(0) &\equiv T_1, \widetilde{u}(L) \equiv T_2
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A 2nd order ODE! (What about the IC?)

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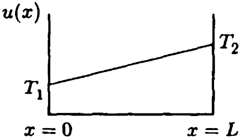
\end{aligned}$$

A 2nd order ODE! (What about the IC?)

Solution:

$$\widetilde{u}(x) = T_1 + \frac{T_2 - T_1}{L} x.$$



Approach to equilibrium

$$\lim_{t \to \infty} u(t, x) = \widetilde{u}(x).$$

2. Insulated BC

$$\partial_t u = \kappa \partial_{xx} u, \quad \text{with } x \in (0,L), t \geq 0$$
$$u(x,0) = f(x)$$
$$\partial_x u(0,t) = 0, \partial_x u(L,t) = 0$$

At equilibrium:

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At equilibrium:

$$\partial_{xx} \widetilde{u} = 0,$$
$$\partial_x \widetilde{u}(0) = \partial_x \widetilde{u}(L) = 0$$

Solution:

$$\widetilde{u}(x) = C$$

Figure?

Arbitrary C?

3. Mixed BC

$$\partial_t u = \kappa \partial_{xx} u, \quad \text{with } x \in (0, L), t \geq 0$$
$$u(x, 0) = f(x)$$
$$u(0, t) = T, u(L, t) + \partial_x u(L, t) = 0$$

At equilibrium: $\partial_{xx}\widetilde{u} = 0, \widetilde{u}(0) = T, \widetilde{u}(L) + \partial_x \widetilde{u}(L) = 0$.

Solution:

$$\widetilde{u}(x) = T(1 - \frac{x}{1 + L})$$

Figure?

Exe1.4.11

1.4.11. Suppose $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x$, $u(x,0) = f(x)$, $\frac{\partial u}{\partial x}(0,t) = \beta$, $\frac{\partial u}{\partial x}(L,t) = 7$.

(a) Calculate the total thermal energy in the one-dimensional rod (as a function of time).

(b) From part (a), determine a value of β for which an equilibrium exists. For this value of β, determine $\lim_{t\to\infty} u(x,t)$.

Outline

Heat equation in 2D and 3D:

$$c\rho\partial_t u = \nabla\cdot\left(K_0\nabla u\right) + Q, \quad x \in D \subset \mathbb{R}^d$$

Sketch of derivation:

1. Energy Conservation Law:

Rate of change of energy = Flow in − out + Generated (per unit time)

$$\frac{d}{dt}\iiint\limits_R c\rho u dV = -\oint\!\!\!\oint\limits_{\partial R} \mathbf{A}\cdot\mathbf{n}dS + \iiint\limits_R Q dV,$$

where $\mathbf{A} = (A_x, A_y, A_z)$ is a vector-valued function.

2. **Gauss' Divergence theorem:**

$$\iiint\limits_V (\nabla\cdot\mathbf{A})dV = \oint\!\!\!\oint\limits_{\partial V} \mathbf{A}\cdot\mathbf{n}dS$$

3. **Fick's law:** $\mathbf{A} = -K_0\nabla u$.

FTC: taking $R = B_\epsilon(x)$ with $\epsilon \to 0$.

Heat equation:

$$c\rho\partial_t u = \nabla\cdot\left(K_0 \nabla u\right) + Q, \quad x \in D \subset \mathbb{R}^d$$

IBVP: with initial and boundary conditions

Laplace's equation (potential equation)

$$\nabla^2 u = 0.$$

Poisson's equation

$$\nabla^2 u = f \quad (\text{ e.g.,} \quad f = \frac{Q}{c\rho K_0}).$$

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Polar and cylindrical coordinates

$$x = r\cos\theta; \quad y = r\sin\theta, \quad z = z$$

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\partial^2 u}{\partial z^2}$$

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Spherical coordinates

$$x = \rho\sin\phi\cos\theta; \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi$$

$$\nabla^2 u = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right) + \frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2}$$

1.2.9. Consider a thin one-dimensional rod without sources of thermal energy whose lateral surface area is not insulated.

(a) Assume that the heat energy flowing out of the lateral sides per unit surface area per unit time is $w(x,t)$. Derive the partial differential equation for the temperature $u(x,t)$.

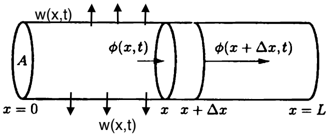
(b) Assume that $w(x,t)$ is proportional to the temperature difference between the rod $u(x,t)$ and a known outside temperature $\gamma(x,t)$. Derive that

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) - \frac{P}{A}[u(x,t) - \gamma(x,t)]h(x), \qquad (1.2.15)$$

where $h(x)$ is a positive x-dependent proportionality, P is the lateral perimeter, and A is the cross-sectional area.

(c) Compare (1.2.15) to the equation for a one-dimensional rod whose lateral surfaces are insulated, but with heat sources.

(d) Specialize (1.2.15) to a rod of circular cross section with constant thermal properties and 0° outside temperature.



Part(a): total energy $=$ flow in-out $+$ generated $(Q = 0)$

$$\frac{d}{dt}\int_x^{x+\Delta x} e(z,t)dzA = A[\phi(x,t) - \phi(x+\Delta x,t)] - P\int_x^{x+\Delta x} w(z,t)dz$$

A problem is **well-posed** when it possesses the following properties:

1. **Existence**: There exists at least one solution;
2. **Uniqueness**: There exists at most one solution;
3. **Continuity**: The solution depends continuously on the data (IC/BC/parameters).

Examples of PDEs:

1. Heat equation or diffusion: $u_t - \kappa \Delta u = 0$;
2. Laplace's or potential equation: $\Delta u = 0$;
3. Wave equation:
$$u_{tt} - c^2 \Delta u = 0;$$
4. Fisher's equation:
$$u_t - \kappa \Delta u = cu(\mu - u);$$
5. Burgers equation:
$$u_t + cuu_x = 0.$$