Final Exam Practice

PDEs and Applications, Spring 2025

Your name: ________________________

This is an open-note Exam, and you are supposed to complete the exam without getting help from others. Please show your work or explain how you reach your answers. Answers without work will receive little credit. Calculators or cell phones are NOT allowed in the exam. Please turn off your cell phone during the exam.

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1. (20 points =10+10). Fourier series and Fourier transform. Let $\alpha > 0$.
(a) Let $f(x) = e^{-\alpha x}$ for $x \in [0, \pi]$. Compute the coefficients a_n in its Fourier sine series $F(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$ and evaluate $\lim_{x\to 0+} F(x)$.
(b) Let $g(x) = e^{-\alpha x}$ for $x \geq 0$. Compute its Fourier sine transform $G(w) = \frac{2}{\pi}\int_0^\infty g(x)\sin(wx)dx$ and evaluate $\lim_{w\to 0+} G(w)$.

ANS: (a) $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$

$I = \int_0^\pi f(x)\sin nx\,dx = \int_0^\pi f(x)\,\frac{-1}{n}\,d\cos nx$

$= \frac{-1}{n} f(x)\cos nx\Big|_0^\pi + \frac{1}{n}\int_0^\pi \cos nx\,(-\alpha)e^{-\alpha x}dx$

$= \frac{1}{n}(1-\cos n\pi\cdot e^{-\alpha\pi}) + \frac{-\alpha}{n}\underbrace{\int_0^\pi e^{-\alpha x}\,\frac{1}{n}\,d\sin nx}$

$= -\frac{\alpha}{n^2}\left[e^{-\alpha x}\sin nx\Big|_0^\pi + \alpha\int_0^\pi e^{-\alpha x}\sin nx\,dx\right]$

$= -\frac{\alpha}{n^2}\left[0 + \alpha I\right]$

$\Rightarrow \quad (1+\frac{\alpha^2}{n^2}) I = \frac{1}{n}\left[1 - e^{-\alpha\pi}(-1)^n\right]$

$I = \frac{n}{\alpha^2+n^2}\left[1-e^{-\alpha\pi}(-1)^n\right]$

$\boxed{a_n = \frac{2}{\pi} I = \frac{2n}{\pi(\alpha^2+n^2)}\left[1-e^{-\alpha\pi}(-1)^n\right]}$

(b) $I = \int_0^\infty e^{-\alpha x}\underline{\sin wx\,dx}$

$= \int_0^\infty e^{-\alpha x}\,\frac{-1}{w}\,d\cos wx$

$= -\frac{e^{-\alpha x}\cos wx}{w}\Big|_0^\infty + \frac{-\alpha}{w}\int_0^\infty e^{-\alpha x}\cos wx\,dx$

$= \frac{1}{w} - \frac{\alpha}{w}\left[\cancel{\frac{e^{-\alpha x}\sin wx}{w}}\Big|_0^\infty - \frac{-\alpha}{w}\underbrace{\int_0^\infty e^{-\alpha x}\sin wx\,dx}\right]$

$= \frac{1}{w} - \frac{\alpha^2}{w^2} I$

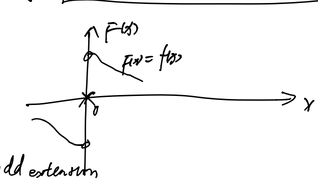
$\Rightarrow \quad I = \frac{1}{w}\cdot\frac{w^2}{w^2+\alpha^2} = \frac{w}{w^2+\alpha^2}$

$\boxed{G(w) = \frac{2}{\pi} I = \frac{2}{\pi}\frac{w}{w^2+\alpha^2}}$

$\boxed{\lim_{w\to 0} G(w) = 0}$

$\boxed{\lim_{x\to 0+} F(x) = \lim_{x\to 0+} f(x) = 1}$

$\lim F(x) = \frac{f(x^+)+f(x^-)}{2} = f(x)$ for $x \in (0,\pi)$



(Note $\frac{f(0+)+f(0-)}{2} = F(0)$ where $f(0-) = -1$ is obtained by odd extension of $f(x) = -e^{+\alpha x}$ when $x<0$)

$\neq \lim_{x\to 0+} F(x)$

2. (20 points) Solve the initial value problem by the method of Fourier transformation: [1]

$$\begin{cases} \partial_t u = \partial_{xx} u - u, & \text{for } -\infty < x < \infty, t > 0; \\ u(x,0) = f(x). \end{cases}$$

ANS: Let $U(w,t) = \mathcal{F}(u(\cdot,t)) = \int_{-\infty}^{\infty} u(x,t) e^{iwx} dx$

Then, $\mathcal{F}(\partial_t u) = \partial_t U$;

$\mathcal{F}(\partial_{xx} u) = (-iw)^2 U = -w^2 U$;

$\Rightarrow \quad \partial_t U = -w^2 U - U$

$\Rightarrow \quad U(w,t) = e^{-(1+w^2)t} U(w,0) = e^{-t} e^{-w^2 t} \hat{f}(w)$

$\Rightarrow \quad u(x,t) = \mathcal{F}^{-1}[U(\cdot,t)](x)$

$= e^{-t} \mathcal{F}^{-1}[e^{-w^2 t} \hat{f}(w)](x)$ by (*) in the table

$= e^{-t} \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) g(x-y) dy$

$g(x-y) = \sqrt{\frac{\pi}{t}} e^{-\frac{|x-y|^2}{4t}}$ b.c. $\mathcal{F}^{-1}[e^{-w^2 t}](x) = \sqrt{\frac{\pi}{t}} e^{-\frac{x^2}{4t}}$

Solution: $\boxed{u(x,t) = e^{-t} \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) \sqrt{\frac{\pi}{t}} e^{-\frac{|x-y|^2}{4t}} dy.}$

[1] Formulas of Fourier transform that you may want to use: $f(x) = \int_{-\infty}^{\infty} F(w) e^{-iwx} dw$, $F(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{iwx} dx$

$f(x)$	$F(w)$	$f(x)$	$F(w)$	$f(x)$	$F(w)$
$e^{-\alpha x^2}$	$\frac{1}{\sqrt{4\pi\alpha}} e^{-w^2/4\alpha}$	f'	$-iwF(w)$	$f(x-\beta)$	$e^{iw\beta}F(w)$
$\sqrt{\frac{\pi}{\beta}} e^{-x^2/4\beta}$	$e^{-\beta w^2}$	f''	$(-iw)^2 F(w)$	$xf(x)$	$-iF'(w)$
$\frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) g(x-y) dy$	F(w)G(w)				

3. (20 points = 8+12) Eigenvalue problems.
(a) Find the values (if any) of β such that $\lambda = 0$ is an eigenvalue to the eigenvalue problem

$$\phi'' + \lambda\phi = 0 \text{ for } x \in (0,1), \text{ with } \phi(0) = \phi'(0) \text{ and } \phi(1) = \beta\phi'(1).$$

(b) Show that $\lambda_1 = 1 + 1/4$ and $\lambda_2 = 4 + 1$ are eigenvalues to the eigenvalue problem

$$\begin{cases} \nabla^2\phi + \lambda\phi = 0, & \text{for } 0 < x < \pi, 0 < y < 2\pi; \\ \phi(0,y) = 0 = \phi(\pi,y); \phi(x,0) = 0 = \phi(x,2\pi). \end{cases}$$

by finding their eigenfunctions. Derive that these eigenfunctions are orthogonal.

ANS: (a). Since $\lambda=0$ is an eigenvalue, then $\varphi'' = -\lambda\varphi = 0$ has a nonzero solution: $\varphi(x) = ax+b$ with $a \neq 0$ or $b \neq 0$

From BC:
$$\left.\begin{array}{l} \varphi(0) = \varphi'(0) \Rightarrow b = a \\ \varphi(1) = \beta\varphi'(1) \Rightarrow a+b = a\beta \end{array}\right\} \Rightarrow \boxed{\beta = 2.}$$

(b) By Separation of variables (b.c. the Eq & BC are homogeneous)

We seek solutions in form of: $\varphi(x,y) = h(x)g(y)$

$$h''(x)g(y) + h(x)g''(y) + \lambda hg = 0$$

$$\frac{h''(x)}{h} = -\lambda - \frac{g''}{g}(y) = -\mu$$

$$\text{BC} \rightarrow \begin{cases} h''(x) = -\mu h(x) \\ h(0) = h(\pi) = 0 \end{cases} \qquad \begin{cases} g''(y) = -\underbrace{(\lambda-\mu)}_{\alpha} g \\ g(0) = g(2\pi) = 0 \end{cases}$$

$$\mu_n = \left(\frac{n\pi}{\pi}\right)^2 = n^2 \qquad \alpha_m = \left(\frac{m\pi}{2\pi}\right)^2 = \frac{m^2}{4}$$

$$h_n(x) = \sin nx \qquad g_m(y) = \sin\frac{m}{2}y$$

$$\Rightarrow \begin{cases} \varphi_{n,m}(x,y) = \sin nx \cdot \sin\frac{m}{2}y, & \lambda_{n,m} = \alpha_m + \mu_n \\ \lambda_{n,m} = n^2 + \frac{m^2}{4}, & n, m \geq 1. \end{cases}$$

Therefore, $\lambda_1 = 1 + \frac{1}{4}$ comes from $n=1, m=1$, $\boxed{\varphi_{1,1} = \sin x \sin\frac{y}{2}.}$

$\lambda_2 = 4+1$ comes from $\begin{cases} n=2, m=2 & \boxed{\varphi_{2,1} = \sin 2x \sin y} \\ \text{or } n=1, m=4, & \boxed{\varphi_{1,4} = \sin x \sin 2y} \end{cases}$

To show orthogonality.. option 1: compute directly:

$$\iint \varphi_{1,1}(x,y)\,\varphi_{2,1}(x,y)\,dxdy = \int_0^\pi \sin x \sin 2x\,dx \underbrace{\int_0^{2\pi} \sin\frac{y}{2} \sin y\,dy}_{0} = 0$$

Similarly, we have $\langle \varphi_{1,1}, \varphi_{1,4} \rangle = 0$; $\langle \varphi_{2,1}, \varphi_{1,4} \rangle = 0$.

Option 2: by Greens formula

$$\nabla^2\varphi_1 = -\lambda_1\varphi_1 \ ; \ \nabla^2\varphi_2 = -\lambda_2\varphi_2$$

$$\iint \nabla^2\varphi_1 \cdot \varphi_2\,dxdy = -\iint \lambda_1\varphi_1\varphi_2\,dxdy$$

$$\parallel$$

$$\iint \nabla^2\varphi_2\,\varphi_1\,dxdy = -\iint \lambda_2\varphi_1\varphi_2\,dxdy$$

Green's formula.

$$\Rightarrow \lambda_1\langle\varphi_1,\varphi_2\rangle = \lambda_2\langle\varphi_1,\varphi_2\rangle$$

$$\Rightarrow \langle\varphi_1,\varphi_2\rangle = 0.$$

Rmk: it works only for $\lambda_1 \neq \lambda_2$

4. (20 points) Solve the initial boundary value problem by the method of eigenfunction expansion:

$$\begin{cases} \partial_{tt}u = \partial_{xx}u - \partial_t u, \text{ for } 0 < x < \pi, t > 0; \\ u(x,0) = f(x), \partial_t u(x,0) = g(x); \\ u(0,t) = 0, \qquad u(\pi,t) = 0. \end{cases}$$

ANS: 1> Determine eigenfunctions. By Sturm-Liouville Thm:

$$\begin{cases} \partial_{xx}\varphi = -\lambda\varphi \\ \varphi(0) = 0 = \varphi(\pi) \end{cases} \Rightarrow \lambda_n = n^2, \ \varphi_n(x) = \sin nx$$

$\{\varphi_n\}_{n=1}^{\infty}$ is a complete, orthogonal basis.

2> Eigenfunction expansion. Assume $u(x,t) = \sum_{n=1}^{\infty} a_n(t)\varphi_n(x)$ satisfies conditions for IBVP:

$$\sum_{n=1}^{\infty} a_n''(t)\varphi_n(x) = \sum_{n=1}^{\infty} \underbrace{a_n(t)\varphi_n''(x) - a_n'(t)\varphi_n(x)}_{(-\lambda_n a_n - a_n')\varphi_n(x)}$$

$$\Rightarrow \quad a_n''(t) = -\lambda_n a_n - a_n' \quad \Leftrightarrow \quad a_n'' + a_n' + \lambda_n a_n = 0$$

characteristic Equ. $z^2 + z + \lambda = 0 \qquad z = \frac{1}{2}[-1 \pm \sqrt{1-4\lambda}]$

Thus, $\quad a_n(t) = e^{-\frac{1}{2}t}\left[c_1 \sin\frac{t}{2}\sqrt{4n^2-1} + c_2 \cos\frac{t}{2}\sqrt{4n^2-1}\right].$

3> Determine c_1 & c_2 from IC's

$$\sum_1^{\infty} a_n(0)\varphi_n(x) = u(x,0) = f(x) \Rightarrow a_n(0) = \overset{\parallel f_n}{\frac{\langle f, \varphi_n\rangle}{\langle \varphi_n, \varphi_n\rangle}} = c_1 \cdot 0 + c_2$$

$$\sum_1^{\infty} a_n'(0)\varphi_n(x) = \partial_t u(x,0) = g(x) \Rightarrow a_n'(0) = \frac{\langle g, \varphi_n\rangle}{\langle \varphi_n, \varphi_n\rangle} = 1\cdot c_1\frac{\sqrt{4n^2-1}}{2}\cdot 1 + c_2 \cdot 0$$

$$(-\tfrac{1}{2})\cdot 1\,[c_1 \cdot 0 + c_2]$$

$$\Rightarrow \quad c_2 = f_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$c_1 = \left(g_n + \tfrac{1}{2}c_2\right)\Big/\frac{\sqrt{4n^2-1}}{2}.$$

The solution is:

$$\boxed{\begin{aligned} &u(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin nx \\ &a_n(t) = e^{-\frac{1}{2}t}\left[\frac{2g_n + f_n}{\sqrt{4n^2-1}}\sin\frac{\sqrt{4n^2-1}}{2}t + f_n\cos\frac{\sqrt{4n^2-1}}{2}t\right] \end{aligned}}$$

5. (20 points) Consider the initial boundary value problem:

$$\begin{cases} \partial_t u = \partial_{xx} u, & \text{for } 0 < x < \pi, t > 0; \\ \partial_x u(0,t) = 0, \ \partial_x u(\pi, t) = 1; \\ u(x,0) = f(x). \end{cases}$$

(a) Find the Green's function.
(b) Find the equilibrium solution if there is one. Otherwise, explain why it does not exist.

(a). 1) We homogenize the BC by letting $u = v + w$ with w satisfying the BC. For example, $w(x) = \frac{x^2}{2\pi}$.

Then $\begin{cases} \partial_t v = \partial_t u - \partial_t w = \partial_{xx} v + \partial_{xx} w - \partial_t w = \partial_{xx} v + \frac{1}{\pi} - 0 \\ \partial_x v(0,t) = 0; \quad \partial_x v(\pi, t) = 0; \\ v(x,0) = f(x) - \frac{x^2}{2\pi} = g(x) \end{cases}$

2). Solve the IBVP by the Method of eigenfunction expansion

$\begin{cases} \partial_{xx} \varphi = -\lambda \varphi \\ \varphi'(0) = \varphi'(\pi) = 0 \end{cases} \Rightarrow \quad \lambda_n = \left(\frac{n\pi}{\pi}\right)^2 = n^2; \ \varphi_n(x) = \cos nx \ ; \ n \geq 0$

seek solution $v(x,t) = \sum_0^\infty a_n(t) \varphi_n(x)$. Assuming conditions for TBTD,

$$\sum_0^\infty a_n'(t) \varphi_n(x) = \sum_0^\infty a_n(t)(-\lambda_n) \varphi_n(x) + \frac{1}{\pi}$$

$\Rightarrow \quad a_n'(t) = -\lambda_n a_n + \frac{\langle \frac{1}{\pi}, \varphi_n \rangle}{\langle \varphi_n, \varphi_n \rangle} \quad \Rightarrow \quad a_n(t) = a_n(0) e^{-\lambda_n t} + \frac{\langle \frac{1}{\pi}, \varphi_n \rangle}{\langle \varphi_n, \varphi_n \rangle} e^{-\lambda_n t}$

$= \frac{\langle g + \frac{1}{\pi}, \varphi_n \rangle}{\langle \varphi_n, \varphi_n \rangle} e^{-\lambda_n t}$

$\Rightarrow \quad v(x,t) = \sum_{n=0}^\infty a_n(t) \varphi_n(x)$

$= \int_0^\pi \left(g(y) + \frac{1}{\pi}\right) \underbrace{\left[\frac{1}{\pi} + \sum_{n=1}^\infty \frac{\varphi_n(y)\varphi_n(x)}{\pi/2} e^{-\lambda_n t}\right]}_{G(x,t;y,0)} dy \qquad a_n(0) = \frac{\langle g, \varphi_n \rangle}{\langle \varphi_n, \varphi_n \rangle} \qquad \begin{matrix} \langle \varphi_0, \varphi_0 \rangle = \pi \\ \langle \varphi_n, \varphi_n \rangle = \frac{\pi}{2} \end{matrix}$

$u(x,t) = v(x,t) + w(x) = w(x) + \int_0^\pi \left(f(y) - \frac{y^2}{2\pi} + \frac{1}{\pi}\right) G(x,t;y,0)\, dy$

The Green's function is $\boxed{G(x,t;y,0) = \frac{1}{\pi} + \sum_{n=1}^\infty \frac{2}{\pi} \varphi_n(x)\varphi_n(y) e^{-\lambda_n t}, \quad \begin{matrix} \lambda_n = n^2 \\ \varphi_n(x) = \cos nx \end{matrix}}$

(b). There is NO equilibrium solution, because $\begin{cases} \partial_{xx} u_E = 0 \\ u_E'(0) = 0, \ u_E'(\pi) = 1 \end{cases}$ has NO solution

(Since $\partial_{xx} u_E = 0 \Rightarrow u_E(x) = ax + b$; $u_E'(0) = a = 0$; $u_E'(\pi) = a = 1$. a contradiction.)

s1

BOUNDARY VALUE PROBLEMS

Boundary conditions	$\phi(0)=0$ $\phi(L)=0$	$\frac{d\phi}{dx}(0)=0$ $\frac{d\phi}{dx}(L)=0$	$\phi(-L)=\phi(L)$ $\frac{d\phi}{dx}(-L)=\frac{d\phi}{dx}(L)$
Eigenvalues λ_n	$\left(\frac{n\pi}{L}\right)^2$ $n=1,\ 2,\ 3,\dots$	$\left(\frac{n\pi}{L}\right)^2$ $n=0,\ 1,\ 2,\ 3,\dots$	$\left(\frac{n\pi}{L}\right)^2$ $n=0,\ 1,\ 2,\ 3,\dots$
Eigenfunctions	$\sin\frac{n\pi x}{L}$	$\cos\frac{n\pi x}{L}$	$\sin\frac{n\pi x}{L}$ and $\cos\frac{n\pi x}{L}$
Series	$f(x)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{L}$	$f(x)=\sum_{n=0}^{\infty}A_n\cos\frac{n\pi x}{L}$	$f(x)=\sum_{n=0}^{\infty}a_n\cos\frac{n\pi x}{L}$ $+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{L}$
Coefficients	$B_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$	$A_0=\frac{1}{L}\int_0^L f(x)\,dx$ $A_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$	$a_0=\frac{1}{2L}\int_{-L}^{L} f(x)\,dx$ $a_n=\frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$ $b_n=\frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$

	Sturm-Liouville	Helmholtz (2-dim)
Eigenvalue Problem	$\frac{d}{dx}\left(p\frac{d\phi}{dx}\right)+(\lambda\sigma+q)\phi=0$	$\nabla^2\phi+\lambda\phi=0$
Operator	$L=\frac{d}{dx}\left(p\frac{d}{dx}\right)+q$	$L=\nabla^2$
Green's Formula	$\int_a^b[uL(v)-vL(u)]\,dx=p\left(u\frac{dv}{dx}-v\frac{du}{dx}\right)\Big\vert_a^b$	$\iint_R(u\nabla^2v-v\nabla^2u)\,dx\,dy$ $=\oint(u\nabla v-v\nabla u)\cdot\hat{\boldsymbol{n}}\,ds$
Rayleigh Quotient	$\lambda=\frac{-p\phi\frac{d\phi}{dx}\vert_a^b+\int_a^b\left[p\left(\frac{d\phi}{dx}\right)^2-q\phi^2\right]dx}{\int_a^b\phi^2\sigma\,dx}$	$\lambda=\frac{-\oint\phi\nabla\phi\cdot\hat{\boldsymbol{n}}\,ds+\iint_R\vert\nabla\phi\vert^2\,dx\,dy}{\iint_R\phi^2\,dx\,dy}$

Solutions to some ODE problems:

$y'(t)=ay(t)+f(t);\ y(0)=y_0$	$y(t)=e^{at}y_0+\int_0^t e^{a(t-s)}f(s)ds$
$y''(t)=0;\ y(0)=A; y(L)=B$	$y(t)=A+\frac{B-A}{L}t$