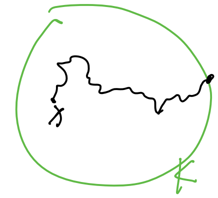
Chp7. Diffusion: Basic properties.



Ito diffusion: $\quad dX_t = \underbrace{b(t,X_t)}_{\text{drift}}dt + \underbrace{\sigma(t,X_t)}_{\text{diffusion coefficient}}dB_t \; ; \quad X_0 = x \in \mathbb{R}^n$

X_t: position of a particle in a moving liquid; $\quad$ molecular bombardments.

Question: ① will X_t exit $K_R = \{y \in \mathbb{R}^n : |y| < R\}$ if $x \in K_R$? $\qquad \tau_K = \inf\{t>0 : X_t \notin K_R\}$

② will X_t hit K_R if $x \notin K_R$? $\qquad \tau_{K^c} = \inf\{t>0 : X_t \notin K_R^c\}$

③ what is the expectation of the exit/hitting time? $\quad$ Distribution of X_{τ_K}? $\quad X_{\tau_R}$

§7.1. Markov property

Time-homogeneous (coefs): $\quad dX_t = b(X_t)dt + \sigma(X_t)dB_t \; ; \quad X_0 = x \in \mathbb{R}^n$

- Global Lipschitz ($\Rightarrow$ linear growth.) $\quad \Rightarrow \exists!$ soln.
- $\{X_t\}_{t\geq 0}$ Time homogeneous: $\{X^{s,x}_{s+h}\}_{h\geq 0}$ and $\{X^{0,x}_h\}_{h\geq 0}$ have the same $\mathbb{P}^0$-distribution, b.c.

$$X^{s,x}_{s+h} = x + \int_s^{s+h} b(X_u)du + \int_s^{s+h} \sigma(X_u)dB_u \qquad X_u = X_u^{s,x}$$

$$= x + \int_0^h b(X^{s,x}_{s+v})dv + \int_0^h \sigma(X^{s,x}_{s+v})d\tilde{B}_v \qquad \tilde{B}_v = B_{s+v} - B_s$$

$$X^{0,x}_h = x + \int_0^h b(X^{0,x}_v)dv + \int_0^h \sigma(X_v)dB_v \qquad \{\tilde{B}_v\}_{v\geq 0} \overset{\mathcal{D}}{=} \{B_v\}_{v\geq 0}$$

$\xrightarrow{\text{! weak soln.}}$ $\{X^{s,x}_{s+h}\}_{h\geq 0} \overset{\mathcal{D}}{=} \{X^{0,x}_h\}_{h\geq 0}$.

- Rmk: flow map $\qquad X^{t,x}_r(\omega) = F(x,t,r,\omega) \quad r \geq t$

$$= F(x, B_{[t,r]}) \approx F(x, B_{t_1:t_k}). \qquad \text{N.N.}$$

Notation $\cdot$ Q^x: probab. law of $\{X_t\}_{t\geq 0}$ w/ $X_0 = x \in \mathbb{R}^n$; $\Rightarrow \mathbb{E}^x$: $\mathbb{E}^x[f_1(X_{t_1})\cdots f_k(X_{t_k})] = \mathbb{E}[f_1(X^x_{t_1})\cdots f_k(X^x_{t_k})]$

$\cdot$ $\mathcal{F}_t = \mathcal{F}_t^B$; $\quad \mathcal{M}_t := \mathcal{F}_t^X \subseteq \mathcal{F}_t^B$

Thm 7.1.2 (Markov property) $\underline{\mathbb{E}^x[f(X_{t+h})|\mathcal{F}_t^B] = \mathbb{E}^{X_t(\omega)}[f(X_h)]}$, $\forall h \geq 0$

f: bdd Borel fn: $\mathbb{R}^n \to \mathbb{R}$.

" Future depends on the current, NOT the past." $= \mathbb{E}[f(X_{t+h})|X_t]$

Proof: $r > t$. $X_r = X_t + \int_t^r b(X_u)du + \int_t^r \sigma(X_u)dB_u \xRightarrow{!\text{ sdu.}} X_r(\omega) = X_r^{t, X_t}(\omega)$

$\mathbb{E}[f(X_r)|\mathcal{F}_t] = \mathbb{E}[f(X_r^{t,X_t})|\mathcal{F}_t] \overset{\text{T.h.}}{=} \mathbb{E}[f(X_h^{0,x})]\Big|_{x=X_t}$ #.

§7.2 Strong Markov property. $\underline{\mathbb{E}^x[f(X_{\tau+h})|\mathcal{F}_\tau^B] = \mathbb{E}^{X_\tau}[f(X_h)]}$, $\forall h \geq 0$

bdd Borel fn. τ: a stopping time w.r.t. $\mathcal{F}_t$, $\tau < \infty$ a.s.

Def. 7.2.1 (Stopping Time) Let $\{N_t\}$ be a family of increasing σ-algebras. A function $\tau: \Omega \to [0,\infty]$ is a stopping time w.r.t. N_t if: $\{\omega: \tau(\omega) \leq t\} \in N_t$, for all $t \geq 0$.

- we can decide wether of not $\tau \leq t$ occured if given N_t.
- $\tau \equiv t_0$ is a trivial stopping time: $\{\omega: \tau(\omega) \leq t\} = \begin{cases} \Omega & \text{if } t_0 \leq t \\ \phi & \text{if } t_0 > t \end{cases}$.

Example 7.2.2 (The first Exit time) Let $U \subseteq \mathbb{R}^n$ open. The the 1st exit time $\tau_U = \inf\{t > 0: X_t \notin U\}$ is a stopping time w.r.t. $\{\mathcal{F}_t^X\}$ since $\{\omega: \tau_U \leq t\} = \bigcap_m \bigcup_{r_m \in \mathbb{Q}, r < t} \{\omega: X_r \notin K_m\} \in \mathcal{F}_t^X$.

$U = \bigcup_m K_m$, K_m closed sets.

- U: open set $\longrightarrow$ any Borel set; (include 0-meas. sets in $\mathcal{F}_t^X$, $\Rightarrow \mathcal{F}_t^X = \mathcal{F}_{t+}^X = \bigcap_{s>t} \mathcal{F}_t^X$.) cts.

Def. 7.2.3 (σ-algebra w/ stopping times). Let τ be a stopping time w.r.t. $\{N_t\}$.

N_∞ be the smallest σ-algebra containing N_t for all $t \geq 0$.

Then the σ-algebra $N_\tau = \sigma\text{-alg}\{N \in N_\infty: N \cap \{\tau \leq t\} \in N_t, \forall t \geq 0\}$

- $\mathcal{F}_\tau^X = \sigma$-alg. generated by $\{X_{\tau \wedge s}, s \geq 0\}$.

Proof of the Strong Markov property: $X_{\tau+h}^{\tau,y} = \overset{X_\tau}{y} + \int_\tau^{\tau+h} b(X_u^{\tau,y})du + \int_\tau^{\tau+h} \sigma(X_u^{\tau,y})dB_u$ $\quad u = v + \tau$

$X_t \triangleq X_t^{0,x}$, $\quad = y + \int_0^h b(X_{\tau+v}^{\tau,y})dv + \int_0^h \sigma(X_v^{\tau,y})d\tilde{B}_v$ $\quad \tilde{B}_v = B_{\tau+v} - B_\tau$ $\downarrow$ Bm.

- strong! $\Rightarrow \{X_{\tau+h}^{\tau,y}\}_{h \geq 0}$ indpt of $\mathcal{F}_\tau^B$;

· weak! ⇒ $\{X^{\tau,y}_{\tau+h}\}_{h\ge0} \overset{\mathcal{D}}{\sim} \{X^{0,y}_h\}_{h\ge0}$. ⇒ $\underline{X_{\tau+h} = X^{0,x}_{\tau+h} = X^{\tau,X_\tau}_{\tau+h}}\Big|_{\mathcal{F}^B_\tau} \overset{\mathcal{D}}{\sim} X^{0,X_\tau}_h$

$$\Rightarrow \quad \mathbb{E}^x[f(X_{\tau+h})\,|\,\mathcal{F}^B_\tau] = \mathbb{E}[f(X^{0,x}_h)]\big|_{x=X_\tau} = \mathbb{E}^{X_\tau}[f(X_h)] \qquad \#.$$

Rmk. · Extension. $\mathbb{E}^x[f_1(X_{\tau+h_1})\cdots f_k(X_{\tau+h_k})\,|\,\mathcal{F}^B_\tau] = \mathbb{E}^{X_\tau}[f_1(X_{h_1})\cdots f_k(X_{h_k})]$, $\forall\, 0\le h_1\le\cdots\le h_k$.

· Shift operator $\theta_t: \mathcal{H}\to\mathcal{H}$, where $\mathcal{H}$ = all real $\mathcal{F}^X_\infty$ meas. functions.

$$\theta_t\big[\underbrace{g_1(X_{t_1})\cdots g_k(X_{t_k})}\big] = g_1(X_{t_1+t})\cdots g_k(X_{t_k+t}) \qquad \cdot \text{ path-wise.}$$

$$\eta = \eta(X_{t_1},\dots,X_{t_k}) \to \theta_t\eta$$

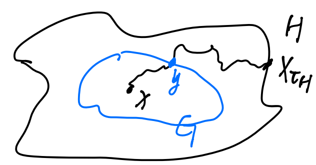
$\xrightarrow{\text{strong Markov}}$ $\mathbb{E}^x[\theta_t\eta\,|\,\mathcal{F}^X_\tau] = \mathbb{E}^{X_\tau}[\eta]$, $\forall\, \eta\in\mathcal{H}$

· Hitting distribution: Let $H\subseteq\mathbb{R}^n$ be a Borel set and τ_H be the 1st exit time from H for X_t.

· The distribution of X_{τ_H} is the hitting distr.

· $F\subseteq\partial H$: $\mu^x_H(F) = Q^x(X_{\tau_H}\in F)$. $x\in H$.

Harmonic meas. of X on ∂H.



· Mean value property: $\boxed{\varphi(x) = \int_{\partial G}\varphi(y)\,d\mu^x_G(y)}$, $\forall\, x\in G\Subset H$ Borel, $\varphi(x) = \mathbb{E}^x[f(X_{\tau_H})]$.
bdd meas.

"Proof": use the strong Markov w/ shift operator.

$$\mathbb{E}^x[f(X_{\tau_H})] = \mathbb{E}^x[\mathbb{E}^{X_{\tau_G}}f(X_{\tau_H})]] = \int_{\partial G}\underbrace{\mathbb{E}^y[f(X_{\tau_H})]}_{\varphi(y)}\cdot\underbrace{Q^x(X_{\tau_G}\in dy)}_{\mu^x_G(dy)} \qquad \#$$

① Dirichlet problem: $\begin{cases} Lu=0 \\ u|_{\partial H}=f \end{cases}$ $u\in C^2$ $\xrightarrow{\Delta\Delta}$ ⇒ $u(x) = \mathbb{E}^x[f(X_{\tau_H})]$

In general. Let α be another stopping time, g a bdd cts function ② Mean value:

$$\eta = g(X_{\tau_H})\,1_{\{\tau_H<\infty\}}, \quad \tau^\alpha_H = \inf\{t>\alpha: X_t\notin H\}$$

Then, $\theta_\alpha\eta\,1_{\{\alpha<\infty\}} = g(X_{\tau^\alpha_H})\,1_{\{\tau^\alpha_H<\infty\}}$.

(b.c. $\eta^{(k)} := \sum_{j=1}^{2^k} g(X_{t_j})\,1_{[t_j,t_{j+1})}(\tau_H)$, $t_j = j\cdot 2^{-k}$;
$\theta_{\underset{=\alpha}{t}} 1_{[t_j,t_{j+1})}(\tau_H) = 1_{[t_j+\underset{=\alpha}{t},\, t_{j+1}+t)}(\tau^t_H)$ $\Big\}$ ⇒ $\theta_{\underset{=\alpha}{t}}\eta = \lim_k \theta_t\eta^{(k)} = g(X_{\underset{\tau^\alpha_H.}{\tau^t_H}})\,1_{\{\tau^t_H<\infty\}}$.

· $\alpha=\tau_G$: $\tau_H<\infty$ a.s. Q^x. Then, $\underline{\tau^\alpha_H = \tau_H}$. $\underline{\theta_{\tau_G} g(X_{\tau_H}) = g(X_{\tau_H})}$.

Back to Q. $\tau \to X_\tau$ $\mathbb{E}[f(X_\tau)]$?

§7.3 The generator of an Itô diffusion. $\Leftrightarrow$ PDE

Def (Generator): Let $\{X_t\}$ be a time-homogeneous Itô diffusion. Its infinitesimal generator is

$$Af(x) = \lim_{t\downarrow 0} \frac{\mathbb{E}^x[f(X_t)] - f(x)}{t}, \quad x \in \mathbb{R}^n.$$

Domain $D_A = \{ f : \mathbb{R}^n \to \mathbb{R}$, the limit exists for all $x \in \mathbb{R}^n \}$.

Thm 7.3.3 For X_t s.t. $dX_t = b(X_t)dt + \sigma(X_t)dB_t$, we have $f \in C_0^2(\mathbb{R}^n) \subseteq D_A$ and

$$Af(x) = b \cdot \nabla f(x) + \frac{1}{2} \underbrace{\sigma\sigma^T : \text{Hess}(f)}_{\sum_{ij} (\sigma\sigma^T)_{ij} \partial_{x_i} \partial_{x_j} f}$$

Proof: Itô formula. $f(X_t) = f(x) + \int_0^t Af(X_s)ds + \int_0^t \nabla f \cdot \sigma dB_s$. (**)

Take $\mathbb{E}^x$: $\mathbb{E}^x[f(X_t)] = f(x) + \mathbb{E}^x \int_0^t Af(X_s)ds + 0$. #

Example: $dX_t = dB_t$; $B_0 = 0 \in \mathbb{R}^n$ $\quad A = \frac{1}{2}\Delta$.

$$\begin{cases} dX_1 = dt \\ dX_2 = dB_t \end{cases} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} dt + \begin{pmatrix} 0 \\ 1 \end{pmatrix} dB_t \Rightarrow A = \partial_t + \frac{1}{2}\partial_{xx}.$$

§7.4. The Dynkin Formula

Thm 7.4.1 $f \in C_0^2(\mathbb{R}^n)$, τ a stopping time s.t. $\mathbb{E}^x[\tau] < \infty$. Then.

$$\mathbb{E}^x[f(X_\tau)] = f(x) + \mathbb{E}^x \int_0^\tau Af(X_s)ds. \quad (*)$$

Rmk. If τ is the 1st exit time of a bdd set, then (*) holds for any $f \in C^2$

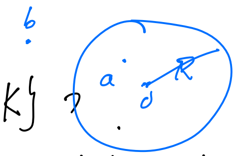
"Proof": (**) holds for any t pathwisely, $\rightarrow$ replace t by τ and take expectation.

Only Need to show that $\mathbb{E}^x \int_0^\tau \nabla f \cdot \sigma dB_s = 0$

$$= \mathbb{E}^x \int_0^t \underbrace{1_{\{s \le \tau\}}}_{\text{are } \mathcal{F}_s^B\text{-measurable}} \nabla f \cdot \sigma dB_s = 0$$

∇f and σ are bounded on supp(f). #.

Example 7.4.2. $\{B_t\}_{t\geq 0}\in\mathbb{R}^n$, Bm ; $K=\{x\in\mathbb{R}^n: |x|<R\}$.



(1) $B_0=a$, $|a|<R$: What is $\mathbb{E}^a[\tau_K]$ where $\tau_K=\inf\{t>0,\ B_t\notin K\}$?

(2) $B_0=b$, $|b|>R$: what is the probability $\{B_t\}$ hits K? (drunk man/bird return home)

ANS: (1) $\forall k\in\mathbb{N}$, apply Dynkin's formula w/ $X=B$, $\tau=\sigma_k=k\wedge\tau_K$, $f\in C_0^2$ s.t. $f(x)=|x|^2$, $|x|\leq R$.

$$\mathbb{E}^a[f(B_{\sigma_k})]=f(a)+\mathbb{E}^a\left[\int_0^{\sigma_k}\tfrac{1}{2}\Delta f(B_s)\,ds\right]$$
$$=|a|^2+\mathbb{E}^a\int_0^{\sigma_k} n\,ds=|a|^2+n\,\mathbb{E}^a[\sigma_k]$$

- by def. of σ_k: $|B_{\sigma_k}|\leq R \Rightarrow f(B_{\sigma_k})\leq R^2 \rightarrow \mathbb{E}^a[\sigma_k]\leq \frac{1}{n}(R^2-|a|^2)$, $\forall k$.
- $k\uparrow\infty$. $\sigma_k\uparrow\tau_K$ a.s. $B_{\sigma_k}\to B_{\tau_K}$ a.s. $\xrightarrow{DCT}$ $R^2=|a|^2+n\,\mathbb{E}^a[\tau_K]$ (monotone)

That is, $\mathbb{E}^a[\tau_K]=\frac{1}{n}(R^2-|a|^2)$.

(2) Let α_k be the first exit time from the annulus: $A_k=\{x: R<|x|<2^kR\}$; $k=1,2,\dots$

Denote $T_K=\inf\{t>0:\ B_t^b\in K\}$ (hitting time).

$$\mathbb{P}^b(T_K<\infty)\underset{as}{=}\lim_{k\uparrow\infty}\mathbb{P}^b(|B_{\alpha_k}|=R)=:p_k$$

Let $f_{n,k}\in C_c^2$ such that $f_{n,k}(x)=\begin{cases}-\log|x|, & \text{when } n=2,\\ |x|^{2-n} & \text{when } n>2\end{cases}$ $\Rightarrow \Delta f=0$ in A_k.

Apply Dynkin's formula, $\mathbb{E}^b[f(B_{\alpha_k})]=f(b)+0$, $\forall k$.

$$\mathbb{E}^b[f(B_{\alpha_k})] = f(R)\,p_k+f(2^kR)\,q_k,\quad q_k=\mathbb{P}^b(|B_{\alpha_k}|=2^kR)=1-p_k.$$

When $n=2$: $-\log R\ p_k-(\log R+k\log 2)(1-p_k)=-\log|b|$, $\Rightarrow (k\log 2)\,p_k=-\log|b|+\log|R|+k\log 2$

$$p_k=\left(\log\tfrac{R}{|b|}+k\log 2\right)/k\log 2\to 1 \text{ as } k\uparrow\infty.$$

$\Rightarrow$ $\mathbb{P}^b(T_K<\infty)=1$ (Bm is recurrent)

When $n>2$: $R^{2-n}\cdot p_k+\underbrace{(2^kR)^{2-n}}_{\to 0}q_k=|b|^{2-n}$, $\forall k$. $\Rightarrow \lim_{k\uparrow\infty}p_k=\left(\frac{|b|}{R}\right)^{2-n}$.

§7.5 The characteristic operator.

The generator A: $\underline{C_0^2} \subseteq \mathrm{Dom}(A)$. $\leftarrow$ Ito: $\mathbb{E}^x[f(X_t)] = f(x) + \mathbb{E}^x\int_0^t Af(X_s)ds$

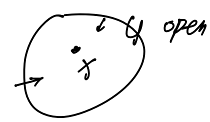
characteristic op. $\mathcal{A}$: $C^2 \subseteq \mathrm{Dom}(\mathcal{A})$ $\leftarrow$ Dynkin: $\mathbb{E}^x[f(X_\tau)] = f(x) + \mathbb{E}^x\int_0^\tau Af(X_s)ds$

for Dirichlet problem in Chp 9.

$$A = \mathcal{A}\big|_{\mathrm{Dom}(A)} \qquad (*)$$

<u>Def. 7.5.1</u> Let X_t be an Ito diffusion. The <u>characteristic operator $\mathcal{A}$</u> of $\{X_t\}$ is

$$\mathcal{A}f(x) = \lim_{U\downarrow x} \frac{\mathbb{E}^x[f(X_{\tau_U})] - f(x)}{\mathbb{E}^x[\tau_U]}$$



- If $\mathbb{E}^x[\tau_U] = \infty$ for all open $U \ni x$, then $\mathcal{A}f(x) = 0$.
- $\mathrm{Dom}(\mathcal{A}) =$ f s.t. the limit exists.

To show $C^2 \subseteq \mathrm{Dom}(\mathcal{A})$, we need to clarify the issue $\mathbb{E}[\tau_{\{x\}}] = \infty$:

<u>Def. (Trap)</u> A point $x \in \mathbb{R}^n$ is a <u>trap</u> for $\{X_t\}$ if $Q^x(\{X_t = x, \forall t\}) = 1$.

- x is a trap iff $\tau_{\{x\}} = \infty$ a.s. e.g. x_0 is a trap if $b(x_0) = \sigma(x_0) = 0$.

<u>Lemma</u> 7.5.3. If x is NOT a trap of X_t, then $\exists$ an open set $U \ni x$ s.t. $\mathbb{E}^x[\tau_U] < \infty$.

<u>Thm 7.5.4</u> Let $f \in \underline{C^2}$. Then $f \in \mathrm{Dom}(\mathcal{A})$ and $\mathcal{A}f = b\cdot\nabla f + \frac{1}{2}\sum_{ij}(\sigma\sigma^T)_{ij}\,\partial_{x_ix_j} f =: Lf$

<u>Proof</u>: If x is a trap: then $\mathcal{A}f(x) = 0$. Let $V \ni x$ be an open set; $f_0 \in C_0^2$ s.t. $f_0|_V = f$.

$\Rightarrow f_0 \in \mathrm{Dom}(A)$ and $0 = \lim_{t\downarrow 0}\frac{\mathbb{E}^x[f(X_t)] - f(x)}{t} = Af_0(x) = Lf_0(x) = Lf(x)$.

If x is NOT a trap: Let $U \ni x$ s.t. open & $\mathbb{E}[\tau_U] < \infty$. Then, by Dynkin:

$$\left|\frac{\mathbb{E}^x[f(X_\tau)] - f(x)}{\mathbb{E}^x[\tau]} - Lf\right| = \frac{\mathbb{E}^x\int_0^\tau [(Lf)(X_s) - Lf(x)]\,ds}{\mathbb{E}^x[\tau]} \le \sup_{y\in U}|Lf(y) - Lf(x)| \downarrow 0$$

as $U \downarrow \{x\}$, since Lf is cts. #

Example 7.5.6 · $D \subseteq \mathbb{R}^n$ open s.t. $\tau_D < \infty$ a.s. Q^x for all $x \in D$.

· Let φ be a bd. measurable fn defined on ∂D and define

$$\tilde{\varphi}(x) = \mathbb{E}^x[\varphi(X_{\tau_D})] \quad \text{called } X\text{-harmonic extension of } \varphi.$$

Then for any $U \underset{\text{open}}{\subset\subset} D$,

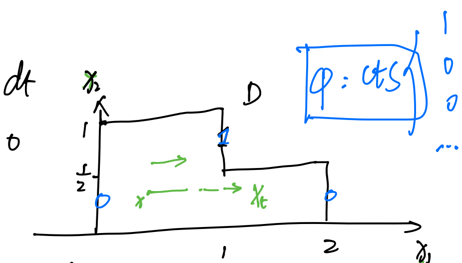
$$\mathbb{E}^x[\tilde{\varphi}(X_{\tau_U})] = \mathbb{E}^x[\mathbb{E}^{X_{\tau_U}}[\varphi(X_{\tau_D})]] = \mathbb{E}^x[\varphi(X_{\tau_D})] = \tilde{\varphi}(x).$$

Thus: $\tilde{\varphi} \in Dom(A)$ and $A\tilde{\varphi} = 0$ in D.

Rmk · Dirichlet problem: $\begin{cases} Lu = 0, & x \in D. \\ u|_{\partial D} = \varphi \end{cases}$ $\longleftarrow$ $\lim\limits_{y \to x \in \partial D} u(y) = \varphi(x)$. If $u \in C^2$ is a solu. $u(x) = \mathbb{E}^x[\varphi(X_{\tau_D})] = \tilde{\varphi}$.

· φ only bdd, meas.; $\tilde{\varphi}$ may NOT even be cts in D: NOT classical BVP.

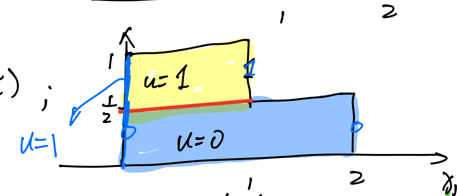
Example 9.2.1: Let $X(t) = \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = X(0) + t\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. $\Leftrightarrow$ $\begin{cases} dX_1(t) = dt \\ dX_2(t) = 0 \end{cases}$



Let $D = ((0,1)\times(0,1)) \cup ((0,2)\times(0,\frac{1}{2}))$

Then

$$u(X) = \mathbb{E}^x[\varphi(X_{\tau_D})] = \begin{cases} 1 & \text{if } x_2 \in (\frac{1}{2}, 1) \\ 0 & \text{if } x_2 \in (0, \frac{1}{2}) \end{cases};$$



· NOT cts. at $x_2 = \frac{1}{2}$

· NOT converge at the boundary $\{0\}\times(\frac{1}{2}, 1)$: $\lim\limits_{x_1 \to 0} u(x_1, x_2) = 1 \neq \varphi(0, x_2) = 0$.

Exe 7.14. (Doob's h-transform). B_t $\mathbb{R}^n$-Bm $D \subseteq \mathbb{R}^n$ bdd open; $h > 0$ and $\Delta h = 0$ in D.

Let $dX_t = \nabla(\ln h)(X_t)\,dt + dB_t$. (*)

More precisely, $\{D_k\} \uparrow$ s.t. open $\overline{D_k} \subseteq D$ and $\cup D_k = D$;

(*) $\exists$ strong sdu. for $t < \tau_{D_k}$ and $t < \tau := \lim_{k\to\infty} \tau_{D_k}$.

(a) show that $Af = \frac{\Delta(hf)}{2h}$, $\forall f \in C_0^2(D)$.

$A = b\cdot\nabla f + \frac{1}{2}\Delta f = \nabla(\ln h)\cdot\nabla f + \frac{1}{2}\Delta f$ $\Big\} \Rightarrow$ #

$\nabla(\ln h)\nabla f = \frac{1}{h}\nabla h\cdot\nabla f = \frac{1}{2h}\Delta(hf) - \frac{1}{2}\Delta f$

$\Delta(hf) = \sum_i \partial_{ii}(hf) = \sum_i \partial_{ii}h\, f + h\partial_{ii}f + 2\partial_i h\,\partial_i f$

$= \cancel{f\Delta h} + h\Delta f + 2\nabla h\cdot\nabla f$

If $f = \frac{1}{h}$, $Af = 0 \Rightarrow Af = 0$.

(b). If $\exists x_0 \in \partial D$ s.t.

$$\lim_{x\to y\in\partial D} h(x) = \begin{cases} 0, & \text{if } y \neq x_0 \\ \infty, & y = x_0 \end{cases}$$

(i.e. h is a kernel function) $h(x) = f_n(x - x_0)$

$f_n(x) = \begin{cases} -\log|x|, & n=2 \\ |x|^{2-n} & n>2 \end{cases}$

then, $\lim_{t\to\tau} X_t = x_0$ a.s.

(We impose a drift on B_t s.t. the process exit at $x_0 \in \partial D$ only.

$\Leftrightarrow$ X_t is obtained by conditioning B_t to exit from D at x_0.)

Proof: Let $\tau_k = \tau_{D_k}$ $f = h^{-1}$. Then

$$\mathbb{E}^x[f(X_{\tau_k})] = f(x) + \underbrace{\mathbb{E}^x\left[\int_0^\tau Af(X_s)\,ds\right]}_{=0}$$

$$\mathbb{E}^x[h^{-1}(X_{\tau_k})] = h(x) \quad \forall k$$

$\tau_k \uparrow \tau$, $\longrightarrow$ $X_{\tau_{n_k}} \to X_\tau = y \in \partial D$ a.s. $\xrightarrow{\text{def. of } h}$ $y = x_0$ (otherwise $\lim_k$ DNE).

$X_k = X_{\tau_k} \in \partial D_k \in D$; D bdd $\Big\} \Rightarrow x_{n_k} \to y \in \partial D$

Q1. generalization: $B_t \to X_t$? $\Delta h = 0 \Rightarrow Lh = 0$, $h > 0$? (maximal principle)

Q2. Apply it to compute $u(x) = \mathbb{E}^x[\phi(X_{\tau_D})]$ $\forall D \subseteq \mathbb{R}^n$ $(n >> 1)$.

or $\mathbb{R}^n \times \mathbb{R}^+$ — time

Exe 7.4. Let B_t be $\mathbb{R}^1$ Bm with $B_0 = x > 0$. Let $\tau = \inf\{t>0,\ B_t = 0\}$.

(a). Show that $\tau < \infty$ a.s., i.e. $\mathbb{P}^x(\tau<\infty) = 1$, for all $x>0$.

(b) Show that $\mathbb{E}^x[\tau] = \infty$, $\forall x > 0$.

Proof: (a) Let $D = (0,a)$ and $\tau_D = \inf\{t : B_t \notin D\}$ for $a > x$. Then, $\mathbb{P}(\tau<\infty) = \lim_{a\uparrow\infty} \mathbb{P}(B_{\tau_D} = 0)$

$= \lim_{a\uparrow\infty} 1 - \mathbb{P}(B_{\tau_D} = a) = 1$

for $\sigma_k = k \wedge \tau_{(0,a)}$, $\forall f \in C_0^2(\mathbb{R})$, $[0,a] \subseteq \text{supp}(f)$:

$$\mathbb{E}^x[f(B_{\sigma_k})] = f(x) + \mathbb{E}\left[\int_0^{\sigma_k} \tfrac{1}{2}\Delta f(B_s)\,ds\right],$$

Let $f(x) = \begin{cases} x & [-1, a+1] \\ \vdots & \\ 0 & [-2, a+2]^c \end{cases}$, $\mathbb{E}[B_{\tau_D}] = x + \mathbb{E}\cdot 0$ [first $\sigma_k = k$, then $k\uparrow\infty$].

$f \in C_0^2$; $\quad \mathbb{E}[B_{\tau_D}] = a\cdot\mathbb{P}(B_{\tau_D} = a) + 0 \quad \Rightarrow \mathbb{P}(B_{\tau_D} = a) = \frac{x}{a} \xrightarrow{a\uparrow\infty} 0$

(b) Let $f(x) = \begin{cases} x^2, & [-1, a+1], \\ \vdots & \\ 0 & [-2, a+2]^c \end{cases}$ Then, $(a+1)^2 \geq \mathbb{E}^x[f(B_{\sigma_k})] = x^2 + \mathbb{E}[\sigma_k]$, $\forall k$.

Sending $k\uparrow\infty$ (by Dominated convergence Thm):

$$\left.\begin{array}{l} \mathbb{E}^x[B_{\tau_D}^2] = x^2 + \mathbb{E}[\tau_D] \\ \mathbb{E}^x[B_{\tau_D}^2] = a^2\,\mathbb{P}(B_{\tau_D} = a) = ax \end{array}\right\} \Rightarrow \mathbb{E}[\tau_D] = x(a-x)$$

Note that $\tau_D \leq \tau$ by definition, $\forall a > x$. Thus, $\mathbb{E}[\tau] \geq \mathbb{E}[\tau_D] = x(a-x) \xrightarrow{a\uparrow\infty} \infty$. #