Chp 6 The Filtering problem

Problem statement.

Given state-space model (SSM) [hidden Markov Model HMM]

state Model, $dX_t = b(t,X_t)dt + \sigma(t,X_t)dU_t, \quad t\geq 0$ $\quad b: \mathbb{R}^+\times\mathbb{R}^n \to \mathbb{R}^n$ · Lipschitz

observation Model: $dZ_t = c(t,X_t)dt + \gamma(t,X_t)dV_t,$ $\quad \sigma: \mathbb{R}^+\times\mathbb{R}^n \to \mathbb{R}^{n\times p}$ · linear growth.

Data: $(Z_s,\ 0\leq s\leq t)$

Goal: BEST Estimator $\hat{X}_t$ "based on data."

Remark1, In applications, the continuous observations are

$$H_t = c(t,X_t) + \gamma(t,X_t)\,\underline{\underline{\dot{V}_t}}$$ —— "white noise", indpt of (U_t) & X_0

In math. $\to$ $Z_t = \int_0^t H_s ds \Rightarrow$ (OM).

Common practice: $\begin{cases} X_{n+1} = A_n(X_n) + B_n(X_n)\,U_n & \text{(discrete time).} \\ Z_n = C_n(X_n) + D_n(X_n)\,V_n & (Z_n \Leftrightarrow \frac{dZ_t}{dt}, \text{ abuse of notation}) \end{cases}$

Remark2, · the goal in math. $\hat{X}_t$ is $\mathcal{G}_t^Z$-measurable

· BEST in the sense that $\hat{X}_t = \arg\min \{\mathbb{E}|X_t - Y|^2 : Y \in K_t\}$

$$K_t = K(Z,t) = \{Y:\Omega\to\mathbb{R}^n;\ Y\in L^2(\mathbb{P}) \text{ and } Y \text{ is } \mathcal{G}_t^Z\text{-measurable}\}.$$

Thm. 6.1.2 $$\boxed{\hat{X}_t = P_{K_t}(X_t) = \mathbb{E}[X_t|\mathcal{G}_t^Z]}$$

Lemma 6.1.1 Let $\mathcal{H}\subseteq\mathcal{F}$ be a σ-algebra and $X\in L^2(\mathbb{P})$ be $\mathcal{F}$-measurable.

Let $N = \{Y\in L^2(\mathbb{P}): Y \text{ is } \mathcal{H}\text{-measurable}\}$. and P_N the orthogonal projection $L^2(\mathbb{P})\to N$.

Then $$P_N(X) = \mathbb{E}[X|\mathcal{H}] \longrightarrow \exists!\, Z \text{ s.t. } \begin{cases} Z \text{ is } \mathcal{H}\text{-meas} \\ \int_A Z\,d\mathbb{P} = \int_A X\,d\mathbb{P},\ \forall A\in\mathcal{H}. \end{cases}$$

Proof (based on definition of conditional Expectation).

$1_A \in L^2$, subspace $\text{span}\{1_A, A\in\mathcal{H}\}$

$P_N(X)$: · $P_N(X)\in\mathcal{H}$.

· $\int Y(X-P_N(X))d\mathbb{P} = 0,\ \forall Y\in N \xrightarrow{Y = 1_A,\ A\in\mathcal{H}} \int_A P_N(X)d\mathbb{P} = \int_A X\,d\mathbb{P},\ \forall A\in\mathcal{H}.$ #

$\Rightarrow$ explicit soln when linear SSM.

§6.2 The 1D linear filtering problem.

. n-D is similar. (Kalman-Bucy filter)

$$(SSM)\quad \begin{cases} dX_t = F_t X_t\,dt + C_t\,dU_t \\ dZ_t = G_t X_t\,dt + D_t\,dV_t \end{cases} \qquad \{U_t, V_t\} \text{ Bm. indpt.}$$

Assumption: · F, G, C, D are bdd on $[0,t]$; $\inf_{t\in[0,T]} D(t) > 0$.

· X_0 Gaussian indpt of $\{U_t\}$ & $\{V_t\}$.

Thm 6.2.8 (1D Kalman-Bucy filter) The solu. $\hat{X}_t = \mathbb{E}[X_t|\mathcal{G}_t^2]$ satisfies

$$\underline{d\hat{X}_t = \left(F_t - \frac{G_t^2 S_t}{D_t^2}\right)\hat{X}_t\,dt + \frac{G_t S_t}{D_t^2}\,dZ_t;\quad \hat{X}_0 = \mathbb{E}[X_0],}$$

where $S_t = \mathbb{E}[|X_t - \hat{X}_t|^2]$ satisfies the deterministic <u>Riccati Equation</u>.

$$\underline{\frac{dS}{dt} = 2F_t S(t) - \frac{G_t^2}{D_t^2} S^2(t) + C_t^2};\quad S(0) = \mathbb{E}|X_0 - \mathbb{E}[X_0]|^2.$$

Proof in 5 steps:

<u>Step1</u>: $\hat{X}_t = P_K(X_t) = \mathbb{E}[X_t|\mathcal{G}_t^2] = P_L(X_t)$: $L = \overline{\{c_0 + c_1 Z_{s_1} + \cdots + c_k Z_{s_k},\ s_j \le t,\ c_j \in \mathbb{R}\}}^{L^2(\mathbb{P})}$

i.e. the best Z-measurable estimate of X_t is the best Z-linear estimate.

<u>Step2</u>, Replace Z_t by the Innovation Process N_t:

$$N_t = Z_t - \int_0^t (GX)_s^\wedge\,ds,\qquad (GX)_s^\wedge = P_{L(Z,s)}(G_s X_s) = G_s \hat{X}_s$$

Then: $\begin{cases} N_t \text{ has orthogonal increments: } \mathbb{E}[(N_{t_1} - N_{s_1})(N_{t_2} - N_{s_2})] = 0, \text{ if } s_1 < t_1 \le s_2 < t_2 \\ L(N,t) = L(Z,t), \text{ so } \hat{X}_t = P_{L(N,t)}(X_t). \end{cases}$

<u>Step3</u>, R_t satisfying $dR_t = D(t)^{-1} dN_t$ is a 1D Bm; $L(N,t) = L(R,t)$

$$\hat{X}_t = P_{L(R,t)} \underset{\triangle}{=} \mathbb{E}[X_t] + \int_0^t \partial_s \mathbb{E}[X_t R_s]\,dR_s$$

<u>Step4</u> solve X_t from the (SM).

<u>Step5</u> 3+4 ⇒ an SDE of $\hat{X}_t$.

Example 1. (Static state: Discrete) (noisy observation of a const process).

$$\begin{cases} X_j \equiv X & \mathbb{E}X=0, \quad \mathbb{E}X^2=a^2 \\ Z_j = X + W_j & \mathbb{E}W_j=0, \quad \mathbb{E}W_j^2=m^2, \quad \text{iid.} \end{cases}$$

Q: The best estimate $\hat{X}_k$ that is linear in $\{Z_j : j\le k\}$; $k=1,\cdots$, streaming data.

$$\hat{X}_k = P_{\mathcal{L}(Z,k)}(X) = c_1 Z_1 + \cdots + c_k Z_k \qquad (\text{No need of } c_0, \text{ b.c. } \mathbb{E}X_0 = 0 = \mathbb{E}W_j)$$

Solution: IDEA 1: Gram-Schmidt procedure $\longrightarrow$ projection on O.N.B.

- $\{A_i\}$ s.t. $\begin{cases} \mathbb{E}[A_i A_j] = \delta_{ij}\cdot ; \\ \mathcal{L}(A,k) = \mathcal{L}(Z,k), \forall k. \end{cases}$ $\xrightarrow{\text{Then}}$ $\hat{X}_k = \sum_{j=1}^k c_j A_j$, $c_j = \dfrac{\mathbb{E}[XA_j]}{\mathbb{E}[A_j^2]}$.

- $A_1 = Z_1$, $A_2 = Z_2 - P_1(Z_2)$; $A_j = Z_j - P_{j-1}(Z_j), \cdots$

Note that $P_{j-1}(W_j) = 0 \longrightarrow \quad = Z_j - P_{j-1}(X+W_j) = Z_j - P_{j-1}(X) = Z_j - \hat{X}_{j-1}$

then,
$$\begin{cases} \mathbb{E}[XA_j] = \mathbb{E}[X(Z_j - \hat{X}_{j-1})] = \mathbb{E}[X(X-\hat{X}_{j-1})] \underset{\uparrow \text{ b.c. } \mathbb{E}[\hat{X}_{j-1}(X-\hat{X}_{j-1})]=0.}{=} \mathbb{E}[(X-\hat{X}_{j-1})^2] \\ \mathbb{E}[A_j^2] = \mathbb{E}[|X+W_j - \hat{X}_{j-1}|^2] = \mathbb{E}[|X-\hat{X}_{j-1}|^2] + \underbrace{\mathbb{E}[W_j^2]}_{=m^2} + \underbrace{2\mathbb{E}[(X-\hat{X}_{j-1})W_j]}_{=0} \end{cases}$$

Hence,
$$\boxed{\hat{X}_k = \hat{X}_{k-1} + \frac{\mathbb{E}[|X-\hat{X}_{k-1}|^2]}{\mathbb{E}[|X-\hat{X}_{k-1}|^2]+m^2}(Z_k - \hat{X}_{k-1}).}$$

$$d\hat{X}_t = \alpha_t \hat{X}_t\, dt + \beta_t\, dZ_t$$

. Rmk: · $\mathbb{E}|X-\hat{X}_{k-1}|^2$ can also be computed recursively. (*)

· A direct computation is tedious:

$$\hat{X}_1 = c_1 A_1 = c_1 Z_1, \quad c_1 = \frac{\mathbb{E}[XZ_1]}{\mathbb{E}Z_1^2} = \frac{a^2}{a^2+m^2} \longleftarrow \begin{cases} \mathbb{E}[XZ_1] = \mathbb{E}[X(X+W_1)] = a^2 \\ \mathbb{E}[Z_1^2] = a^2 + m \end{cases}$$

$$\hat{X}_2 = c_1 A_1 + c_2 A_2, \quad c_2 = \frac{\mathbb{E}[XA_2]}{\mathbb{E}A_2^2} = \frac{a^2}{2a^2+m^2}$$

$$= c_1 Z_1 + c_2(Z_2 - c_1 Z_1)$$

$$= c_1(1-c_2)Z_1 + c_2 Z_2$$

$$A_2 = Z_2 - \hat{X}_1 = Z_2 - c_1 Z_1 = (1-c_1)X + W_2 - c_1 W_1$$

$$\mathbb{E}[XA_2] = (1-c_1)a^2 = \frac{m^2}{a^2+m^2}a^2$$

$$\mathbb{E}[A_2^2] = (1-c_1)^2 a^2 + m^2(1+c_1^2)$$

$$= \frac{m^4}{(a^2+m^2)^2}a^2 + m^2\left(1+\frac{a^4}{(a^2+m^2)^2}\right)$$

$$= \frac{m^2}{(a^2+m^2)^2}\left[m^2a^2 + (a^2+m^2)^2 + a^4\right]$$

$$\cancel{(a^2+m^2)}(a^2+a^2+m^2)$$

$$= \frac{m^2}{a^2+m^2}(2a^2+m^2)$$

(*): $X - \hat{X}_{k+1} = X - \hat{X}_k - \dfrac{\beta_k}{\beta_k+m^2}(\overset{X+W_{k+1}}{Z_{k+1}} - \hat{X}_k)$

$$\Rightarrow \mathbb{E}|X-\hat{X}_{k+1}|^2 = \mathbb{E}|X-\hat{X}_k|^2 + \left(\frac{\beta_k}{\beta_k+m^2}\right)^2\left(\mathbb{E}|X-\hat{X}_k|^2+m^2\right) - 2\frac{\beta_k}{\beta_k+m^2}\mathbb{E}[(X-\hat{X}_k)(Z_{k+1}-\hat{X}_k)]$$

$$\beta_{k+1} = \beta_k + \left(\frac{\beta_k}{\beta_k+m^2}\right)^2(\beta_k+m^2) - 2\left(\frac{\beta_k}{\beta_k+m^2}\right)\cdot\beta_k = \beta_k - \frac{\beta_k^2}{\beta_k+m^2}.$$

- Simplified representation:

$$\hat{X}_k = \frac{a^2}{a^2+m^2/k}\bar{Z}_k, \qquad \bar{Z}_k = \frac{1}{k}\sum_{j=1}^k Z_j$$

(derivation: let $\alpha_k = .$ $U_k = \alpha_k \bar{Z}_k$. Then $\begin{cases} U_k \in \mathcal{L}(Z,k); \\ X-U_k \perp \mathcal{L}(Z,k); \end{cases} \Rightarrow \hat{X}_k = U_k$

$$\mathbb{E}[(X-U_k)Z_i] = \mathbb{E}[XZ_i - U_kZ_i] = a^2 - \alpha_k \frac{1}{k}\sum_j \mathbb{E}Z_jZ_i = a^2 - \alpha_k \frac{1}{k}(ka^2+m) = 0.\ )$$

Rmk: ① when k is large, $\hat{X}_k \approx \bar{Z}_k = X + \frac{1}{k}\sum_{j=1}^k W_j \to X$ a.s. by LLN. (CLT).
when k is small, m^2 is important; ($m^2 \uparrow \infty$, $\Rightarrow \hat{X}_k \to 0$.).

☆ ② The Gram-Schmidt $\Rightarrow$ $\underbrace{\text{orthogonal "new contributions" from new obs.}}$ A_j

Innovation process $N_t = Z_t - \int_0^t (GX)^\wedge_s ds$

linear estimate $\{N_t\}$.

IDEA0 (Regression) $\quad \hat{c} = \arg\min_{c\in\mathbb{R}^k} \mathbb{E}[|X - \sum_{i=1}^k c_i z_i|^2] \Rightarrow \partial_{c_j} \mathcal{E}(c) = 2\mathbb{E}[(X-\sum_{i=1}^k c_i z_i) z_j]$, $j=1\cdots k$

$$\sum_{i=1}^k \mathbb{E}[z_iz_j]\, c_i = \mathbb{E}[Xz_j]$$

$$A_k\vec{c} = \vec{b}_k \Rightarrow \vec{c} = A_k^{-1}\vec{b}_k$$

$$\left.\begin{aligned} \mathbb{E}[Z_iZ_j] &= \mathbb{E}[(X+W_i)(X+W_j)] = \mathbb{E}X^2 + \mathbb{E}[W_iW_j] \\ &= a^2 + m^2\delta_{ij}. \\ \mathbb{E}[XZ_j] &= a^2 \end{aligned}\right\}$$

$$\begin{pmatrix} a^2+m^2, & a^2 \cdots & a^2 \\ & a^2+m^2 & \\ & & \ddots \end{pmatrix}\vec{c} = \begin{pmatrix} a^2 \\ \vdots \\ a^2 \end{pmatrix}$$

$$(a^2+m^2)c_1 + a^2\sum_{i=2}^k c_i = a^2$$

$$\left.\begin{aligned} c_1 m^2 + a^2\sum_{i=1}^k c_i &= a^2 \\ c_1 = c_2 = \cdots &= c_k \end{aligned}\right\} \Rightarrow c_i^k = \frac{a^2}{ka^2+m^2}$$

$$\boxed{\hat{X}_k = \sum_{i=1}^k \hat{c}_i Z_i = \frac{a^2}{ka^2+m^2}\sum_{i=1}^k Z_i}$$

IDEA2 (Maximal a posterior)

$$\begin{aligned} p(x|z_1\cdots,z_k) &= \frac{p(z_1\cdots z_k|x)\,p(x)}{p(z_1\cdots z_k)} \\ &\propto \prod_{i=1}^k p(z_i|x)\ p(x) \\ &= \exp\left(-\frac{1}{2m^2}\sum_{i=1}^k |x-z_i|^2 - \frac{1}{2a^2}x^2\right) \end{aligned}$$

$$\frac{1}{m^2}\sum_{i=1}^k (x^2 + z_i^2 - 2xz_i) + \frac{1}{a^2}x^2$$

$$= \left(\frac{k}{m^2}+\frac{1}{a^2}\right)x^2 - 2x\frac{1}{m^2}\sum_{i=1}^k z_i + \cdots$$

$$= \frac{ka^2+m^2}{m^2a^2}\left(x - \frac{a^2}{ka^2+m^2}\sum_{i=1}^k z_i\right)^2 + \cdots$$

$$\hat{X}_k = \max_x p(x|z_1\cdots z_k) = \alpha_k \bar{Z}_k$$

Rmk Extension of IDEA0 & 2 to continuous time? General linear Eqn.? $\to$ recursive ☐ is better.

Proof of Thm.

Step 1 Z-linear & Z-measurable estimates.

Lemma 6.2.2 Let $X, (Z_s)_{s\le t}$ be r.v. in $L^2(\mathbb{P})$ and assume that

$$(X, Z_{s_1}, Z_{s_2}, \cdots, Z_{s_n}) \in \mathbb{R}^{n+1}$$

has normal distribution $\forall s_1, s_2, \cdots, s_n \le t$, $n \ge 1$. (i.e. Gaussian process)

Then $$P_{\mathcal{L}}(X) = \mathbb{E}[X|\mathcal{G}^Z] = P_{\mathcal{K}}(X).$$

i.e. the best Z-linear estimate for X coincides w/ the best Z-meas. estimate.

Proof: Put $\tilde{X} = X - P_{\mathcal{L}}(X)$. Then $\mathbb{E}[\tilde{X} Z_{s_j}] = 0$, $\forall Z_{s_j}$; $\Rightarrow$ $\tilde{X}$ indpt of (Z_{s_j}).

normal $\Longrightarrow$ $\tilde{X}, Z_{s_j}$ is normal, $\forall s_j$ $\Rightarrow$ $\tilde{X}$ indpt of $\mathcal{G}^Z$.

Then, $$\mathbb{E}[\tilde{X} 1_G] = \mathbb{E}[1_G]\,\mathbb{E}[\tilde{X}] = 0, \ \forall G \in \mathcal{G}^Z \Rightarrow P_{\mathcal{L}}(X) = \mathbb{E}[X|\mathcal{G}^Z].$$

$$\mathbb{E}[\tilde{X} 1_G] = \mathbb{E}[X 1_G] - \mathbb{E}[P_{\mathcal{L}}(X) 1_G]$$

#

Lemma 6.2.3 $M_t = \begin{pmatrix} X_t \\ Z_t \end{pmatrix} \in \mathbb{R}^2$ is a Gaussian Process.

Proof: From the SSM: $$dM_t = H(t) M_t\, dt + K(t)\, dB_t, \quad M_0 = \begin{pmatrix} X_0 \\ 0 \end{pmatrix}$$

$H(t), K(t) \in \mathbb{R}^{2\times 2}$; B_t: $\mathbb{R}^2$-Bm. $M_s^{(0)} \equiv 0$

By Picard iteration, $$M_t^{(n+1)} = M_0 + \int_0^t H(s) M_s^{(n)} ds + \int_0^t K(s)\, dB_s, \quad n = 0, \cdots$$

Thus, $(M_t^{(n)})$ is GP $\forall n$; $M_t^{(n)} \to M_t$ in $L^2(\mathbb{P})$ $\Rightarrow$ (M_t) is Gaussian

$$\Rightarrow \quad P_{\mathcal{L}(Z,t)}(X) = \mathbb{E}[X|\mathcal{G}_t^Z]$$

Lemma 6.2.4 $$\mathcal{L}(Z,t) = \{ c_0 + \int_0^t f(s)\, dZ_s,\ f \in L^2[0,t],\ c_0 \in \mathbb{R} \}.$$

Proof: a) RHS $\subseteq L(Z,t)$ = closure in $L^2(\mathbb{P})$ of linear combinations.

b) RHS contains all linear combinations.

c) RHS is closed b.c. $L^2[0,t]$ is complete.

Rmk: $\hat{X}_t = P_{\mathcal{L}(Z,t)}(X) = c_0(t) + \int_0^t f(s)\, dZ_s \xrightarrow{\mathbb{E} Z_s = 0} c_0(t) = \mathbb{E}[\hat{X}_t] = \mathbb{E}[\mathbb{E}[X|\mathcal{G}_t^Z]] = \mathbb{E}X.$

Step 2 The innovation process.

Let $\quad N_t = Z_t - \int_0^t (GX)^\wedge_s \, ds, \qquad (GX)^\wedge_s = P_{\mathcal{L}(Z,s)}(G(s)X_s) = G(s)\underbrace{P_{\mathcal{L}(Z,s)}(X)}_{\hat{X}_s}$

Then $\quad dN_t = dZ_t - G(t)\hat{X}_t \, dt$

$\qquad = G(t)(X_t - \hat{X}_t)dt + D(t)dV_t$

Lemma 6.2.5 (i) N_t has orthogonal increments

(ii) $\mathbb{E}[N_t^2] = \int_0^t D^2(s)ds$

(iii) $\mathcal{L}(N,t) = \mathcal{L}(Z,t)$

(iv) N_t is a Gaussian process.

Proof: (i) If $s<t$, $\quad N_t - N_s \perp N_s$? $\quad \forall Y \in \mathcal{L}(Z,s)$

$$\mathbb{E}[(N_t - N_s)Y] = \mathbb{E}\left[\left(\int_s^t G(r)(X_r - \hat{X}_r)dr + \int_s^t D(r)dV_r\right)Y\right]$$

$$= \int_s^t G(r)\underbrace{\mathbb{E}[(X_r - \hat{X}_r)Y]}_{=0}dr + \underbrace{\mathbb{E}\left[\int_s^t D(r)dV_r \, Y\right]}_{=0}$$

$=0 \quad X_r - \hat{X}_r \perp \mathcal{L}(Z,r) \supset \mathcal{L}(Z,s)$, $\qquad V$ has indep't increments

#

(ii) Ito formula with $g(x) = x^2$:

$$dN_t^2 = 2N_t dN_t + \tfrac{1}{2}2D(t)^2 dt \qquad \mathbb{E}[AB]=0$$

$$\mathbb{E} N_t^2 = \mathbb{E}\int_0^t 2N_s dN_s + \int_0^t D(s)^2 ds \qquad \int_0^t N_s dN_s = \lim \sum N_{t_i}(N_{t_{i+1}} - N_{t_i})$$

$$= 0 + \int_0^t D(s)^2 ds. \qquad \Rightarrow \mathbb{E}\int_0^t N_s dN_s = 0.$$

(iii) clearly, $\mathcal{L}(N,t) \subseteq \mathcal{L}(Z,t)$. Need the opposite direction: $Z_t \overset{??}{=} \alpha(t) + \int_0^t f(s)dN_s$?

By Lemma 6.2.4: $\quad (GX)^\wedge_r = c(r) + \int_0^r g(r,s)dZ_s$, for some $g(r,\cdot) \in L^2[0,r]$, $c(r) \in \mathbb{R}$.

Then, $\quad \int_0^t f(s)dN_s = \int_0^t f(s)dZ_s - \int_0^t f(r)(GX)^\wedge_r dr \qquad \int_0^t f(r)\int_0^r g(r,s)dZ_s dr = \int_0^t \int_s^t f(r)g(r,s)dr \, dZ_s$

$$= \int_0^t \underbrace{\left[f(s) - \int_s^t f(r)g(r,s)dr\right]}dZ - \int_0^t f(r)c(r)dr.$$

Theory of Volterra integral eq: $\forall h \in L^2[0,t]$, $\exists f \in L^2[0,t]$, s.t.

$$f(s) - \int_0^t f(r)g(r,s)dr = h(s)$$

Let $h(s) = \mathbb{1}_{[0,t_1]}(s)$, $0 \le t_1 \le t$, then $\exists f$ (depending on t_1).

$$\int_0^t f(s)dN_s + \int_0^t f(r)c(r)dr = \int_0^t h(s)dZ_s = Z_{t_1} \qquad \#$$

(iv): clear.

Step3. The innovation process and Bm.

Fact: $R_t = \int_0^t D(s)^{-1} dN_s$ is a Bm.

(Proof: $dR_t = \frac{1}{D(t)} dN_t = \frac{1}{D(t)} G(t)(X_t - \hat{X}_t)dt + dV_t$

(i) cts path; ✓ (ii) orthogonal increments (since N_t has) ✓

(iii) R_t is Gaussian ✓; (iv) $\mathbb{E}R_t = 0$, $\mathbb{E}[R_t R_s] = t \wedge s$.

$dR_t^2 = 2R_t dR_t + dt \Rightarrow \mathbb{E}R_t^2 = \mathbb{E}\underbrace{\int_0^t 2R_s dR_s}_{=0} + t = t.$

$\mathbb{E}[R_t R_s] \overset{t>s}{=} \underbrace{\mathbb{E}[(R_t - R_s)R_s]}_{=0} + \mathbb{E}[R_s^2] = s.$)

Lemma 6.2.7. $\hat{X}_t = \mathbb{E}[X_t] + \int_0^t \partial_s \mathbb{E}[X_t R_s]\, dR_s$ (*)

Proof: $L(N,t) = L(Z,t)$ and Lemma 6.2.4 $\Rightarrow$ $\hat{X}_t = c_0(t) + \int_0^t g(s) dR_s$, $g \in L^2[0,t]$, $c_0(t) \in \mathbb{R}$

$\quad\;\, \| $

$\mathcal{L}(R,t)$

· $c_0(t) = \mathbb{E}[\hat{X}_t] = \mathbb{E}[X_t]$

· $X - \hat{X}_t \perp \int_0^t f(s) dR_s$, $\forall f \in L^2[0,t]$

$\Rightarrow \mathbb{E}[X_t \int_0^t f(s) dR_s] = \mathbb{E}[\hat{X}_t \int_0^t f(s) dR_s] = \mathbb{E}[\int_0^t g(s) dR_s \int_0^t f(s) dR_s] = \int_0^t g(s) f(s) ds$, $\forall f$

$\xrightarrow{f(s) = 1_{[0,r]}(s),\ r \le t} \mathbb{E}[X_t R_r] = \int_0^r g(s) ds$ i.e. $g(r) = \partial_r \mathbb{E}[X_t R_r]$. #

Step4. Explicit solu. for X_t from the SM:

$$X_t = e^{\int_0^t F(s)ds} X_0 + \int_0^t e^{\int_s^t F(u)du} C(s) dU_s;$$

$$X_t = e^{\int_r^t F(s)ds} X_r + \int_r^t e^{\int_s^t F(u)du} C(s) dU_s.$$

$c_0(t) = \mathbb{E}X_t = e^{\int_0^t F(s)ds} \mathbb{E}X_0.$

Step5. SDE for $\hat{X}_t$: $\hat{X}_t = \mathbb{E}X_t + \int_0^t f(s,t) dR_s$ · $f(s,t) = \partial_s \mathbb{E}[X_t R_s]$.

To do: explicit expression for $f(s,t)$: $R_s = \int_0^t \frac{G(r)}{D(r)} (X_r - \hat{X}_r) dr + V_s$

$\mathbb{E}[X_t R_s] = \int_0^s \frac{G(r)}{D(r)} \mathbb{E}[X_t (X_r - \hat{X}_r)] dr + 0$ ✓ (V_s) indpt. of $X_t \in \{(U_s)$ and $X_0\}$

· $= \mathbb{E}[e^{\int_r^t F(u)du} \tilde{X}_r \tilde{X}_r]$ ✓ $(U_s)_{s \ge r}$ indpt $X_r, \mathcal{G}_r^Z \ni \tilde{X}_r$

· $X_r = \tilde{X}_r + \hat{X}_r$, $\mathbb{E}[X_r \tilde{X}_r] = \mathbb{E}(\tilde{X}_r)^2 = S(r)$

$$= \int_0^s \frac{G(r)}{D(r)} e^{\int_r^t F(u)du} S(r)\, dr$$

$$\Rightarrow \quad f(s,t) = \frac{G(s)}{D(s)} e^{\int_s^t F(u)du} S(s).$$

<u>Claim:</u> $S(t)$ satisfies the deterministic ODE.

$$S'(t) = 2F(t)S - \frac{G^2}{D^2}S^2 + C^2$$

Proof (basic.)

SDE for $\hat{X}_t$: $d\hat{X}_t = C_0'(t)dt + f(t,t)dR_t + \left(\int_0^t \partial_t f(s,t)\,dR_s\right)dt$

$$\int_0^u \int_0^t \partial_t f(s,t)\,dR_s\,dt = \int_0^u \int_s^u \partial_t f(s,t)\,dt\,dR_s$$

$$= \int_0^u [f(s,u) - f(s,s)]dR_s$$

$$= \hat{X}_u - C_0(u) - \int_0^u f(s,s)\,dR_s$$

$f(t,t) = \frac{G}{D}S(t)$;

$\partial_t f(s,t) = \frac{G}{D}S(s)F(t)e^{\int_s^t F(v)dv} = F(t)f(s,t)$

$\int_0^t \partial_t f(s,t)dR_s = F(t)\int_0^t f(s,t)\,dR_s = F(t)[\hat{X}_t - C_0(t)]$

$= C_0'(t)dt + F(t)[\hat{X}_t - C_0(t)]dt + \frac{G}{D}S(t)\,dR_t$ $\qquad C_0'(t) = F(t)C_0(t)$

$= F(t)\hat{X}_t dt + \frac{G}{D}S(t)\,dR_t$

$$dR_t = \frac{1}{D(t)}[dZ_t - G(t)\hat{X}_t dt]$$

$$\Rightarrow \quad d\hat{X}_t = \left[F(t) - \frac{G^2}{D^2}S(t)\right]\hat{X}_t dt + \frac{G(t)S(t)}{D^2(t)}dZ_t.$$

#.

<u>Example</u> 6.2.9 (Noisy observation of a constant process)

$$\begin{cases} dX_t = 0, \quad X_t \equiv X_0, \quad \mathbb{E}[X_0] = \hat{X}_0, \quad \mathbb{E}[|X_0 - \hat{X}_0|^2] = a^2 \\ dZ_t = X_t dt + m\,dV_t, \quad Z_0 = 0, \end{cases}$$

$H_j = X + W_j$

$$\Leftrightarrow \quad H_t = \frac{dZ}{dt} = X_t + m\dot{V}_t$$

- Riccati equation for $S(t) = \mathbb{E}[|X_t - \hat{X}_t|^2]$: $\qquad F \equiv 0, \quad C \equiv 0$

$$\frac{dS}{dt} = -\frac{1}{m^2}S^2; \quad S(0) = a^2 \qquad G \equiv 1, \quad D(t) \equiv m.$$

$\Rightarrow S(t) = \frac{a^2m^2}{m^2 + a^2t}$ $\qquad \leftarrow$ $S^{-2}dS = -\frac{1}{m^2}dt$

$d\hat{X}_t = -\frac{a^2}{m^2 + a^2t}\hat{X}_t dt + \frac{a^2}{m^2+a^2t}dZ_t$ $\qquad -(S^{-1})\Big|_0^t = -t/m^2$

$-S(t)^{-1} = -S(0)^{-1} - t/m^2$

$\Rightarrow \underline{\hat{X}_t = \frac{m^2}{m^2+a^2t}\hat{X}_0 + \frac{a^2}{m^2+a^2t}Z_t, \ t \geq 0.}$ $\qquad S(t) = \frac{1}{S(0)^{-1} + t/m^2} = \frac{a^2m^2}{a^2t + m^2}$