Chp 5 SDE

1. Examples
2. Existence + Uniqueness
3. weak / strong solu.

HW: 5.1 (ii) & (iii)
5.7; 5.16(c).

$$dX_t = b(t,X_t)\,dt + \sigma(t,X_t)\,dB_t$$

§5.1 Examples.

<u>Example</u> (Ornstein-Uhlenbeck process) (Langevin equation in physics).

$$dX_t = \mu X_t dt + \sigma dB_t \qquad Y_t = e^{-\mu t} X_t$$

solution: $X_t = e^{\mu t} X_0 + \sigma \int_0^t e^{\mu(t-s)} dB_s$ $\qquad dY_t = -\mu e^{-\mu t} X_t\, dt + e^{-\mu t}(\mu X_t dt + \sigma dB_t)$

- $X_t | X_0 \sim N\left(e^{\mu t} X_0,\ \frac{-1}{2\mu}\sigma^2(1-e^{2\mu t})\right)$; $\mathbb{E}[X_t X_s] = e^{\mu(t-s)} \mathbb{E}[X_s^2]$
- stationary: if $\mu<0$, $N(0, \frac{-1}{2\mu}\sigma^2)$ is the stationary distribution; ergodic.
- Moment closure: $M_n(t) = \mathbb{E}[X_t^n]$;
  $M_n(t)' = \mu n M_n + \frac{1}{2}\sigma^2 n(n-1) M_{n-2}$
  $d X_t^n = n X_t^{n-1} dX_t + \frac{1}{2} n(n-1) X_t^{n-2} \cdot \sigma^2 dt$
  $= n X_t^n \mu dt + n(n-1) X_t^{n-2} \frac{\sigma^2}{2} dt + \cdots dB_t$
- $dX_t = (\mu X_t + m(t))\,dt + \sigma dB_t$:
  - $m(t) \equiv m$: $d(X_t + \frac{m}{\mu}) = \mu(X_t + \frac{m}{\mu})dt + \sigma dB_t$ $\qquad X_t + \frac{m}{\mu} = e^{\mu t}(X_0 + \frac{m}{\mu}) + \sigma\int_0^t e^{\mu(t-s)} dB_s$
    $\Rightarrow \mu = -1,\ m(t) \equiv m$: mean-reserving: $\mathbb{E} X_t - m = e^{\mu t}\cdot 0 + 0$.
  - $m(t)$ non-constant: $Y_t = e^{-\mu t} X_t \Rightarrow dY_t = \cancel{-\mu e^{-\mu t} X_t dt} + e^{-\mu t}[\cancel{\mu X_t dt} + m(t)dt + \sigma dB_t]$
    $= e^{-\mu t} m(t)\,dt + e^{-\mu t}\sigma dB_t$
    $Y_t = Y_0 + \int_0^t e^{-\mu s} m(s)\,ds + \sigma \int_0^t e^{-\mu s} dB_s$
    $\Rightarrow X_t = e^{\mu t} X_0 + \int_0^t e^{\mu(t-s)}[m(s)ds + \sigma dB_s]$.

- Is $Y_t = e^{-t} B_{e^{2t}}$ an OU process? By definition of FDD.

Example 2. $LQ'' + RQ' + C^{-1}Q = G_t + \alpha W_t$ $\quad\downarrow\quad X_t = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} Q_t \\ Q_t' \end{pmatrix}$

$$\begin{cases} X_1' = X_2 \\ LX_2' = -RX_2 - C^{-1}X_1 + G_t + \alpha W_t \end{cases}$$

$$\begin{pmatrix} dX_1 \\ dX_2 \end{pmatrix} = \begin{pmatrix} 0 & L^{-1} \\ -\frac{1}{CL} & -L^{-1}R \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + \begin{pmatrix} 0 \\ L^{-1}G_t \end{pmatrix} + \begin{pmatrix} 0 \\ \alpha L^{-1} \end{pmatrix} dB_t$$

$$dX = AX\,dt + H(t)\,dt + K\,dB_t$$

$$d(e^{-At}X) = -A\cancel{e^{-At}X\,dt} + e^{-At}(\cancel{AX_t\,dt} + H\,dt + K\,dB)$$

$$X(t) = e^{At}X(0) + e^{At}\int_0^t e^{-As}(H(s)\,ds + K\,dB_s)$$

Example 3. Linear SDEs; $\quad dX_t = -AX_t\,dt + \sigma\,dW_t \qquad \overset{PD}{A > 0},\ \Sigma \geq 0$ PSD

$$\Big(dX_i(t) = -\sum_j A_{ij} X_j\,dt + \sum_j \sigma_{ij}\cdot dW_j(t),\quad i = 1, \dots, n \Big). \qquad \mathbb{R}^{n\times n} \quad \Sigma = \sigma\sigma^T$$

$$\Rightarrow \quad X_t = e^{-At}X_0 + \int_0^t e^{-A(t-s)}\sigma\,dW_s$$

- $\mathbb{E}[X_t] = e^{-At}\,\mathbb{E}X_0$ $\qquad$ Assume $\mathbb{E}X_0 = 0$
- $R(t,s) = \mathbb{E}[X_t X_s^T] = \mathrm{Cov}(X_t, X_s)$ $\qquad \mathbb{R}^{n\times n}$ Auto-correlation matrix.

$$= \mathbb{E}\Big[\Big(e^{-A(t-s)}X_s + \int_s^t e^{-A(t-r)}\sigma\,dW_r\Big)X_s^T\Big] = e^{-A(t-s)}\mathbb{E}[X_s X_s^T]$$

$$R(t,s) = \mathbb{E}\Big[\Big(e^{-At}X_0 + \int_0^t e^{-A(t-r)}\sigma\,dW_r\Big)\Big(e^{-As}X_0 + \int_0^s e^{-A(s-u)}\sigma\,dW_u\Big)^T\Big]$$

$$= e^{-At}\,\mathbb{E}[X_0X_0^T]\,e^{-A^Ts} + \int_0^t\int_0^s e^{-A(t-r)}\,\mathbb{E}[\sigma\,dW_r\,dW_u^T\sigma^T]\,e^{-A^T(s-u)}$$

$\Sigma(t) = R(t,t)$

$$= e^{-At}R_0e^{-A^Ts} + e^{-At}\int_0^{t\wedge s} e^{Ar}\,\Sigma\, e^{A^Tr}\,dr\; e^{-A^Ts}$$

- $\frac{d}{dt}\Sigma(t) = -A\Sigma(t) - \Sigma(t)A^T + \Sigma$
- $\Sigma(\infty) = \int_0^\infty e^{Ar}\,\Sigma\, e^{A^Tr}\,dr$ $\qquad$ covariance of invariant measure.

Example 4 (Stochastic Heat equation)

$$\begin{cases} \partial_t u = \partial_x^2 u + \partial_t W(x,t) & [0,\pi] \\ u(0,t) = u(\pi,t) = 0; \end{cases} \qquad W(x,t) = \sum_{k=1}^{\infty} e_k(x) W_k(t)$$

$u(x,t) = \sum_{k=1}^{\infty} u_k(t) e_k(x)$, $\quad du_k(t) = -k^2 u_k(t)\,dt + dB_k(t)$, $k \geq 1$. $\quad e_k(x) = \sin(kx)$

$$d\vec{u} = \underbrace{\begin{pmatrix} \ddots & & \\ & -k^2 & \\ & & \ddots \end{pmatrix}}_{A} \vec{u}\,dt + dB(t) \qquad \partial_{xx} e_k(x) = -k^2 e_k(x)$$

Exp 5 (The Brownian Bridge) from a to b:

$$\begin{cases} dY_t = \frac{b - Y_t}{1-t}\,dt + dB_t \;; & 0 \leq t < 1. \\ Y_0 = a \end{cases}$$

Verify that $Y_t = a(1-t) + bt + (1-t)\int_0^t \frac{1}{1-s}\,dB_s$, $0 \leq t < 1$ solve the equ. & $\lim_{t \nearrow 1} Y_t = b$ a.s.

Proof: $\quad Y_0 = a$; ✓ $\qquad (1-t)^{-1}\left(Y_t - a(1-t) - bt\right) = \int_0^t \frac{1}{1-s}\,dB_s$

$$(1-t)^{-2}(-1)[\;]\,dt + (1-t)^{-1}[dY_t + (a-b)dt] = (1-t)^{-1} dB_s$$

$$(1-t)^{-1}(-1)[Y_t - a(1-t) + bt] + [dY_t + (a-b)dt] = dB_s$$

$$dY_t - \frac{b - Y_t}{1-t}\,dt = dB_t.$$

To show that $\lim_{t \nearrow 1} Y_t \overset{a.s.}{=} b$, it suffices to show that $(1-t)\int_0^t (1-s)^{-1} dB_s \to 0$.

Note: $\quad \mathbb{E}\left|\int_0^t (1-s)^{-1} dB_s\right|^2 = \int_0^t (1-s)^{-2} ds \overset{r=1-s}{=} \int_{1-t}^{1} r^{-2} dr = (1-t)^{-1} - 1$

$$\mathbb{E}\left[(1-t)^2 \left|\int_0^t (1-s)^{-1} dB_s\right|^2\right] = (1-t)[1-(1-t)] = (1-t)t \to 0. \qquad \#$$

Other ~~examples~~ (Nonlinear) $\qquad$ Existence & Uniqueness ?

Cox-Ingersoll-Ross (CIR) $\quad dX_t = \alpha(b - X_t)\,dt + \sigma\sqrt{X_t}\,dB_t$

Stochastic Verhulst (population) $\quad dX_t = (\lambda X_t - X_t^2)\,dt + \sigma X_t\,dB_t$

Coupled Lotka-Volterra: $\quad dX_i(t) = X_i(t)\left(a_i + \sum_j b_{ij} X_j\right)dt + \sigma_i X_i\,dW_i(t)$

Protein kinetics: $\quad dX_t = (\alpha - X_t + \lambda X_t(1-X_t))\,dt + \sigma X_t(1-X_t) \circ dW_t$

Tracer particle (turbulent diffusion) $\quad \begin{cases} dX_t = u(X_t, t)\,dt + \sigma\,dW_t\,; \\ \nabla \cdot u(x,t) = 0. \end{cases}$

Example 1' (Geometric Bm). $dN_t = rN_t\,dt + \alpha N_t\,dB_t$, $r>0$, $\alpha>0$ constant

sol: ① $N_t \equiv 0$ ✓

② $N_t > 0$ $(N_0>0)$: $\quad \frac{dN_t}{N_t} = r\,dt + \alpha\,dB_t$

Let $g(t,x) = \ln x$. $\quad d\ln N_t = \frac{1}{N_t}dN_t + \left(-\frac{1}{N_t^2}\alpha^2 N_t^2\,dt\right)$

$$= r\,dt + \alpha\,dB_t - \frac{1}{2}\alpha^2\,dt$$

$\int_0^t \Rightarrow \quad \ln\frac{N_t}{N_0} = \ln N_t - \ln N_0 = (r-\frac{1}{2}\alpha^2)t + \alpha B_t$

$\Rightarrow \quad N_t = N_0\, e^{(r-\frac{1}{2}\alpha^2)t + \alpha B_t}$ $\quad \mathbb{E}\,e^{\alpha B_t} = e^{\frac{1}{2}\alpha^2 t}$

- N_0 indpt of B_t: $\quad \mathbb{E}[N_t] = \mathbb{E}[N_0]\,e^{rt} \longrightarrow +\infty$ as $t\uparrow\infty$.

Q: Does this mean if $r>0$, $N_t \uparrow \infty$ a.s.? No

- $\begin{cases} N_t \xrightarrow{t\uparrow\infty} \infty \text{ a.s.} & \text{if } r > \frac{1}{2}\alpha^2 \\ N_t \xrightarrow{t\uparrow\infty} 0 \text{ a.s.} & \text{if } r < \frac{1}{2}\alpha^2 \\ N_t \sim \pm\infty & \text{if } r = \frac{1}{2}\alpha^2 \end{cases}$

b.c. the Law of iterated Logarithm $\quad \limsup_{t\to\infty} \frac{B_t}{\sqrt{2t\log\log t}} = 1$ a.s.

(For any i.i.d seq. $\{\xi_i\}_{i=1}^\infty$, $S_n = \sum_{i=1}^n \xi_i$

Exe 5.16 (Integrating factor method for non linear equ. with linear multiplicative noise.)

$$dX_t = f(t, X_t)\,dt + C(t) X_t\,dB_t; \quad X_0 = x$$

(a) Let $F_t = e^{-\int_0^t C_s dB_s + \frac{1}{2}\int_0^t C_s^2 ds} = e^{Z_t}$

Then $dF_t = F_t (dZ_t + \langle dZ_t \rangle) = F_t (- C_t dB_t + C_t^2 dt)$

$$d(F_t X_t) = X_t dF_t + F_t dX_t + dF\,dX_t$$

$$= X_t F_t (\cancel{-C_t dB_t + C_t^2 dt}) + F_t (f(t,X_t) dt + \cancel{C_t X_t dB_t}) + F_t (\cancel{-C_t^2 X_t dt})$$

$Y_t = F_t X_t$ $\quad = F_t f(t, X_t) dt$

$\Rightarrow$ $\underline{dY_t = F_t f(t, F_t^{-1} Y_t) dt}$ ODE ✓

$$Y_t = Y_0 + \int_0^t F_s f(s, F_s^{-1} Y_s) ds ;$$

$$Y_0 = F_0 X_0 = X_0 ;$$

$$X_t = F_t^{-1} X_0 + F_t^{-1} \int_0^t F_s f(s, X_s) ds$$

(C). Solve $dX_t = X_t^{-1} dt + \alpha X_t dB_t ;\ X_0 = x > 0.$ α constant.

Sdu: $F_t = e^{-\alpha B_t + \frac{1}{2}\alpha^2 t}$; $Y_t = F_t X_t$ $\quad f(t,X) = x^{-1}$

$$dY_t = F_t (F_t^{-1} Y_t)^{-1} dt = F_t^2 Y_t^{-1} dt$$

$$Y_t = \left(2\int_0^t F_s^2 ds + Y_0^2\right)^{\frac{1}{2}}$$

$$\boxed{X_t = F_t^{-1} Y_t = e^{\alpha B_t - \frac{1}{2}\alpha^2 t}\left(x^2 + 2\int_0^t F_s^2 ds\right)^{\frac{1}{2}}}$$

| $yy' = F_t^2$
| $\frac{1}{2}(y^2 - y_0^2) = \int_0^t F_s^2 ds$
| $y = \sqrt{2\int_0^t F_s^2 ds + y_0^2}$

(d) $dX_t = X_t^{\gamma} dt + \alpha X_t dB_t ;\quad X_0 = x > 0$

Sdu. $dY_t = F_t (F_t^{-1} Y_t)^{\gamma} dt = F_t^{1-\gamma} Y_t^{\gamma} dt$

$$Y_t = \left[x^{1-\gamma} + (1-\gamma)\int_0^t F_s^{1-\gamma} ds\right]^{\frac{1}{1-\gamma}}$$

$$X_t = F_t^{-1} Y_t = e^{\alpha B_t - \frac{1}{2}\alpha^2 t} \Big[\qquad \Big]^{\frac{1}{1-\gamma}}$$

| $y^{-\gamma} y' = F_t^{1-\gamma}$
| $\frac{1}{1-\gamma}(y^{1-\gamma} - y_0^{1-\gamma}) = \int_0^t F_s^{1-\gamma} ds$
$y^{1-\gamma} = y_0^{1-\gamma} + (1-\gamma)\int_0^t \ldots$
$y = (y_0^{1-\gamma} + (1-\gamma)\int_0^t \quad]^{\frac{1}{1-\gamma}}$

· If $\boxed{\gamma > 1,}$ then $1-\gamma < 0$, and $\exists\, t_* \ s.t$

$$x^{1-\gamma} + (1-\gamma)\int_0^{t_*} F_s^{1-\gamma} ds = 0 \quad , \text{ blow up.}$$

· $\alpha = 0,\ \gamma = \frac{2}{3}$: $\quad dX_t = 3 X_t^{2/3} dt ;\ X_0 = 0 ;\quad X_t^a = \begin{cases} 0 & t \le a \\ (t-a)^3, & t > a \end{cases},\ \forall a > 0 .$

§5.2 Existence & Uniqueness

<u>Thm 5.2.1</u> The SDE $\quad dX_t = b(t,X_t)\,dt + \sigma(t,X_t)\,dB_t$, $\quad 0\le t\le T$, $X_0 = Z \qquad (1)$

has a unique t-cts sln. X_t $\begin{cases} \text{adapted to } \mathcal{F}_t^{Z,B} \leftarrow \{Z, B_s, s\le t\} \\ \mathbb{E}[\int_0^T X_t^2\,dt] < \infty \end{cases}$

if $b : [0,T]\times\mathbb{R}^n$, $\sigma : [0,T]\times\mathbb{R}^n \to \mathbb{R}^{n\times m}$ are measurable fns satisfying

$\begin{cases} \text{linear growth: } |b(t,x)| + |\sigma(t,x)| \le C(1+|x|), \quad \forall t\in[0,T],\ x\in\mathbb{R}^n \\ \text{global Lipschitz: } |b(t,x)-b(t,y)| + |\sigma(t,x)-\sigma(t,y)| \le D|x-y|, \ \cdots \end{cases}$

for some constants $C>0$, $D>0$.

<u>Rmk</u>: Intuition from Deterministic Equ.

a). $\frac{dX_t}{dt} = X_t^2$; $X_0 = 1$; $\qquad \frac{dx}{x^2} = dt \Rightarrow -\frac{1}{x}\Big|_0^t = t \Rightarrow -\frac{1}{x(t)} + 1 = t$, $x(t) = \frac{1}{1-t}$, $0\le t<1$

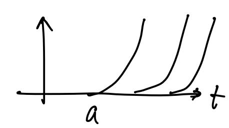
$\Rightarrow$ No global sln. on $[0,\infty)$. $\qquad b(x) = x^2$; NOT linear growth.

linear growth ensures the sln. does NOT explode in finite time $\Rightarrow$ global sln.

b). $\frac{dX_t}{dt} = 3X_t^{2/3}$; $X_0 = 0$ has many solutions. $\quad X_t^a = \begin{cases} 0 & \text{for } t\le a \\ (t-a)^3 & \text{for } t>a \end{cases}$, $\forall a>0$

$b(x) = 3x^{2/3}$, $b'(x) = 2x^{-\frac{1}{3}}$, NOT Lipschitz at $x=0$

(Global) Lipschitz condition ensures !.



Proof of Thm 5.1:

1. Uniqueness (by Lipschitz). Let X_t and $\hat{X}_t$ be two cts sln. w/ initial condition X_0

i.e.
$$X_t = X_0 + \int_0^t b(s,X_s)\,ds + \int_0^t \sigma(s,X_s)\,dB_s \qquad (*)$$
$$\hat{X}_t = X_0 + \int_0^t b(s,\hat{X}_s)\,ds + \int_0^t \sigma(s,\hat{X}_s)\,dB_s$$

$\Delta b(s) = b(s,X_s) - b(s,\hat{X}_s)$

$\Delta\sigma(s) = \sigma(s,X_s) - \sigma(s,\hat{X}_s)$

$$\begin{aligned}
\mathbb{E}|X_t - \hat{X}_t|^2 &= \mathbb{E}\left|\int_0^t \Delta b\,ds + \int_0^t \Delta\sigma\,dB_s\right|^2 \\
&\le 2\,\mathbb{E}\left|\int_0^t \Delta b\,ds\right|^2 + 2\,\mathbb{E}\left|\int_0^t \Delta\sigma\,dB_s\right|^2 \qquad (a+b)^2 \le 2(a^2+b^2) \\
&\le 2t\,\mathbb{E}\int_0^t |\Delta b(s)|^2\,ds + 2\,\mathbb{E}\int_0^t |\Delta\sigma(s)|^2\,ds \\
&\le 2(1+t)D^2 \int_0^t \mathbb{E}|X_s - \hat{X}_s|^2\,ds
\end{aligned}$$

$V(t) \le 2(1+t)D^2 \int_0^t V(s)ds \le 2(1+T)D^2 \int_0^t V(s)ds.$ rational numbers.

Gronwall Ineq. $\Rightarrow$ $V(t)=0$ for all t. $\Rightarrow$ $\mathbb{P}(|X_t-\hat{X}_t|=0$ for all $t\in\mathbb{Q}\cap[0,T])=1$

By continuity, $\mathbb{P}($ - - - - - $t\in[0,T])=1$.

2. Existence (Convergent Picard Iteration). $Y_t^{(0)}\equiv X_0$;

$$Y_t^{(k+1)} = X_0 + \int_0^t b(s,Y_s^{(k)})\,ds + \int_0^t \sigma(s,Y_s^{(k)})\,dB_s. \qquad (**)$$

Then, $\mathbb{E}|Y_t^{(k+1)}-Y_t^{(k)}|^2 \le 2(1+T)D^2\int_0^t \mathbb{E}|Y_s^{(k)}-Y_s^{(k-1)}|^2 ds$ for $k\ge 1$, $t\le T$.

& $\mathbb{E}|Y_t^1-Y_t^0|^2 \le \mathbb{E}\left|\int_0^t b(s,X_0)ds + \int_0^t \sigma(s,X_0)dB_s\right|^2$ $k=0$

$$\le 2t\int_0^t \mathbb{E}\, b(s,X_0)^2 ds + 2\int_0^t \mathbb{E}\,\sigma(s,X_0)^2 ds$$

$$\le \underbrace{2tC^2(1+\mathbb{E}X_0^2)t + 2C^2(1+\mathbb{E}X_0^2)t}\le A_1 t.$$

By induction, $\mathbb{E}|Y_t^2-Y_t^1|^2 \le 2(1+T)D^2\int_0^t \underbrace{\mathbb{E}|Y_s^1-Y_s^0|^2}_{\le A_1 s}ds \le \underbrace{2(1+T)D^2}_{\le A_2^2 = (1+T)^2D^2 2A_1}A_1\frac{1}{2}t^2$

⋮

$\Rightarrow$ $\mathbb{E}|Y_t^{k+1}-Y_t^k|^2 \le \dfrac{A_2^{k+1}t^{k+1}}{(k+1)!}$, $k\ge 0$, $t\in[0,T]$

$$\|Y_t^{(m)}-Y_t^{(n)}\|_{L^2([0,T]\times\Omega)} \le \sum_{k=n}^{m-1}\underbrace{\|Y_t^{k+1}-Y_t^k\|_{L^2}}_{\left(\int_0^T \mathbb{E}|Y_t^{k+1}-Y_t^k|^2dt\right)^{\frac12}\le\left(\int_0^T A_2^{k+1}t^{k+1}/(k+1)!\,dt\right)^{\frac12}} \le \sum_{k=n}^{m-1} A_2^{k+1}\frac{T^{k+2}}{(k+2)!}\xrightarrow{m,n\uparrow\infty}0.$$

$\left\|\sum_{k=n}^{m-1}(Y_t^{k+1}-Y_t^k)\right\|$

$= A_2^{k+1}\dfrac{T^{k+2}}{(k+2)!}$

Then, $\{Y_t^{(k)}\}_k$ is a Cauchy seq. in $L^2([0,T]\times\Omega)$. Hence,

$Y_t^{(k)}\xrightarrow{k\uparrow\infty} X_t$ in $L^2([0,T]\times\Omega)$, $\xrightarrow{(**)}$ X_t is a soln. (*)

· X_t is $\mathcal{F}_t^Z$-measurable $\forall t$, b.c. $Y_t^{(k)}$ is.

3. X_t has a continuous version: The right hand side has a cts version $\tilde{X}_t$. (Itô Integral)

$\tilde{X}_t = X_0 + \int_0^t b(s,X_s)ds + \int_0^t \sigma(s,X_s)dB_s$ a.s. $\Rightarrow X_t=\tilde{X}_t$ a.s.

$= \tilde{X}_0 + \int_0^t b(s,\tilde{X}_s)ds + \int_0^t \sigma(s,\tilde{X}_s)dB_s$. a.s.

#

§5.3 Weak and Strong solutions.

Strong: Given a (B_t) trajectory, $\to X_t$ ($\mathcal{F}_t^Z$-adapted) "path wise"

Weak: $b, \sigma \to ((\tilde{X}_t, \tilde{B}_t), \mathcal{H}_t)$ $\tilde{X}_t, \tilde{B}_t$ both $\mathcal{H}_t$ adapted "distribution-wise".

Karatzas - Shreve 91: P_{285} & P_{300}

Def. (Strong solution) A strong solu. of SDE (1) on $(\Omega, \mathcal{F}, \mathbb{P})$, with a fixed Bm & ,

is a process w/ cts paths s.t.

(i) X_t is $\mathcal{F}_t^Z$-adapted; $\mathbb{P}(X_0 = Z) = 1$;

(ii) $\mathbb{P}\left(\int_0^t [|b(s, X_s)| + \sigma^2(s, X_s)] ds < \infty \right) = 1$.

(iii) The integral version (*) holds a.s.

Def (weak solu.) A weak solu. of SDE(1) is a triple $(X, B), (\Omega, \mathcal{F}, \mathbb{P}), \{\mathcal{F}_t\}$, where

(i) $(\Omega, \mathcal{F}, \mathbb{P})$ is a prob. sp. $\{\mathcal{F}_t\}$ is a filtration, $\subseteq \mathcal{F}$.

(ii) $\{X_t\}$ is cts, $\mathcal{F}_t$-adapted; $\{B_t\}$ is a Bm w.r.t. $\{\mathcal{F}_t\}$

(iii) Eq. (*) holds a.s.

- $\{\mathcal{F}_t\}$ may NOT be $\{\mathcal{F}_t^{B \cup Z}\}$; X_t may NOT be in $\mathcal{F}_t^{B \cup Z}$ (i.e. X_t may not be a functional of $(B_s\ s \le t)$ & Z.)
-

1. Strong $\Rightarrow$ weak; weak $\not\Rightarrow$ strong (No strong solu. but $\exists$ weak.) $\downarrow$
2. uniqueness: pathwise (strong) or in distribution (weak)

Lemma 5.3.1 (Weak uniqueness) A solu. (weak or strong) is $\underbrace{\text{weakly unique}}$.

(identical in law, i.e. same FDD.)

Rmk: · modeling: it is natural to use weak solu. (b.c. Bm is unknown.)

· weak solu: the distribution of the process.

Example 5.3.2 (The Tanaka Equ.) $\quad dX_t = \mathrm{sign}(X_t)\,dB_t;\ X_0 = 0 \qquad \mathrm{sign}(x) = \begin{cases} +1, & x>0 \\ -1, & x\le 0. \end{cases}$

- $\sigma(x) = \mathrm{sign}(x)$: NOT Lipschitz.
- No strong solu. Sps X_t is a strong solu. Then by thm 8.4.2, X_t is a Bm.
  - Note that $dB_t = \mathrm{sign}(X_t)\,dX_t$ b.c. $\mathrm{sign}(x)^2 = 1$.
  - Tanaka formula: $B_t = \int_0^t \mathrm{sign}(X_s)\,dX_s = |X_t| - |X_0| - L_t(\omega)$ (trajectory-wise)

$$\tilde{L}_t \overset{L^2}{=} \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{\{|X_s|<\varepsilon\}}\,ds$$

$\Rightarrow$ B_t is measurable w.r.t a filtration generated by $(|X_s|, s\le t)$

$\Rightarrow$ The filtration $\mathcal{F}_t^B$ is $\boxed{\text{strictly}}$ contained in $\mathcal{F}_t^X$. But $\mathcal{F}_t^X \subseteq \mathcal{F}_t^B$. #.

- weak solu $\exists$!: for any Bm $\hat{B}_t$, the pair $((\hat{B}_t, \tilde{B}_t), \mathcal{F}_t^{\hat{B}})$ is a weak solu.

where $\tilde{B}_t := \int_0^t \mathrm{sign}(\hat{B}_s)\,d\hat{B}_s \Leftrightarrow d\tilde{B} = \mathrm{sign}(\hat{B}_t)\,d\hat{B}_t$,

$\Rightarrow \left.\begin{cases} d\hat{B}_t = \mathrm{sign}(\hat{B}_t)\,d\tilde{B}_t \\ \xrightarrow{\text{Thm 8.4.2,}}\ \tilde{B}_t \text{ is a Bm.} \end{cases}\right\} \Rightarrow \hat{B}_t$ is a weak solu.

weak ! follows directly b.c. X_t must be a Bm.