Chp4. Ito formula and the Martingale representation Theorem.

HW 4.4; 4.6; 4.15
(Read: 4.3, 7, 9, 10, 17)

Ito formula $\downarrow$ calculation of Ito integral

Martingale representation Theorem $\downarrow$ any mG be represented as an Ito integral

§4.1 1D Ito formula

Calculus:

$$\int_0^t s\,ds = \tfrac{1}{2}s^2$$

$$\int_0^t f(s)\,ds = F(t)-F(0) \quad \text{if } F'=f$$

$$\int_0^t g(F(s))F'(s)\,ds = \int_{F(0)}^{F(t)} g(r)\,dr = g(F(t))-g(F(0)).$$

Ito integral

$$\int_0^t B_s\,dB_s = \tfrac{1}{2}B_t^2 - \tfrac{1}{2}t$$

$$\int_0^t f(B_s)\,dB_s = F(B_t)-F(B_0) \ ??$$

$$d\,B_t^2 = 2B_t\,dB_t + dt$$

Differential: "closed" in ds; dB_s & ds:

$$\tfrac{1}{2}B_t^2 = \int_0^t B_s\,dB_s + \int_0^t \tfrac{1}{2}\,ds.$$

$$dF(B_t) = F'(B_t)\,dB_t + \tfrac{1}{2}F''(B_t)\,dt?$$

Def. 4.1.1 (1D Ito process) (X_t) is an Ito process if it is in the form

$$X_t = X_0 + \int_0^t u(s,\omega)\,ds + \int_0^t v(s,\omega)\,dB_s \;\overset{\text{Notation}}{\Longleftrightarrow}\; dX_t = u\,ds + v\,dB.$$

w/. $v \in \mathcal{W}_{\mathcal{H}} = \bigcap_{T>0} \mathcal{W}_{\mathcal{H}}(0,T)$, σ-algebra of Filtration $(\mathcal{H}_t)_{t\ge 0}$;

$\mathcal{H}_t$-adapted; $\quad \mathbb{P}\{\int_0^t v(s,\omega)^2\,ds < \infty,\ \forall t\ge 0\} = 1.$

u: $\mathcal{H}_t$-adapted, $\quad \mathbb{P}\{\int_0^t |u(s,\omega)|\,ds < \infty,\ \forall t\ge 0\} = 1.$

Thm 4.1.2 (Ito formula) Let (X_t) be $\quad dX_t = u\,dt + v\,dB_t.$

Let $g(t,x) \in C^2([0,\infty)\times\mathbb{R})$. Then $Y_t = g(t,X_t)$ is again an Ito process:

$$dY_t = \partial_t g(t,X_t)\,dt + \partial_x g(t,X_t)\,dX_t + \tfrac{1}{2}\partial_{xx} g(t,X_t)\,(dX_t)^2, \tag{4.1.7}$$

where $(dX_t)^2 = d(X_t)\cdot d(X_t) = d\langle X\rangle_t$ is computed according to the rules:

$$dt\cdot dt = dt\,dB_t = 0; \qquad dB_t\cdot dB_t = dt. \qquad \text{(a.s.)}.$$

Example:

$$d(B_t^2) = 2B_t\,dB_t + dt \;\Rightarrow\; \int_0^t B_s\,dB_s = \tfrac{1}{2}B_t^2 - \tfrac{1}{2}t.$$

$$dF(B_t) = F'\cdot dB_t + \tfrac{1}{2}F''(B_t)\,dt \;\Rightarrow\; \int_0^t F'(B_s)\,dB_s = F(B_t)-F(B_0) - \tfrac{1}{2}\int_0^t F''(B_s)\,ds$$

$$d(tB_t) = B_t\,dt + t\,dB_t + 0 \;\Rightarrow\; \int_0^t s\,dB_s = tB_t - 0 - \int_0^t B_s\,ds \quad \text{IBP.}$$

Thm 4.1.5 (Integration by parts) Sps $f(s,\omega)$ is cts and of bounded total variation w.r.t. $s\in[0,t]$ for a.s. ω. Then $\int_0^t f(s)\,dB_s = f(t)B_t - \int_0^t B_s\,df_s$

Proof. $I_t = \lim_{\Delta t_j\downarrow 0} \sum_{j\ge 0} f(t_j)\Delta B_j = \underbrace{\sum_{j\ge 0} \Delta[fB]_j}_{=\,f(t)B_t - 0} - \underbrace{\sum_{j\ge 0} B_{t_{j+1}}\Delta f_j}_{\downarrow\, \pi\downarrow 0:\ \int_0^t B_s\,df_s}$ b.c. B cts, f bdd TV.

#.

Proof of Ito's formula

$$(4.1.7) \Leftrightarrow \quad g(t,X_t) = g(0,X_0) + \int_0^t \left[\partial_t g(s,X_s) + \partial_x g\,(u_s\,ds + v_s\,dB_s) + \tfrac{1}{2}\partial_{xx} g\, v_s^2\right] ds \quad (4.1.9)$$

- WLOG, assume g, $\partial_t g$, $\partial_x g$, $\partial_{tx} g$, $\partial_{xx} g$, $\partial_{tt} g$ are bounded

[otherwise, consider $g_n(s,x) = g(s,x)$ for $s\le t$, $|x|\le n$

stopping time $\tau_n = \tau_n(\omega) = \inf\{s>0 : |X_s(\omega)| \ge n\}$;

and prove (4.1.9) with t replaced by $\tau_n\wedge t$. ; $P(\tau_n > t)\to 1$ as $n\uparrow\infty$ $\Rightarrow$ $\lim_{n\to\infty}$ to get (4.1.9).]

Start from Taylor expansion (for each ω)

$$g(t,X_t) = g(0,X_0) + \underbrace{\sum_j \Delta g(t_j,X_j)}_{\parallel} = g(0,X_0) + \sum_j \left[\partial_t g\,\Delta t_j + \partial_x g\,\Delta X_j\right]$$

① $\to \int_0^t \partial_t g\,ds$ + ② $\to \int_0^t \partial_x g\,dX_s$ $L^2(\Omega)$ $\Delta t_j\downarrow 0$.

$$+ \sum_j \left[\tfrac{1}{2}\partial_{tt} g\,(\Delta t_j)^2 + \partial_{tx} g\,\Delta t_j\Delta X_j + \tfrac{1}{2}\partial_{xx} g\,(\Delta X_j)^2\right] + \sum_j R_j$$

$$g(t_{j+1},X_{j+1}) - g(t_j,X_j) = \partial_t g\,\Delta t + \partial_x g\,\Delta X_j + \underbrace{\tfrac{1}{2}\partial_{tt} g\,(\Delta t_j)^2}_{③\,\to\,0} + \underbrace{\partial_{tx} g\,\Delta t_j\Delta X_j}_{④\,\to\,0} + \underbrace{\tfrac{1}{2}\partial_{xx} g\,(\Delta X_j)^2}_{⑤\,\to\,\frac{1}{2}\int_0^t \partial_{xx} g\, v_s^2\,ds} + \underbrace{o(|\Delta t_j|^2 + |\Delta X_j|^2)}_{R_j\,\to\,0}$$

①: Riemann sum; ② Riemann & Ito, in $L^2(\Omega)$; ③ $\to 0$.

④ $\sum_j \partial_{tx} g\,\Delta t_j\Delta X_j = \sum_j \partial_{tx} g\,\Delta t_j\,(u_j\Delta t_j + v_j\Delta B_j) = \sum_j \partial_{tx} g\,[u_j\,\Delta t_j\Delta t_j + v_j\Delta t_j\Delta B_j] \to 0.$

Assume u & v are elementary, WLOG.

$$\mathbb{E}\left|\sum_j \partial_{tx} g\, v_j\,\Delta t_j\Delta B_j\right|^2 = \mathbb{E}\sum_{i,j}\partial_{tx} g_i\,\partial_{tx} g_j\, v_i v_j\,\Delta t_j\Delta t_i\Delta B_j\Delta B_i = \sum_j \partial_{tx} g_j^2\, v_j^2\,\Delta t_j^3 \to 0.$$

⑤ $\sum_j \partial_{xx} g\,(\Delta X_j)^2 = \sum_j \partial_{xx} g\,(\underbrace{u_j^2\Delta t_j^2}_{\downarrow\,0} + \underbrace{u_j v_j\,\Delta t_j\Delta B_j}_{\downarrow\,0 \text{ as } ④} + v_j^2\Delta B_j^2) \to \int_0^t \partial_{xx} g\, v_s^2\,ds.$

$$\mathbb{E}\left(\sum a_j\Delta B_j^2 - \sum a_j\Delta t_j\right)^2 = \sum_{i,j}\mathbb{E}\left[a_i a_j\,(\Delta B_j^2 - \Delta t_j)(\Delta B_i^2 - \Delta t_i)\right] \to 0.$$

#

4.2. Ito formula on $\mathbb{R}^n$ $\quad g(t, X_t)$; $dX_t = u\,dt + v\,dB_t$

$$\underline{dg(t,X_t) = \partial_t g(t,X_t)\,dt + \nabla g \cdot dX_t + \frac{1}{2}\sum_{ij} \partial_{x_i x_j} g \,\langle dX_i, dX_j \rangle}$$

with $dB_i dB_j = 0$; $dB_i\,dt = 0$; .

Example: $\quad g(x,y) = xy$

$$d(X_t \cdot Y_t) = \partial_x g\, dX_t + \partial_y g\, dY_t + \partial_{xy} g\, dX_t \cdot dY_t$$

$$= Y_t\,dX_t + X_t\,dY_t + dX_t \cdot dY_t .$$

$\Rightarrow$ Integration by parts: $\quad \int_0^t X_s\,dY_s = X_t Y_t - X_0 Y_0 - \int_0^t Y_s\,dX_s - \int_0^t dX_s \cdot dY_s .$

Example: (Exponential martingale)

Let $\theta(t,\omega) \in \mathbb{R}^n$, $\theta_k \in \mathcal{V}[0,T]$ for $k = 1, \cdots, n$, $\qquad dX_t = -\frac{1}{2}|\theta_t|^2 dt + \theta_s\,dB_s$

show $\quad Z_t = \exp\left(\underbrace{\int_0^t \theta(s,\omega)\,dB_s - \frac{1}{2}\int_0^t |\theta_s|^2\,ds}\right)$ is a martingale if

$\mathbb{E}\left[e^{\int_0^t \theta_s\,dB_s}\right] < \infty, \quad \forall t \le T \qquad$ (Kazamaki)

or $\quad \mathbb{E}\left[e^{\frac{1}{2}\int_0^T \theta_s^2\,ds}\right] < \infty, \qquad$ (Novikov)

Proof: 1>. Z_t is $\mathcal{F}_t$-adapted. ✓

2> $\mathbb{E}[|Z_t|] = \mathbb{E}[Z_t] \le \mathbb{E}\left[e^{\int_0^t \theta_s\,dB_s}\right] \underset{\Delta_1}{=} \mathbb{E}\, e^{\frac{1}{2}\int_0^t \theta_s^2\,ds} < \infty$

3> $\mathbb{E}[Z_t | \mathcal{F}_s] = Z_s$?

$$dZ_t = e^{X_t}\,dX_t + \frac{1}{2} e^{X_t} \langle dX_t, dX_t \rangle$$

$$= Z_t\left(\cancel{-\frac{1}{2}\theta_t^2\,dt} + \theta_t\,dB_t + \cancel{\frac{1}{2}\theta_s^2\,dt}\right) = Z_t\,\theta_t\,dB .$$

$$Z_t - Z_s = \int_s^t Z_r \theta_r\,dB_r \Rightarrow \mathbb{E}[Z_t | \mathcal{F}_s] = Z_s + 0 .$$

(Δ_1: 1> Note that for $N \sim N(0,\sigma^2)$, $\mathbb{E}[e^N] = e^{\frac{1}{2}\sigma^2}$; $\quad \mathbb{E}\, e^{B_t - \frac{1}{2}t} = 1$

2> start from elementary process $\theta_s = \theta_0 \mathbb{1}_{[0,t_1]}(s)$; $\quad \int_0^t \theta_s\,dB_s = \theta_0 \Delta B_1$

$$\mathbb{E}\left[e^{\theta_0 \Delta B_1}\right] = \mathbb{E}\left(\mathbb{E}\left[e^{\theta_0 \Delta B_1} | \mathcal{F}_0\right]\right) = \mathbb{E}\left[e^{\theta_0^2 \Delta t_1}\right] .$$
)

HW 4.4; 4.6; 4.15
(Read: 4.3, 7, 9, 10, 17)

4.3 The Martingale representation Theorem.

We have $X_t = X_0 + \int_0^t v(s,\omega)\, dB_s$ is a mG. wrt. $\mathcal{F}_t$

i.e. Ito integral $\Rightarrow$ mG. of $v \in \mathcal{V}$

To show: $\Leftarrow$.

Thm 4.3.3 (The Ito representation theorem)

Let $F \in L^2(\mathcal{F}_T, \mathbb{P})$. Then, $\exists!$ $f(t,\omega) \in \mathcal{V}[0,T]$ s.t.

$$F(\omega) = \mathbb{E}[F] + \int_0^T f(t)\, dB_t .$$

- Proof later. Application

Thm 4.3.4 (mG Representation) Let B_t be an $\mathbb{R}^n$-Bm; (M_t) be an $\mathcal{F}_t$ mG. and $M_t \in L^2, \forall t \geq 0$.

then, $\exists!$ $f(s,\omega) \in \mathcal{V}(0,t)$ for all t s.t.

$$M_t = \mathbb{E}[M_0] + \int_0^t f(s)\, dB_s \qquad (*)$$

Proof: Apply Thm 4.3.3 with $F = M_t$, $T = t$, we have that $\exists!$ $f^{(t)} \in L^2(\mathcal{F}_t, \mathbb{P})$

$$M_t = \underbrace{\mathbb{E}[M_t]}_{=\mathbb{E}[M_0]} + \int_0^t f_s^{(t)}\, dB_s$$

For $t_1 < t_2$, we have

$$M_{t_1} = \mathbb{E}[M_{t_2} | \mathcal{F}_{t_1}] = \mathbb{E}[M_0] + \mathbb{E}\left[\int_0^{t_2} f_s^{(t_2)} dB_s \,\Big|\, \mathcal{F}_{t_1}\right]$$

$$\mathbb{E}[M_0] + \int_0^{t_1} f_s^{(t_1)} dB_s \overset{L^2}{=} \mathbb{E}[M_0] + \int_0^{t_1} f_s^{(t_2)} dB_s$$

$$0 = \mathbb{E}\left|\int_0^t (f_s^{(t_1)} - f_s^{(t_2)})\, dB_s\right|^2 = \mathbb{E}\left[\int_0^t |f_s^{(t_1)} - f_s^{(t_2)}|^2 ds\right].$$

$\Rightarrow f_s^{(t_1)} = f_s^{(t_2)}$ a.s.

We can define $f(s,\omega) = f_s^{(T)}$ a.s. for $s \leq T$ (for any T). $\Rightarrow (*)$. #

To prove Ito Rrepresentation Thm, two lemmas:

Lemma 4.3.1 Fix $T>0$. the set of random variables $\{\varphi(B_{t_1},\cdots,B_{t_n});\ t_i\in[0,T],\ \varphi\in C_0^\infty(\mathbb{R}^n),\ n\in\mathbb{N}\}$ is dense in $L^2(\mathcal{F}_T,\mathbb{P})$

$\mathcal{H}_n \leftarrow (B_{t_1}\cdots B_{t_n})$

Proof: mG. convergence: $g\in L^2(\mathcal{F}_T)$: $g=\mathbb{E}[g|\mathcal{F}_T]=\lim_{n\to\infty}\mathbb{E}[g|\mathcal{H}_n]$

Doob-Dynkin: $\mathbb{E}[g|\mathcal{H}_n]=g_n(B_{t_1},\cdots,B_{t_n})$. #

☆ Lemma 4.3.2 the linear span $\mathrm{span}\{e^{\int_0^T h(t)\,dB_t-\frac{1}{2}\int_0^T h^2 ds},\ h\in L^2[0,T]\}\overset{\text{dense}}{\subseteq} L^2(\mathcal{F}_T)$.

Proof: · $G(\lambda)=\int\exp(\lambda_1 B_{t_1}(\omega)+\cdots+\lambda_n B_{t_n}(\omega))\,g(\omega)\,dP(\omega)$ is analytic in $\lambda\in\mathbb{R}^n$, $\forall g\in L^2(\mathcal{F}_T)$

$\Downarrow$

$\lambda\in\mathbb{C}^n$

· Fourier transform.

Proof of Thm Ito Representation WLOG, $n=1$.

1>. F is an exponential form: $F(\omega)=e^{\int_0^T h(t)\,dB_t(\omega)-\frac{1}{2}\int_0^T h^2 dt}$, $h\in L^2[0,T]$. det.

Let $Y_t(\omega)=e^{\int_0^t h(s)\,dB_s(\omega)-\frac{1}{2}\int_0^t h(s)^2 ds}$

Ito formula.

Then $dY_t=Y_t h(t)\,dB_t$

$\Leftrightarrow\ Y_t=\underbrace{Y_0}_{1}+\int_0^t Y_s h(s)\,dB_s\ \Rightarrow\ F=Y_T=1+\int_0^T Y_s h(s)\,dB_s$.

2>. General $F\in L^2(\mathcal{F}_T,\mathbb{P})$. By Lemma 4.3.2, $F_n\to F$ in L^2 s.t.

$$F_n=\sum_{j=1}^n c_j\exp\left(\int_0^T h_j\,dB_t-\frac{1}{2}\int_0^T h^2 dt\right)=\underbrace{\sum_{j=1}^n c_j\cdot 1}_{\mathbb{E}[F_n]}+\int_0^T\underbrace{\sum_{j=1}^n Y_s^j h_j(s)}_{:=f_n(t,\omega)}dB_s\ \in\mathcal{V}[0,T].$$

$$\underset{0}{\mathbb{E}[(F_n-F_m)^2]}=\mathbb{E}\Big[\big|\underbrace{\mathbb{E}F_n-\mathbb{E}F_m}_{A}+\underbrace{\int_0^T(f_n-f_m)\,dB_s}_{B}\big|^2\Big]\quad \text{Ito isometry;}\ \ \mathbb{E}AB=0.$$

$$=\underbrace{|\mathbb{E}(F_n-F_m)|^2}_{\to 0}+\int_0^T\mathbb{E}[|f_n-f_m|^2]\,ds$$

$\Rightarrow\{f_n\}$ Cauchy in $L^2([0,T]\times\Omega)$. $\Rightarrow f_n\to f\in L^2([0,T]\times\Omega)$, in $\mathcal{V}[0,T]$

($\mathcal{F}_t$-adapted: $f_{n_k}\to f$ a.s. $\Rightarrow\mathcal{F}_t$-measurable, a.a. $t\to$ modification in t.)

$\Rightarrow F=\lim_{n\to\infty}F_n=\lim_{n\to\infty}\mathbb{E}F_n+\int_0^T f_n\,dB_t=\mathbb{E}F+\int_0^T f\,dB_t$.

3>. Uniqueness. Follows from Ito isometry. #.

Rmk: Alternative proof by Davis (1980). Clark representation formula.

- By Lemma 4.3.1, it suffices to consider $Y = \varphi(B_{t_1}, \cdots, B_{t_n})$, $\varphi \in C_0^\infty(\mathbb{R}^n)$

$$Y = \mathbb{E}[Y] + \int_0^T f(t)\, dB_t$$

a). A basic fact: $W = W(t, x_1, \cdots, x_k) : [t_{k-1}, t_k] \times \mathbb{R}^k$ $\qquad t \in [t_{k-1}, t_k]$.

$$W(t, B_{t_1}, \cdots, B_{t_{k-1}}, \underline{B_t}) - W(t, B_{t_1}, \cdots, B_{t_{k-1}}, \underline{B_{t_{k-1}}})$$

$$= \int_{t_{k-1}}^t \partial_{x_k} W(s, B_{t_1 : t_{k-1}}, B_s)\, dB_s + \int_{t_{k-1}}^t [\partial_t W + \partial_{x_k}^2 W](s, B_{t_1 : t_{k-1}}, B_s)\, ds,$$

b). Inductively: $V_k : [t_{k-1}, t_k] \times \mathbb{R}^k \to \mathbb{R}$

$n \downarrow n-1$

$$\begin{cases} \partial_t V_n + \frac{1}{2} \partial_{x_n}^2 V_n = 0 & t \in [t_{n-1}, t_n]. \\ V_n(t_n, x_{1:n}) = \varphi(x_{1:n}); & \end{cases}$$

$$\begin{cases} \partial_t V_k + \frac{1}{2} \partial_{x_k}^2 V_k = 0 \\ V_k(t_k, x_{1:k}) = V_{k+1}(t_k, x_{1:k}, x_k) \end{cases}$$

c). $f(t, \omega) = \partial_{x_k} V_k(t, B_{t_1 : t_{k-1}}, B_t)$ for $t \in [t_{k-1}, t_k]$. $\forall k$.