Chp 3 Ito integrals.

HW: 3.1, 3.4(iii) 3.7ab.
(read 3.16, 3.17 conditional expectation)

§3.1 Construction of the Ito integral

"white noise": $\frac{dX}{dt} = b(t, X_t)\,dt + \sigma(t, X_t)\, \underset{\approx}{W_t}$ (3.1.2)

W_t has the properties: (i) $t_1 \neq t_2 \Rightarrow W_{t_1}$ and W_{t_2} are indpt.

(ii) $\{W_t\}$ is stationary

(iii) $\mathbb{E}[W_t] = 0$;

$\boxed{\mathbb{E}[W_t W_s] = \delta_{t-s}. \quad G.P. \ \mathbb{E}[W_t] = 0.}$

- W_t w/ (i) & (ii) cannot have cts paths; If $\mathbb{E}\, W_t^2 = 1$; NOT even $([0,\infty), \mathcal{B})$ mearable.
- It is a generalized SP: a prob. measure on S': the space of tempered distributions on $[0,\infty)$.

(NOT a prob. measure on $\mathbb{R}^{[0,\infty)}$).

Follow the need in applications. (3.1.2) ↓ $X_k = X(t_k)$, $0 = t_0 < t_1 \cdots < t_m = t$.

$$X_{k+1} - X_k = b(t_k, X_k)\Delta t + \sigma(t_k, X_k)\, \underset{\approx}{\underline{W_k \Delta t}}_{V_{t_k}} \qquad \Delta V_{t_k} = W_{t_k}\Delta t$$

$\{V_t\}_{t\geq 0}$: stationary, indpt, mean-0 increments. $\quad V_t = V_{t_m} = \sum_{k=1}^{m} W_{t_k}\Delta t$, "i.i.d" ↓ CLT Normal

must be Bm. ← Knight (1981)

$$X_k = X_0 + \sum_{j=0}^{k-1} b(t_j, X_j)\Delta t + \sum_{j=0}^{k-1} \sigma(t_j, X_j)\Delta B_j$$

$$\xrightarrow{\Delta t \downarrow 0} \quad X_t = X_0 + \int_0^t b(s, X_s)\,ds + \text{“}\int_0^t \sigma(s, X_s)\,dB_s\text{”}$$

limit exist? in what sense?

This chapter: for general random integrand $\int_0^t f(s,\omega)\,dB_s(\omega)$

- pathwise integral: $\forall \omega$: Riemann-Stieltjes? $\int_0^T f(t)\,dg(t)$ C_b, Bounded TV. not for $B.(\omega)$ ↓

Young's integral $f \in C^\alpha, g \in C^\beta$ $\alpha + \beta > 1$. ✓.

- Stochastic integral. Return to the above "Riemann Sum / Euler"

To define: $\int_0^T f(t,\omega)\, dB_t(\omega)$

$$\stackrel{??}{=} \lim_{\Delta t \to 0} \sum_{j \geq 0} f(t_j^*, \omega)\, dB_{t_j}(\omega) \quad \text{as Riemann–Stieltjes ?}$$

<u>An example</u>: $f(t,\omega) = B_t(\omega)$, $\int_0^T B_t(\omega)\, dB_t(\omega)$

- The path of B has a variation that is too large for R-S. (Exe 2.17 $\|B\|_{TV} = \infty$.)

<u>Start.</u> <u>simple functions</u>: $\left\{ \varphi : \varphi(t,\omega) = \sum_{j=0}^{2^n T} e_j(\omega) \cdot \chi_{[\frac{j}{2^n}, \frac{j+1}{2^n}]}(t) \right\}$ $\quad t_k = t_k^{(n)} = \begin{cases} \frac{k}{2^n}, & \text{if } 0 \leq \frac{k}{2^n} \leq T \\ T, & \text{if } \frac{k}{2^n} > T. \end{cases}$

$$\int_0^T \varphi(t,\omega)\, dB_t(\omega) = \sum_{j \geq 0} e_j(\omega) \left[B_{t_{j+1}} - B_{t_j} \right](\omega).$$

Not enough: Specification of relation between $e_j(\omega)$ & $(B_t(\omega))_{t \geq 0}$ is important:

<u>Example 3.1.1.</u> Consider three examples of φ (all approximate $B_t(\omega)$, a.c. $\to \int_0^T B_t\, dB_t$)

$t_j^* = t_j$ ① $e_j(\omega) = B_{t_j}(\omega)$, $\Rightarrow$ $\mathbb{E} \int_0^T \varphi(t,\omega)\, dB_t(\omega) = \sum_{j \geq 0} \mathbb{E}[B_{t_j}(B_{t_{j+1}} - B_{t_j})] = 0$

$t_j^* = t_{j+1}$ ② $e_j(\omega) = B_{t_{j+1}}(\omega)$, $\Rightarrow$ $\mathbb{E}$ ---- $= \sum_{j \geq 0} \underbrace{\mathbb{E}[(B_{t_{j+1}} - B_{t_j} + B_{t_j})(B_{t_{j+1}} - B_{t_j})]}$

$= \underbrace{\mathbb{E}[(B_{t_{j+1}} - B_{t_j})^2]} = t_{j+1} - t_j$

$= T - 0 = T.$

$t_j^* = \frac{1}{2}(t_j + t_{j+1})$ ③ $e_j(\omega) = B_{\frac{1}{2}(t_{j+1}+t_j)}(\omega)$ $\Rightarrow$ $\mathbb{E}[\cdots] = \mathbb{E}[B_{t_j^*}(B_{t_{j+1}} - B_{t_j})] = \mathbb{E}[(B_{t_j^*} - B_{t_j})^2]$

$\Rightarrow \mathbb{E} \int_0^T \varphi(t,\omega)\, dB_t(\omega) = \sum_{j \geq 0} \frac{1}{2} \Delta t = \frac{1}{2} T.$

$\Rightarrow$ the choice of t_j^* matters. which one to use?

<u>Rmk 1</u> Two mostly used ones (later)

$\begin{cases} \text{Ito integral}: & t_j^* = t_j \quad (\text{left pt}) & \int_0^T f(t,\omega)\, dB_t(\omega) \\ \text{Stratonovich} & t_j^* = \frac{1}{2}(t_j + t_{j+1}) \quad (\text{mid pt}) & \int_0^T f(t,\omega) \circ dB_t(\omega) \end{cases}$

<u>Rmk 2</u> The increment dB_t is "independent" of the current + past : ① $\Rightarrow$ Ito integral.

Filtration

Filtration & adapted process.

Def 3.1.2 (Filtration of B) Let $B_t(\omega)$ be n-d Bm. The filtration generate by $\{B\}$, $\mathcal{F}_t = \mathcal{F}_t^B$, is the σ-algebra generated by the r.v. $\{B_j(s)\}_{0\le s\le t}^{1\le j\le n}$, i.e., it is the smallest σ-algebra containing all sets:

$$\{\omega: B_{t_1}(\omega)\in F_1, \cdots, B_{t_k}(\omega)\in F_k\}, \quad \forall t_i \le t,\ F_i \subseteq \mathbb{R}^n \text{ Borel}.$$

- $\mathcal{F}_t$ is the history of B up to time t.
- Y is $\mathcal{F}_t$-measurable $\Leftrightarrow$ Y = lim sum of $g_1(B_{t_1})g_2(B_{t_2})\cdots g_k(B_{t_k})$ $g_1\cdots g_k\in C_b$, $t_1\cdots t_k$

  i.e. Y is a functional of $B_{[0,t]}$; ; eg. $B_{t_1}\in\mathcal{F}_t$, $t_1\le t$. $B_{2t}\notin\mathcal{F}_t$.

  Proof: see exe 3.14: $Y=\lim_n \mathbb{E}[Y|\mathcal{H}_n]$ + Doob–Dynkin Lemma.
- $\mathcal{F}_t \nearrow$: $\mathcal{F}_s\subseteq\mathcal{F}_t$ if $s<t$; $\mathcal{F}_t\subseteq\mathcal{F}$.
- Filtration of a process X_t: $\mathcal{F}_t^X$

eg. $\mathcal{F}_t$.

Def 3.1.3 ($\mathcal{N}_t$-adapted) Let $\{\mathcal{N}_t\}$ be an increasing family of σ-algebras of subsets of Ω. A process $\{X_t(\omega)\}$ is $\mathcal{N}_t$-adapted if for each $t\ge 0$, the r.v. $X_t(\omega)$ is $\mathcal{N}_t$ measurable.

E.g. · $B_{t/2}$ or B_t is $\mathcal{F}_t^B$-adapted; $\frac{1}{2}(B_t+B_{t+h})$ is NOT $\mathcal{F}_t$-adapted

· left pt appr. $\int f(t,\omega)\,dB_t \approx \sum_i f(t_i)(B_{t_{i+1}}-B_{t_i})$; middle/right point: NOT.

Def 3.1.4 (Function space $\mathcal{V}$)

$$\mathcal{V}=\left\{ f:[0,\infty)\times\Omega\to\mathbb{R} : \begin{cases} (1)\ f \text{ is } \mathcal{B}\times\mathcal{F} \text{ measurable;} \\ (2)\ f_t \text{ is } \mathcal{F}_t \text{ adapted} \\ (3)\ \mathbb{E}\left[\int_0^T f(t,\omega)^2\,dt\right]<\infty \end{cases}\right\}.$$

Elementary function: $\phi(t,\omega)=\sum_j e_j(\omega)\,\mathbb{1}_{[t_j,t_{j+1}]}(t) \in \mathcal{V}$

- $\phi\in\mathcal{V} \Rightarrow e_j$ is $\mathcal{F}_{t_j}$-measurable.

The Ito integral (left pt)

Lemma 3.1.5 If $\varphi(t,\omega)$ is bounded and elementary. Then $\mathbb{E}\left[\left|\int_0^T \varphi(t)\,dB_t\right|^2\right] = \mathbb{E}\left[\int_0^T |\varphi_t|^2 dt\right]$

Proof: $\mathbb{E}\left|\int_0^T \varphi(t)\,dB_t\right|^2 = \mathbb{E}\left|\sum_j e_j \Delta B_j\right|^2 = \mathbb{E} \sum_{ij} e_i e_j \Delta B_i \Delta B_j$

$= \sum_{ij} \mathbb{E}[e_i^2]\,\Delta t \quad \leftarrow \quad \mathbb{E}[\underbrace{e_i e_j \Delta B_i}_{\text{indpt}} \Delta B_j] = 0 \quad \text{if } i<j$

$= \mathbb{E}\int_0^T |\varphi(t)|^2 dt$ $\quad$ indpt; $\mathbb{E}[\Delta B_j]=0$. #.

For any $f \in V$, construct $\{\varphi_n\}$ bdd and elementary s.t.

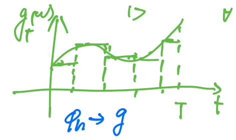
$$\mathbb{E}\int_0^T |f(t)-\varphi_n(t)|^2 dt \to 0$$

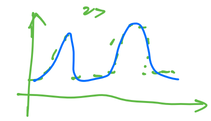
Then, $\{\int_0^T \varphi_n(t)\,dB_t\}_n$ is a Cauchy sequence b.c.

$$\int \varphi_n\,dB_t - \int_0^T \varphi_m\,dB_t = \int (\varphi_n - \varphi_m)\,dB_t$$

$$\mathbb{E}\left|\int \varphi_n\,dB_t - \int_0^T \varphi_m\,dB_t\right|^2 = \mathbb{E}\int_0^T |\varphi_n-\varphi_m|^2 dt \xrightarrow{m,n\to\infty} 0$$

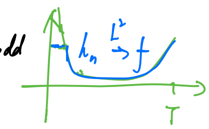
1>. bdded cts g: ω-wise, left pt. $L^2(\Omega\times[0,T])$





2>. bdd: smoother $\to$ cts + bdd $\quad g_n \xrightarrow{L^2} h$

3>. unbdd: truncation $\to h_n$ bdd



Def. 3.1.6 (The Ito integral) Let $f \in V$. Then the Ito integral is defined by

$$\int_0^T f(t,\omega)\,dB_t(\omega) = \lim_{n\to\infty} \int_0^T \varphi_n(t,\omega)\,dB_t(\omega) \quad \text{in } L^2(P), \qquad (*)$$

where $\{\varphi_n\}$ is a seq. of elementary funs s.t. $\mathbb{E}\left[\int_0^T |f(t,\omega)-\varphi_n(t,\omega)|^2 dt\right] \to 0$ as $n\to\infty$.

Q: Does the limit depend on the selection of $\{\varphi_n\}$?

Cor 3.1.7 (Ito Isometry) $\mathbb{E}\left[\left(\int_0^T f_t\,dB_t\right)^2\right] = \mathbb{E}\left[\int_0^T f(t)^2 dt\right], \quad \forall f \in V$

Cor 3.1.8. If $f, f_n \in V$ and $\mathbb{E}\left[\int_0^T |f_n(t)-f(t)|^2 dt\right] \xrightarrow{n\to\infty} 0$, then $\int_0^T f_n(t)\,dB_t \xrightarrow{L^2(P)} \int_0^T f(t)\,dB_t$.

Example 3.1.9 $B_0 = 0$. $\quad \int_0^t B_s\,dB_s = \frac{1}{2}B_t^2 - \frac{1}{2}t$

Proof: $\varphi_n(s,\omega) = \sum_j B_{t_j}(\omega)\, 1_{[t_j,t_{j+1}]}(s)$.

$$\mathbb{E}\left[\int_0^T |\varphi_n(s)-B_s|^2 ds\right] = \sum_j \int_{t_j}^{t_{j+1}} \underbrace{\mathbb{E}[|B_{t_j}-B_s|^2]}_{s-t_j} ds = \sum_j \frac{1}{2}(t_{j+1}-t_j)^2 \xrightarrow{\Delta\to 0} 0.$$

$\Rightarrow \int_0^t B_s\,dB_s = \lim_{n\to\infty} \int_0^t \varphi_n(s)\,dB_s = \lim_{\delta\to 0} \sum_j B_j \Delta B_j$

$= \lim_{\Delta\to 0} \frac{1}{2}B_t^2 - \sum_j (\Delta B_j)^2$

$= \frac{1}{2}B_t^2 - t \quad L^2(P)$

$\to$ $B_j B_{j+1} - B_j^2 = \frac{1}{2}\left[B_{j+1}^2 - B_j^2 - (\Delta B_j)^2\right]$

$\sum_j B_j \Delta B_j = \frac{1}{2}\left[\sum_j B_{j+1}^2 - B_j^2 - (\Delta B_j)^2\right]$

$= \frac{1}{2}\left[B_t^2 - B_0^2 - \sum_j (\Delta B_j)^2\right]$

Properties of Ito integral

Thm 3.2.1 $f, g \in \mathcal{V}(0,T)$, (1) $\int_0^T = \int_0^\alpha + \int_\alpha^T$ $0<\alpha<T$

(2) linear ; (3) $\mathbb{E}\int_0^T f(t)dB_t = 0$.

(4) $\int_0^t f(s)dB_s$ is $\mathcal{F}_t$-measurable. (5) mG.

Def 3.2.2 (Filtration and Martingale)

- A filtration is a family of increasing σ-algebras $\mathcal{M}_t \subseteq \mathcal{F}$, $\forall t$.
- A process $\{M_t\}_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is a martingale w.r.t. $\{\mathcal{M}_t\}$ if
  - (i) M_t is $\mathcal{M}_t$-measurable for all t;
  - (ii) $\mathbb{E}[|M_t|] < \infty$, $\forall t$
  - (iii) $\mathbb{E}[M_s | \mathcal{M}_t] = M_t$ for all $s \geq t$. ("current" is the mean of "future").

Example: Bm is a mG. w.r.t. $\mathcal{F}_t^B$: (i), (ii) ✓ ; (iii) $\mathbb{E}[\underset{B_t + B_s - B_t}{B_s} | \mathcal{F}_t] = B_t$

Theorem 3.2.4 (Doob's mG inequality) Let $\{M_t\}$ be a mG cts a.s. Then

$$\mathbb{P}\left(\sup_{0\leq t\leq T} |M_t| \geq \lambda\right) \leq \lambda^{-p}\, \mathbb{E}[|M_T|^p]$$

Thm 3.2.5 (cts) $\cdot$ $\int_0^t f_s dB_s$ has a cts version, $\forall f \in \mathcal{V}$. $\downarrow p=2$

Cor 3.2.6 $\underline{M_t = \int_0^t f_s dB_s \text{ is a mG w.r.t } \mathcal{F}_t \text{ and } \mathbb{P}(\sup_{0\leq s\leq t} |M_t| \geq \lambda) \leq \lambda^{-2}\mathbb{E}[\int_0^t f_s^2 ds]}$.

Proof of Thm 3.2.5: $\{\varphi_n\} \subseteq \mathcal{V}$ elementary, $\mathbb{E}\int_0^T |f - \varphi_n|^2 ds \to 0$, $n \nearrow \infty$.

- $I_n(t) = \int_0^t \varphi_n dB_s = \sum_j e_j \Delta B_j \mathbb{1}_{t_{j+1} < t < t_{j+2}} + e_{j+1}(B_t - B_{t_{j+1}})$, $\Rightarrow$ cts.
- $I_n(t)$ is a mG: $\mathbb{E}[I_n(s)|\mathcal{F}_t] = \mathbb{E}[\int_0^t + \int_t^s | \mathcal{F}_t]$

$$= \underbrace{\int_0^t \varphi_n(s) dB_s}_{I_n(t)} + \underbrace{\mathbb{E}[\sum_{t_j > t} e_j \Delta B_j | \mathcal{F}_t]}_{\mathbb{E}[e_j \Delta B_j | \mathcal{F}_t] = \mathbb{E}[e_j\, \mathbb{E}[\Delta B | \mathcal{F}_{t_j}] | \mathcal{F}_t]} = 0$$

$\Rightarrow$ $I_n - I_m$ is a mG and cts..

$$\mathbb{P}\Big(\underbrace{\sup_{0\leq t\leq T} |I_n(t) - I_m(t)| > \varepsilon}_{A_k}\Big) \leq \varepsilon^{-2}\mathbb{E}[|I_n(T) - I_m(T)|^2] = \varepsilon^{-2}\mathbb{E}[\int_0^T |\varphi_n - \varphi_m|^2 ds] \to 0$$

n_{k+1}, n_k s.t. $\varepsilon_k = 2^{-k}$, $\mathbb{P}(A_k) \leq 2^{-k} \xrightarrow{\text{Borel-Cantelli}} \mathbb{P}(A_k \text{ i.o.}) = 0$

$\Rightarrow \forall \omega, \exists k(\omega)$: $\sup_{0\leq t\leq T} |I_{n_{k+1}} - I_{n_k}|(\omega) \leq 2^{-k}$, for $k > k(\omega)$ a.s. $\Rightarrow I_{n_k}(t) \to J_t$ cts. a.s. $I_t = J_t$. #

§3.3. Extensions of Ito integral.

1. Larger function space (than $\mathcal{V}$): $\mathcal{F}_t^B \to \mathcal{H}_t$:

$\begin{cases} B_t \text{ is } \underline{\underline{\text{a mG}}} \text{ w.r.t. } \mathcal{H}_t \\ f(t,\omega) \text{ is } \mathcal{H}_t\text{-adapted.} \end{cases}$

2. Integration w.r.t. mG w/ finite 2-variation.

   (KS91: chp3)

$\int B_2 \, dB_1$

$\downarrow$

Not in $\mathcal{F}_t^{B_1}$, but $\mathcal{F}_t^B$

3. $\mathbb{R}^1 \to \mathbb{R}^n$-valued Bm: $\int_0^T B_t^i \, dB_t^j$

   $\int_0^T V_t \, d\vec{B}_t = \sum_{j=1}^n \int_0^T V_{t,ij} \, dB_t^j$

A comparison of Ito & Stratonovich Integrals.

Ito: $\int_0^T f(t,\omega) \, dB_t = \lim_{\Delta \downarrow 0} \sum_{t_k \in T} f(t_j, \omega)(B_{t_{j+1}} - B_{t_j})$

Stra: $\int_0^T f(t,\omega) \circ dB_t = \lim_{\Delta \downarrow 0} \sum_{t_k \in T} f(\frac{1}{2}(t_j + t_{j+1}), \omega)(B_{t_{j+1}} - B_{t_j})$

Comparison,

- Stra $\Leftrightarrow$ Ito

$$dX_t = b(t,X_t)\,dt + \sigma(t,X_t) \circ dB_t$$

$$\Leftrightarrow dX_t = b(t,X_t)\,dt + \sigma' \sigma(t,X_t)\,dt + \sigma \, dB_t$$

- NOT looking into the future: Ito. ✓ Stra. ✗.
- martingale: Ito ✓, Stra. ✗.
- Stra: ordinary chain rule

  Ito: new chain rule (Ito formula) (Chp4)

HW 3.1: Prove directly from the definition that

$$I_t = \int_0^t s\, dB_s = tB_t - \int_0^t B_s\, ds \qquad (*)$$

Proof: Let $0 \le t_0 < t_1 \cdots < t_N = t$ be a partition of $[0,t]$. Let $\tau = \max_j |t_{j+1} - t_j|$.

By Def., we need $I_t = \lim_{\tau \downarrow 0} \sum_{j \ge 0} t_j \Delta B_j$.

Note that $\Delta(tB_t)_j := t_{j+1} B_{t_{j+1}} - t_j B_{t_j} = (t_{j+1} - t_j) B_{t_{j+1}} + t_j \Delta B_{t_j}$

Then $$tB_t - 0B_0 = \sum_{j \ge 0} \Delta(tB_t)_j = \sum_j (\Delta t)_j B_{t_{j+1}} + \sum_j t_j \Delta B_{t_j}. \qquad (**)$$

as a Riemann integral. $\downarrow N \uparrow \infty$

b.c. (B) is cts a.s. $\quad \int_0^t B_s\, ds$ #.

HW 3.4 (ii) Check if $X_t = B_t^2$ is a mG.

ANS: No, it is NOT.

- X_t is $\mathcal{F}_t^B$-measurable;
- $\mathbb{E}|X_t| = \mathbb{E} B_t^2 = t < \infty$.
- $\mathbb{E}[X_t | \mathcal{F}_s^B] = \mathbb{E}[B_t^2 - B_s^2 + B_s^2 | \mathcal{F}_s^B] = \underbrace{\mathbb{E}[B_t^2 - B_s^2 | \mathcal{F}_s^B]}_{Z_t} + B_s^2 \ne B_s^2$

b.c. $\mathbb{E}[Z_t] = \mathbb{E}[B_t^2 - B_s^2] = t - s$.

HW 3.7 (a) Since the integrands are $f_n(t,\omega) = \frac{t^{n/2}}{n!} h_n\left(\frac{B_t}{\sqrt{t}}\right)$, they are $\left\{\begin{array}{l} \mathcal{B} \times \mathcal{F} \text{ measurable} \\ \mathcal{F}_t\text{-adapted.} \end{array}\right.$

To check the integrability, note that (wiki)

$$\mathbb{E}[h_n(N) h_m(N)] = \delta_{n,m} \sqrt{2\pi}\, n! \qquad \text{for } N \sim N(0,1).$$

Thus, $\mathbb{E}[f_n^2(t,\omega)] = \sqrt{2\pi}\, t^n$ b.c. $B_t/\sqrt{t} \sim N(0,1)$.

Then $\mathbb{E}\left[\int_0^T f_n(t,\omega)^2 dt\right] = \sqrt{2\pi} \frac{1}{(n+1)} T^{n+1} < \infty$.

(b). $n=1$: $\int_0^t 1\, dB_u = B_t - 0 = t^{\frac{1}{2}} \frac{B_t}{\sqrt{t}}$ ✓.

$n=2$: $\int_0^t B_u\, dB_u = \frac{1}{2} B_t^2 - \frac{1}{2} t = \frac{1}{2!} t\, h_2\left(\frac{B_t}{\sqrt{t}}\right)$

$n=3$: $\int_0^t \left(\frac{1}{2} B_u^2 - \frac{1}{2} u\right) dB_u = \frac{1}{2} \int_0^t B_u^2\, dB_u - \frac{1}{2} \int_0^t u\, dB_u$

$$\overset{\text{exe 3.2}}{=} \frac{1}{6} B_t^3 - \cancel{\frac{1}{2} \int_0^t B_s\, ds} - \frac{1}{2} t B_t + \cancel{\frac{1}{2} \int_0^t B_s\, dB_s} = \frac{1}{3!} t^{\frac{3}{2}} h_3\left(\frac{B_t}{\sqrt{t}}\right).$$