An introduction by discussion.

Population evolution

$\frac{dN}{dt} = \alpha(t) N(t)$; $\quad N(0)=N_0$. $\quad \frac{dN}{N} = \alpha(t)$

$N(t) - N(0) = \int_0^t \alpha(s) N(s) ds$ $\quad \ln N(t) - \ln N(0) = \int_0^t \alpha(s) ds$, $\alpha \in L^1$

$N(t) = N_0 e^{\int_0^t \alpha(s) ds}$

- randomness.

			Q1	2	3
①	$\alpha(t) = X(\omega) r(t)$ or $N_0 = N_0(\omega)$:	$N(t,\omega) = N_0(\omega) e^{\int_0^t r(s)ds X(\omega)}$	✓	✓	✓
②	$\alpha(t) = r(t) + X(\omega)$:	$N(t,\omega) = N_0 e^{\int_0^t r(s)ds + tX(\omega)}$	✓	✓	?
③	$\alpha(t) = r(t) + X(\omega,t)$:	$N(t,\omega) = N_0 e^{\int_0^t r(s)ds + \int_0^t X(\omega,s)ds}$	✓	?	?
④	$RHS = \alpha(t) N(t) + X(\omega,t)$;	$N(t,\omega) = N_0 + \int_0^t \alpha(s) N(s,\omega) ds + \int_0^t X(\omega,s) ds$	?	?	?

Q1: How to solve the equation?

Q2: How to describe the randomness (X & N)?

Q3: when $t \nearrow \infty$?

Q4: If observe $Z(t) = h(N(t)) + \text{Noise}$. How to estimate $N(t)$?

Chp2. 1. Probability space, random variables, stochastic processes.

Def 2.1.1 <u>Probability space</u> $(\Omega, \mathcal{F}, \mathbb{P})$. Ω = a set. $\mathcal{F}$ = a <u>σ-algebra</u>, $\mathbb{P}$ = a probab. meas.

(a) $\mathbb{P}(\phi) = 0$, $\mathbb{P}(\Omega) = 1$

(b) If $A_1, A_2 \cdots \in \mathcal{F}$ & $\{A_i\}$ is disjoint, then

$$\mathbb{P}(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

↓ $A_i \cap A_j = \phi, \ \forall i \neq j$

(i) $\phi \in \mathcal{F}$

(ii) $F \in \mathcal{F} \Rightarrow F^C \in \mathcal{F}$ complement

(iii) $A_1, A_2 \cdots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

- <u>complete</u> if $\mathcal{F}$ contains all subsets G of Ω w/ $\mathbb{P}$-<u>outer meas.</u> zero.

  (i.e. $\mathbb{P}^*(G) = \inf\{\mathbb{P}(F);\ G \subseteq F \in \mathcal{F}\}$).

- $\mathcal{F}$-measurable sets: $F \in \mathcal{F}$

- $\mathcal{U}$ = a family of subsets of Ω: $\mathcal{H}_{\mathcal{U}} = \bigcap\{\mathcal{H} : \mathcal{H}$ is a σ-algebra of Ω, $\mathcal{U} \subseteq \mathcal{H}\}$

  <u>σ-algebra generated by $\mathcal{U}$.</u>

  <u>Example1</u> Borel σ-algebra $\mathcal{B}$ for $\Omega = \mathbb{R}^n$: generated by all open subsets.

  <u>Example2</u>. $\mathcal{H}_X$ = σ-algebra generated by $X: \Omega \to \mathbb{R}^n$

  = the smallest σ-algebra on Ω containing all sets $X^{-1}(U)$, $U \subseteq \mathbb{R}^n$ open.

  ↓ $\{\omega \in \Omega : X(\omega) \in U\}$

<u>Random variable</u>: $= \{X^{-1}(B) : B \in \mathcal{B}\}$

- A function $X: \Omega \to \mathbb{R}^n$ is $\mathcal{F}$-measurable if $X^{-1}(U) \in \mathcal{F}$, $\forall U \in \mathcal{B}$. <u>random variable</u>

  Distribution of X: $\mu_X(B) = X^{-1}(B)$, $B \in \mathcal{B}$.

  Expectation: $\mathbb{E}[X] = \int_\Omega X(\omega)\, d\mathbb{P}(\omega) = \int_{\mathbb{R}^n} x\, d\mu_X(x)$

  $L^p(\Omega)$: $\mathbb{E}[|X|^p] < \infty$.

  $L^p(\mu_X) = \{f: \mathbb{R}^n \to \mathbb{R}$ Borel measurable s.t. $\mathbb{E}[|f(X)|^p] = \int_{\mathbb{R}^n} |f(x)|^p\, d\mu_X(x) < \infty.\}$

- $\mathbb{E}[1_A] = \mathbb{P}(A)$

Independence

Lemma 2.1.2 (Doob-Dynkin Lemma) If $X, Y: \Omega \to \mathbb{R}^n$ are two given fcns. then Y is $\mathcal{H}_X$-meas.
IFF $\exists$ a Borel measurable function $g: \mathbb{R}^n \to \mathbb{R}^n$ st. $Y = g(X)$.

Def 2.1.3 Two set $A, B \in \mathcal{F}$ are independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

Two families $\mathcal{H}_1, \mathcal{H}_2 \in \mathcal{F}$ are indpt if $\mathbb{P}(H_{i_1} \cap H_{i_2}) = \mathbb{P}(H_{i_1})\,\mathbb{P}(H_{i_2})$, $\forall H_{i_k} \in \mathcal{H}_k$.

A collection of families $\{\mathcal{H}_i, i \in I\}$ are indpt. if

$$\mathbb{P}(H_{i_1} \cap \cdots \cap H_{i_k}) = \mathbb{P}(H_{i_1}) \cdots \mathbb{P}(H_{i_k}), \quad \forall H_{i_k} \in \mathcal{H}_k.$$

A collection of r.v. $\{X_i, i \in I\}$ is indpt. if $\{\mathcal{H}_{X_i}, i \in I\}$ is indpt.

$\Rightarrow$ If X & Y are indpt, then $\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$, if $\mathbb{E}[|X|] < \infty$, $\mathbb{E}[|Y|] < \infty$.

Conditional expectation (Later)

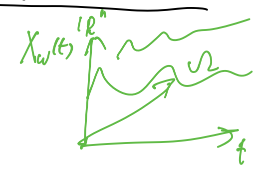
§ Chp 2: Mathematical Preliminaries & Notations.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

Definition 2.1.4 A <u>stochastic process</u> is a parametrized collection of random variables $\{X_t\}_{t\in T}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. [$\mathbb{R}^n$-valued]

(a set, a σ-algebra, a probab.)

- parameter space T: $[0,\infty)$, $[a,b]$; set;



— random variable view: $\forall t$, $X_t: \Omega \to \mathbb{R}^n$ is a random variable.

function view: $X: T\times\Omega \to \mathbb{R}^n$ (jointly measurable)

path view: $\forall \omega$, $X_\cdot(\omega): T \to \mathbb{R}^n$ is a path / trajectory.

↓ identify ω with the path $X_\cdot(\omega)$ $\Rightarrow$ $\Omega \hookrightarrow \tilde{\Omega} = (\mathbb{R}^n)^T \supset C(T, \mathbb{R}^d)$

measure on path space: $(\tilde{\Omega}, \mathcal{B}, \mathbb{P})$ ← $\mathcal{B}$ = Borel σ-algebra on $\tilde{\Omega}$.

$\{\omega = X_\cdot(\omega) : \omega(t_1)\in F_1, \cdots, \omega(t_k)\in F_k\}$ with $F_i \in \mathbb{R}^n$ Borel sets.

<u>Q1</u>: How do we "specify" an SP?

<u>Finite-dimensional distributions</u> of $\{X_t\}_{t\in T}$ are the distr. of $(X_{t_1}, \cdots, X_{t_k})$, $\forall k$.

i.e. the measures $\mu_{t_1\cdots t_k}$ defined on $\mathbb{R}^{nk}$ by

$$\mu_{t_1\cdots t_k}(F_1\times F_2 \times\cdots\times F_k) = \mathbb{P}(X_{t_1}\in F_1, \cdots, X_{t_k}\in F_k), \quad t_i \in T$$

where $\{F_i\}$ are Borel sets in $\mathbb{R}^n$.

"From experiments or numerical simulations we have information only about the FDDs of an SP."

<u>Q1'</u>: Given a family of FDDs, can we construct an SP? Yes ↓ Kolmogorov.

<u>Thm 2.1.5</u> (Kolmogorov's extension theorem) (consistency conditions required)

For all $t_1\cdots t_k \in T$, $k\in\mathbb{N}$, let $\nu_{t_1\cdots t_k}$ be prob. measures on $\mathbb{R}^{nk}$ s.t.

(i) permutation consistent: $\nu_{t_{\sigma(1)}\cdots t_{\sigma(k)}}(F_1\times\cdots\times F_k) = \nu_{t_1\cdots t_k}(F_{\sigma(1)}\cdots F_{\sigma(k)})$, $\forall\sigma$;

(ii) marginal consistent: $\nu_{t_1\cdots t_k}(F_1\times\cdots\times F_k) = \nu_{t_1\cdots t_k t_{k+1}\cdots t_{k+m}}(F_1\times\cdots\times F_k\times\mathbb{R}^n\times\cdots\times\mathbb{R}^n)$, $\forall m$

Then $\exists$ a prob. space $(\Omega, \mathcal{F}, \mathbb{P})$ and an SP $\{X_t\}$ s.t. $\nu_{t_1\cdots t_k} \sim X_{t_1\cdots t_k}$.

Q2: Can the FDDs determine an SP uniquely? { Not for general SP. / Yes for cts SP. → [Version & cts. ↓]

same FDDs / Distribution, different path properties.

Def. 2.2.2 Sps that $\{X_t\}$ & $\{Y_t\}$ are SPs on $(\Omega, \mathcal{F}, \mathbb{P})$. We say that $\{X_t\}$ is a version of (or a modification of) $\{Y_t\}$ if

$$\mathbb{P}(\{\omega : X_t(\omega) = Y_t(\omega)\}) = 1 \quad \text{for all } t.$$

Note: X & Y have the same FDPs ⇒ the two SPs are the same, but path properties may be different.

Exe 2.9: Let $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,\infty), \mathcal{B}, \mu)$ Borel σ-algebra, μ: No mass on single pt.

Define $X_t(\omega) = \begin{cases} 1 & \text{if } t = \omega \\ 0 & \text{otherwise,} \end{cases}$ ← discontinuous for each trajectory.

$Y_t(\omega) \equiv 0$, $\forall t, \omega$. → cts.

Then: $\mathbb{P}(X_t = Y_t) = \mathbb{P}(X_t = 0) = \mathbb{P}(\omega : t \neq \omega) = 1$

Thm 2.2.3 (Kolmogorov's continuity theorem)

Sps that the process $X = \{X_t\}_{t \geq 0}$ satisfies the moment condition: $\forall T$, $\exists D, \alpha, \beta > 0$.

$$\mathbb{E}[|X_t - X_s|^{\alpha}] \leq D\, |t-s|^{1+\beta}, \quad \forall\, 0 \leq s,t \leq T.$$

Then there exists a cts version of X.

[Proof: see e.g. Nualart Appendix.]

Typical SP:

1. Gaussian Processes (GP).

Def: A cts-time GP is an SP whose FDDs are Gaussian, i.e.

$$X_{t_1:t_k} \sim N(m_k, C_k) \iff \exists\, m_k \,\&\, C_k \geq 0 \text{ s.t. } \mathbb{E}[e^{i\sum_{j=1}^{k} u_j X_{t_j}}] = e^{-\frac{1}{2}\sum_{j,l=1}^{k} u_j u_l C_k(j,l) + i\sum_j u_j m_k^j}$$

($m_k \in \mathbb{R}^{nk}$, $C_k \in \mathbb{R}^{nk\times nk}$)

Fact: A GP is completely characterized by its mean $m(t) = \mathbb{E}[X_t]$ &

covariance $C(s,t) = \mathbb{E}[(X_t - m(t)) \otimes (X_s - m(s))]$.

Numerical simulation of GP (low-D) $t_1, \cdots, t_N$; $m(t) \to m_N \in \mathbb{R}^N$,

$X_{t_1:t_N} = m_N + \Lambda N(0, I_N)$; $C(s,t) \to C_N \in \mathbb{R}^{N\times N}$

Example 1. (Brownian motion)

Robert Brown (1828) pollen grains in water; irregular motion Diffusion

: average of random collisions with the water molecular.

Bachelier (1900): quantitative work; stock price fluctuation;

Einstein (1905): derive the transition density from molecular-kinetic theory of heat

Wiener (1923, 24): rigorous math, $\exists$

Levy (1939, 1948): construction of Bm; (interpolation; path) passage time & related functionals

Kolmogorov (1933) construction: FDDs

Daniell (1918/19) integral on a space of sequences;

Loève (1978) Centsov (1956). Holder continuity.

Definition (Bm). A Bm $\{B_t\}_{t\ge0}$ starting from $B_0=x$ is an SP whose FDDs satisfy:

$$\mathbb{P}^x(B_{t_1}\in F_1,\cdots,B_{t_k}\in F_k)=\int_{F_1\times F_2\times\cdots\times F_k} p(t_1,x,x_1)\cdots p(t_k-t_{k-1},x_{k-1},x_k)\,dx_{1:k}$$

$\forall\ 0<t_1<t_2<\cdots<t_k$ and $F_1\cdots F_k$ being Borel sets in $\mathbb{R}^n$,

where $p(t,x,y)=(2\pi t)^{-n/2}e^{-\frac{1}{2t}|x-y|^2}$, $\forall\ x,y\in\mathbb{R}^n,\ t>0$.

- transition density

  $B_t\in dy\,|\,B_s=x\sim p(t,y|s,x)\sim N(0,t-s)$

- $p(t,x_0,x)$ x_0 fixed

  $\begin{cases}\partial_t p=\frac{1}{2}\Delta p, & x\in\mathbb{R}^d,\ t>0\\ p(0)=\delta_{x_0}\end{cases}$

  Green's Fun; $\begin{cases}\partial_t u=\frac{1}{2}\Delta u\\ u(0,x)=f(x)\end{cases}$

  $u(t,x)=\mathbb{E}[f(B_t^x)]$

Basic properties:
1. Bm is a GP.
2. Indpt increments.
3. has a cts version.

Rmk1. Bm defined by FDDs is NOT unique;

Bm paths are cts a.s. ← Kolmogorov's continuity theorem

$\mathbb{E}[|B_t-B_s|^4]=3|t-s|^2$

choose the version w/ cts paths.

Path space view: Bm is just the space $C([0,\infty),\mathbb{R}^n)$ equiped w/ a meas. $\mathbb{P}^x$.

$\rightarrow(\tilde{\Omega},\mathcal{B},\mathbb{P}^x)$ canonical Bm.

Rmk2. other definitions: ① Bm is a centered GP w/ cts sample paths & covariance $\mathbb{E}[B_tB_s]=t\wedge s$.

② see KS 91 for a definition w/ $\mathcal{F}_t$

③ Levy characterization: $\begin{cases}X_t \text{ is a martingle (w.r.t. its own filteration)}\\ X_{t,i}X_{t,j}-\delta_{ij}t \text{ is a mG.}\end{cases}$ $=X_t^2-\langle X_t,X_t\rangle$

Rmk3 Construction of Bm.

① rescaled random walk: Let $\{X_i\}$ i.i.d. mean 0 variance 1,

Let $S_n=\sum_{j=1}^n X_j$, $n\ge0$. Define

$$W_t^n=\frac{1}{\sqrt{n}}S_{[nt]}+(nt-[nt])\frac{1}{\sqrt{n}}X_{[nt]+1}$$

② Levy's construction based on interpolation (→ path)

③ Payley-Wiener $\quad B_t=\frac{\sqrt{2}}{\pi}\sum_{n=0}^{\infty}\frac{\sin\pi t(n+\frac{1}{2})}{n+1/2}\xi_n$, $\quad\xi_n\sim N(0,1)$ i.i.d

Other properties: (Prop 1.5 Pav. 14)

- scaling $\hat{B}_t := c^{-\frac{1}{2}} B_{ct}$ is also a Bm (Exe 2.16)
- Shifting $(B_{t+c} - B_c)$ is a Bm, $\forall c > 0$. (Exe 2.12)
- Time reversal: $(B_{1-t} - B_1)_{t \in [0,1]}$ is a Bm.
- Inversion: $(X_0 = 0,\ X_t = t W_{1/t})_{t \geq 0}$ is a Bm.

Example 2. fBm B_t^H is a GP. with its sample paths whose covariance is given by

$$\mathbb{E}[B_t^H B_s^H] = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}), \quad H \in (0,1)$$

- When $H = \frac{1}{2}$, we have Bm.

Example 3 Brownian Bridge $W_t = B_t - t B_1 \quad t \in [0,1])$

Example 4: Ornstein-Uhlenbeck process.

$$X_t = e^{-\theta t} X_0 + \int_0^t e^{-\theta(t-s)} dB_s \quad (1)$$

$$X_t = e^{-\theta t} B_{e^{2\theta t}} \leftarrow \text{time change of Bm.} \quad (2)$$

- A definition without using the integral? $(X_t, t \geq 0)$, $X_0 = x_0$.

a GP
$$\begin{cases} \mathbb{E}[X_t] = x_0 e^{-\theta t} \\ Cov(X_t, X_s) = \frac{1}{2\theta}(1 - e^{-2\theta s}) e^{-\theta(t-s)}, \quad t \geq s \end{cases} \quad (3)$$

1). Compute (3) from (1):

$$Var(X_t) = \int_0^t e^{-2\theta(t-s)} ds = \frac{1}{2\theta}(1 - e^{-2\theta t}) \ ; \quad \mathbb{E} X_t^2 = e^{-2\theta t} \mathbb{E} X_0^2 + \frac{1}{2\theta}(1 - e^{-2\theta t})$$

$$Cov(X_t, X_s) = \mathbb{E}[(X_t - \mathbb{E}X_t)(X_s - \mathbb{E}X_s)] = \mathbb{E}[X_t X_s] - \mathbb{E}[X_t]\mathbb{E}[X_s] = \frac{1}{2\theta}(1 - e^{-2\theta s}) e^{-\theta(t-s)}$$

$$X_t = e^{-\theta(t-s)} X_s + \int_s^t e^{-\theta(t-u)} dB_u$$

$$\Rightarrow \mathbb{E}[X_t X_s] = e^{-\theta(t-s)} \mathbb{E}[X_s^2] + 0 = e^{-\theta(t-s)} \cdot [e^{-2\theta s} x_0^2 + \frac{1}{2\theta}(1 - e^{-2\theta s})]$$

$$\mathbb{E}[X_t]\mathbb{E}[X_s] = x_0 e^{-\theta t} x_0 e^{-\theta s} = x_0^2 e^{-\theta(t+s)}$$

$\Rightarrow$

2). stationary: $t \uparrow \infty$.

$$\begin{cases} \lim_{t \uparrow \infty} \mathbb{E}[X_t] = 0 \\ \lim_{t \uparrow \infty} \mathbb{E}[X_t^2] = \frac{1}{2\theta} \end{cases} \Rightarrow \text{stationary } X_t \sim N(0, \tfrac{1}{2\theta})$$

$Cov(X_t, X_s)$ = same as above.

(dynamic properties.)

2. Stationary Processes.

Definition (Strong stationary process) An SP is strong stationary if all FDDs are invariant under time translation: $X_{t_1:t_k} \sim X_{t_1+h:t_k+h}$, $\forall t_1, \cdots t_k, h$.

Example 1: Let Z be a r.v. & let $X_n \equiv Z$, $\forall n$. Then (X_n) is stationary

Example 2, i.i.d seq.

Example 3: OU w/ IC being the stationary distribution: GP w/ $\begin{cases} \mathbb{E} X_t \equiv 0, & X_0 \sim N(0, \frac{1}{2\theta}) \\ \text{cov}(X_t, X_s) = \frac{1}{2\theta}(1-e^{-2\theta s}) e^{-\theta(t-s)} \end{cases}$.

Def: (Weak stationary / 2nd-order stationary) if $\mathbb{E} X_t \equiv \mu$

$$\mathbb{E}[(X_t-\mu)(X_s-\mu)] = C(t-s)$$

Prop 1.3 (Pav14) (Ergodicity of stationary process) $(X_t)_{t\geq 0}$ weak stationary with μ & C above. Assume $C \in L^1(0,+\infty)$. Then $\lim_{T\to\infty} \mathbb{E}\left|\frac{1}{T}\int_0^T X_s ds - \mu\right|^2 = 0$.

3. Karhunen-Loeve Expansion.

$\{X_t, t\in[0,1]\}$ be an L^2-process w/ 0 mean & cts. corr. $R(t,s)$

Let $\{\lambda_n, e_n(t)\}$ be the eigen-pairs of R-integral operator on $L^2[0,1]$. Then

$$X_t = \sum_{n=1}^{\infty} \xi_n e_n(t), \quad t\in[0,1], \qquad \xi_n = \int_0^1 X_t e_n(t)\, dt$$

converges in L^2 to X_t, uniformly in t. $\qquad \mathbb{E}\xi_n = 0; \quad \mathbb{E}[\xi_n \xi_m] = \lambda_n \delta_{nm}$.

Homework.

2.8. Let B_t be Bm on $\mathbb{R}$, $B_0=0$. Put $E=E^0$.

(a) Use $\mathbb{E}^x[e^{i\sum_{j=1}^{nk} u_j Z_j}] = e^{-\frac{1}{2}\sum_{j,m} u_j C_{jm} u_m + i\sum_j u_j M_j}$ $\quad C\in\mathbb{R}^{nk\times nk}$, $u\in\mathbb{R}^{nk}$

to prove that $\quad M=\mathbb{E}^x[Z]\in\mathbb{R}^{nk}$,

$$\mathbb{E}[e^{iuB_t}] = e^{-\frac{1}{2}u^2 t}, \quad \forall u\in\mathbb{R}$$

$C_{jm}=\mathbb{E}^x[(Z_j-M_j)(Z_m-M_m)]$

(b) Use the power series expansion of the exponential fn on both sides, compare the terms w/ the same power of u and deduce that

$$\mathbb{E}[B_t^4] = 3t^2$$

$$\mathbb{E}[B_t^{2k}] = \frac{(2k)!}{2^k\cdot k!}t^k, \quad k\in\mathbb{N}.$$

(c) Alternative of (b): Prove that (2.2.2):

$$\mathbb{P}^x(B_{t_1}\in F_1, \cdots, B_{t_k}\in F_k) = \int_{F_1\times\cdots\times F_k} p(t_1,x,x_1)\cdots p(t_k-t_{k-1},x_{k-1},x_k)\,dx_{1:k}$$

implies that $\mathbb{E}[f(B_t)] = \frac{1}{\sqrt{2\pi t}}\int f(x)e^{-\frac{x^2}{2t}}dx$

for all functions f s.t. the integral on the right exists. Then apply this to $f(x)=x^{2k}$ and use integration by parts and induction on k.

(d) Prove $\quad \mathbb{E}^x[|B_t-B_s|^4] = n(n+2)|t-s|^2$ by using b) & induction

2.8. (a), (b), (c), follow the direction.

(d). $n=1$, $\mathbb{E}[B_t^4] = 3t^2 = n(n+2)t^2$

$n=2$ $\mathbb{E}[|(B_{t,1}, B_{t,2})|^4] = \mathbb{E}[(B_{t_1}^2 + B_{t_2}^2)^2] = \mathbb{E}[B_{t_1}^4 + B_{t_2}^4 + 2B_{t_1}^2 B_{t_2}^2]$

$= 3t^2 + 3t^2 + 2t^2 = 8t^2 = n(n+2)t^2$

$n \geq 2$: $\mathbb{E}[|(B_{t_1}, \cdots B_{t_n})|^4] = \mathbb{E}[(B_{t_1}^2 + \sum_{i=2}^n B_{t_i}^2)^2]$

$= \mathbb{E}[B_{t_1}^4 + 2B_{t_1}^2 \sum_{i=2}^n B_{t_i}^2 + (\sum_{i=2}^n B_{t_i}^2)^2]$

$= 3t^2 + 2t(n-1)t + \underbrace{\mathbb{E}[(\sum_{i=2}^n B_{t_i}^2)^2]}_{(n-1)(n+1)t^2}$

$= [3 + 2(n-1) + n^2 - 1]\, t^2 = (n^2 + 2n)\, t^2$.

2.16. By GP definition; verify the covariance.

2.17. Show that Bm has unbdd TV a.s. from $\mathbb{E}[|\sum_{t_k \leq t} (\Delta B_{t_k})^2 - t|^2] = 2\sum_{t_k \leq t} (\Delta t_k)^2 \leq 2t\Delta \to 0$ $\quad \Delta = \max_k \Delta t_k$

$\sum_{t_k}[(\Delta B_{t_k})^2 - \Delta t_k]$

$\Rightarrow Y_t^\Delta := \sum_{t_k \leq t} |\Delta B_{t_k}(\omega)|^2 \to t$ in $L^2(\mathbb{P})$. $(*)$

Proof: ① Note that (a) implies $Y_t^\Delta := \sum_{t_k \leq t} |\Delta B_{t_k}(\omega)|^2 \to t$ a.s.

(You can also use $\langle B \rangle_t = t$ a.s. here).

② Let $V_t(\omega) = \lim_{\Delta t_k \to 0} \sum_{t_k \leq t} |B_{t_{k+1}}^{(\omega)} - B_{t_k}^{(\omega)}| < \infty$ for some ω.

Then, noting that $\sup_k |\Delta B_{t_k}| \xrightarrow[\Delta \downarrow 0]{} 0$ b/c $B_\cdot(\omega)$ is cts,

we have, as $\Delta \downarrow 0$,

$Y_t^\Delta = \sum_{t_k \leq t} |\Delta B_{t_k}(\omega)|^2 \leq \underbrace{\sup_\Delta |\Delta B_{t_k}|}_{\to 0} \underbrace{\sum_{t_k \leq t} |\Delta B_{t_k}|}_{\to V_t(\omega)} \to 0$.

$\xRightarrow{①\&②}$ $\mathbb{P}(\omega: V_t(\omega) < \infty) = 0$.

$\leftarrow$ If $X_n \to X$ in $L^2(\mathbb{P})$

Then by Chebyshev's Ineq:

$\mathbb{P}(|X_n - X| > \varepsilon) \leq \varepsilon^{-2}\mathbb{E}[|X_n - X|^2] \xrightarrow{n \uparrow \infty} 0$

① i.e. $X_n \to X$ in prob.

② $\Delta_k = 2^{-k} \cdot 2^{-k}$, $\varepsilon_k = 2^{-k/2}$,

$\mathbb{P}(\underbrace{|Y_t^{\Delta_k} - t| > 2^{-k}}_{A_k}) \leq 2^{+k}\, 2t\, 2^{-2k} = 2^{-k} \cdot 2t$

$\sum_{k=1}^\infty 2^{-k} 2t < \infty$

Borel-Cantelli $\Rightarrow$

$\mathbb{P}(\bigcap_{m=1}^\infty \bigcup_{k=m}^\infty A_k) = 0$

$= \mathbb{P}(\lim_{k \uparrow \infty} |Y_t^{\Delta_k} - t| > 0)$

i.e. $Y_t^{\Delta_k} \to t$ a.s.