Chp 11. Application to stochastic control.

Problem 6 (An optimal portfolio problem) Example 11.2.5

A person w/ two investment possibilities: $\begin{cases} \text{(i) safe asset: } dX_0 = bX_0\,dt & b<a \\ \text{(ii) risky asset: } dX_1 = aX_1\,dt + \alpha X_1\,dB_t\ ; & \alpha>0 \end{cases}$

At each time, a fraction $u(t)$ is in the risky asset, $1-u(t)$ in the safe asset. Total Z_t

$$dZ_t = auZ_t\,dt + \alpha uZ_t\,dB_t + (1-u)bZ_t\,dt = Z_t[au+b(1-u)]\,dt + \alpha uZ_t\,dB_t.$$

- No borrowing, $u\le 1$; No short selling: $u\ge 0$. $\quad u\in[0,1]$.
- Given a utility function $N:[0,\infty)\to[0,\infty)$; $N(0)=0$. increasing and concave.

Problem: starting w/ $Z_s=x$, to maximize the expected utility of the wealth at $t_0>s$:

find a control $u^*=u^*(t,Z_t)$ and the value function:

$$\Phi(s,x) = \sup_u \{ J^u(s,x) : u=u(t,Z_t),\ 0\le u\le 1\} = J^{u^*}(s,x).$$

$$J^u(s,x) = \mathbb{E}^{s,x}[N(Z^u_{\tau_G})],\quad \tau_G = \text{the first exit time from } G=\{(r,z): r<t_0, z>0\}$$

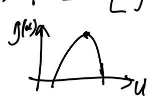
Some solutions:

$N(x)=x^\gamma,\quad 0<\gamma<1:\qquad u^*(t,x)=\dfrac{a-b}{\alpha^2(1-\gamma)}$

$N(x)=\log x,\qquad u^*(t,x)=\dfrac{a-b}{\alpha^2}$

By Dynkin's formula: $\quad L^u\phi(t,x) = \partial_t\phi + x(au+b(1-u))\partial_x\phi + \frac12\alpha^2u^2x^2\partial_{xx}\phi.\quad \forall\phi\in C_b^{1,2}.$

$$\mathbb{E}[N(Z_{\tau_G})] = N(x) + \mathbb{E}^{s,x}\Big[\int_0^{\tau_G}\big\{\underbrace{[(a-b)u+b]Z_t\partial_xN + \tfrac12\alpha^2u^2Z_t^2\partial_{xx}N}\big\}\,dt\Big]$$

- when $\partial_xN>0$, $\partial_{xx}N<0$,  $\Rightarrow\quad u^*(t,x) = -\dfrac{(a-b)\partial_xN}{x\alpha^2\partial_{xx}N}.$

- $\Phi(s,x) = J^{u^*}(s,x) = \mathbb{E}^{s,x}[N(Z^{u^*}_{\tau_G})] = ?$

1). statement of the problem (general)

An Ito process X_t: $dX_t = b(t, X_t, u_t)\,dt + \sigma(t, X_t, u_t)\,dB_t$; $X_s = x \in \mathbb{R}^n$

- $u_t = u(t,\omega) \in U \subseteq \mathbb{R}^k$, $\mathcal{F}_t^B$ adapted

Given profit rate fun: $f: \mathbb{R}\times\mathbb{R}^n\times U \to \mathbb{R}$, both continuous,

bequest fun.: $g: \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$,

a fixed domain: $G \subseteq \mathbb{R}\times\mathbb{R}^n$, $\hat{T}$ = the 1st exit time $= \inf\{r>s: (r, X_r^{s,x}) \notin G\}$

Goal: Find u^*, maximizing the performance function:

$$J^u(s,x) = \mathbb{E}^{s,x}\left[\int_s^{\hat{T}} f^{u_r}(r, X_r)\,dr + g(\hat{T}, X_{\hat{T}})\,\mathbb{1}_{\{\hat{T}<\infty\}}\right]$$

$$\Phi(s,x) = \sup_{u\in A} J^u(s,x) = J^{u^*}(s,x) \qquad . \ u^*: \text{ optimal control.}$$

- A = admissible controls $= \{u: u_t \in \mathcal{F}_t$, with values in $U\}$.

1. Deterministic/open loop control: $u(t,\omega) = u(t)$: NOT dependent on ω.
2. Feedback/closed loop control: $u(t,\omega) \in \mathcal{F}_t^X$
3. Partial information control: $u(t,\cdot) \in \mathcal{F}_t^R$, $\{R_t\}$ is a noisy observation of X_t. $dR_t = a(X_t)dt + \gamma(X_t)d\hat{B}_t$. Filtering.
4. Markov control: $u(t,\omega) = u(t, X_t(\omega))$.

$\to$ X_t is an Ito diffusion, a Markov process.

Notation: $Y_t = (s+t, X_{s+t})_{t\ge 0}$; $Y_0 = (s,x)$. Then

$$dY_t = b(Y_t, u_t)\,dt + \sigma(Y_t, u_t)\,dB_t \qquad (*)$$

- $\tau_G = \inf\{t>0: Y_t \notin G\} = \hat{T} - s$; $g(\hat{T}, X_{\hat{T}}) = g(Y_{\tau_G})$
- $J^u(y) = J^u(s,x) = \mathbb{E}^y\left[\int_0^{\tau_G} f^u(Y_t)\,dt + g(Y_{\tau_G})\,\mathbb{1}_{\{\tau_G<\infty\}}\right]$.

Rmk: we only need to consider $Y_t \in \mathbb{R}^n$ in Eq. (*) with $u_t = u(Y_t)$, and $G \subseteq \mathbb{R}^n$.

$$\Phi(y) = \sup_{u\in A} J^u(y) = J^{u^*}(y).$$

2. The Hamilton-Jacobi-Bellman Equ.

For any $v \in \mathcal{U}$ and $\phi \in C_0^2$, define $[L^v \phi](y) = \partial_s \phi(y) + \sum_i b_i(y,v)\partial_i \phi + \sum_{ij} a_{ij}(y,v)\partial_{ij}\phi$.

$= $ generator of $(*)$

Thm 11.2.1 (HJB1) Define $\Phi(y) = \sup\{J^u(y) : u = u(Y) \text{ a Markov control}\}$.

(i) Sps that $\Phi \in C^2(G)\cap C(\bar{G})$ satisfies: $\mathbb{E}^y[\Phi(Y_\alpha) + \int_0^\alpha L^v \Phi(Y_t)dt] < \infty$, $\forall \alpha \le \tau_G$, $\forall y \in G$, $v \in \mathcal{U}$.

(ii) Moreover, sps that an optimal control u^* exists and ∂G is regular for $Y_t^{u^*}$ (i.e. $Q^y(\tau_D = 0) = 1$, $\forall y \in \partial G$).

Then,
$$\boxed{\begin{aligned} &\sup_{v\in\mathcal{U}}\{f^v(y) + (L^v\Phi)(y)\} = 0, \quad \forall y \in G. \\ &\Phi|_{\partial G} = g \end{aligned}} \qquad f^v(y) = f(y,v) \qquad (2)$$

with the supremum obtained when $v = u^*$: $f(y, u^*(y)) + [L^{u^*(y)}\Phi](y) = 0$, $\forall y \in G$.

Rmk: ① Stochastic control $\Rightarrow$ optimization w/ PDE. (deterministic)

② Need the assumptions (i)(ii); (Example 11.2.6)

③ HJB1: it is necessary that u^* is the maximizer of (2).

HJB2: it is sufficient under some conditions.

$\Rightarrow$ We only need to find u^* & Φ satisfying (2) above.

Thm 11.2.2 (HJB2) Let $\phi \in C^2(G)\cap C(\bar{G})$ s.t.

(i) $\forall v \in \mathcal{U}$, $\begin{cases} f^v(y) + [L^v\phi](y) \le 0, & \forall y \in G. \\ \lim_{t\to\tau_G}\phi(Y_t) = g(Y_{\tau_G})1_{\{\tau_G<\infty\}}, & \text{a.s. } Q^y \end{cases}$

(ii) $\{\phi(Y_\tau) : \tau \text{ stopping time}; \tau \le \tau_G\}$ is uniformly Q^y-integrable $\forall$ Markov controls u and all $y \in G$.

Then, $\phi(y) \ge J^u(y)$, $\forall u$ Markov control, $\forall y \in G$.

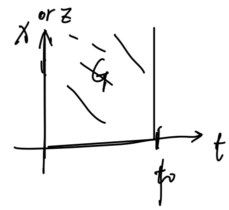
Moreover, if for each $y \in G$, we have $u_0(y)$ s.t. $f^{u_0(y)}(y) + L^{u_0(y)}\phi(y) = 0$

and $\{\phi(Y_\tau^{u_0}) : \tau \text{ stopping time}; \tau \le \tau_G\}$ is uniformly Q^y-integrable, $\forall y \in G$.

Then, $u_0 = u_0(y)$ is a Markov control s.t. $\phi(y) = J^{u_0}(y)$. i.e. an optimal control.

Rmk: an optimal Markov control is also an optimal $\mathcal{F}_t$-adapted control (Thm 11.2.3)

Application to the Portfolio selection problem



- $f=0$; $g=N(z)$; $\quad G=\{(r,z): r<t_0,\ z>0\}$
- The operator L^v; $dZ_t = Z_t[au+b(1-u)]dt + \alpha u Z_t dB_t$

$$L^v\Phi(t,x) = \partial_t\Phi + x(av+b(1-v))\partial_x\Phi + \frac{1}{2}\alpha^2 v^2 x^2 \partial_{xx}\Phi.$$

- The HJB Eq. $\quad \sup\limits_{v\in U} L^v\Phi(t,x) = 0, \quad \forall (t,x)\in G.$

$$\Phi|_{\partial G} = N: \begin{cases} \Phi(t,x) = N(x), & \text{for } t=t_0, \quad x>0 \\ \Phi(t,0) = N(0) & \text{for } t<t_0, \quad x=0 \end{cases}$$

$\Rightarrow \quad \forall (t,x)$, we find the $v=u(t,x)$ that maximizes

$$g(v) = L^v\Phi = \partial_t\Phi + x((a-b)v+b)\partial_x\Phi + \frac{1}{2}\alpha^2 v^2 x^2 \partial_{xx}\Phi, \qquad \text{quadratic in } v \downarrow$$

- If $\partial_x\Phi>0$, $\partial_{xx}\Phi<0$, the soln is: $\quad v = u(t,x) = -\dfrac{(a-b)\partial_x\Phi}{x\alpha^2\partial_{xx}\Phi}$

$$\Rightarrow \quad \begin{cases} \partial_t\Phi + xb\partial_x\Phi - \dfrac{(a-b)^2(\partial_x\Phi)^2}{2\alpha^2\partial_{xx}\Phi} = 0, & t<t_0,\ x>0. \\ \Phi(t,x) = N(x), & \text{for } t=t_0 \text{ or } x=0. \end{cases} \qquad (3)$$

- (3) is a $\boxed{\text{fully nonlinear PDE,}}$ hard to solve in general.
- Special cases:

1). $N(x) = x^\gamma$, $0<\gamma<1$. Separation of variables. $\Phi(t,x) = f(t)x^\gamma$

$$f'(t)x^\gamma + f(t)\left[xb\gamma x^{\gamma-1} - \frac{(a-b)^2\gamma^2 x^{(\gamma-1)2}}{2\alpha^2\gamma(\gamma-1)x^{\gamma-2}}\right] = 0$$

$$[f'(t) + \lambda f(t)]x^\gamma = 0, \qquad \lambda = b\gamma + \frac{(a-b)^2}{2\alpha^2(1-\gamma)}$$

$$\Rightarrow \Phi(t,x) = e^{\lambda(t_0-t)}x^\gamma.$$

$$u^*(t,x) = \frac{a-b}{\alpha^2(1-\gamma)} \qquad (\text{if it is in } (0,1)).$$

2). Similarly, $N(x) = \log x$

<u>Remark</u>: Such simple $N(x)$ can be solved directly by Dynkin's formula (Aase 1984).