Homework assignment, week 5: numerical SDEs

1. Consider the SDE with smooth bounded a, b:

$$dX_t = a(X_t)dt + b(X_t)dW_t \tag{1}$$

Derive the strong order 1.5 Ito-Taylor scheme (see note or Kloeden+Platen Chapter 10). (Hint: you will need to use the multiple integrals and to pay attention to their relations:

$$I_1 = \int_0^h dW_s,\ I_{10} = \int_0^h \int_0^s dW_{s_1} ds,\ I_{11} = \int_0^h \int_0^s dW_{s_1} dWs,$$
$$I_{111} = \int_0^h \int_0^s \int_0^r dW_u dW_r dWs$$

Optional: write a code to test the order of convergence for your IT1.5 scheme.)

2. Consider the Ornstein-Uhlenbeck equation with $\lambda < 0$:

$$dX_t = \lambda X_t dt + \sigma dW_t,$$

(a) Find the range of the time step size δ such that the Euler-Maruyama scheme

$$Y_{n+1} = Y_n + \lambda Y_n \delta + \sigma\sqrt{\delta}\xi_n; \quad Y_0 = 0;\ \text{where } \xi_n \sim \mathcal{N}(0,1)$$

is stable in the sense that $\mathbb{E}\left[Y_n^2\right] < \infty$ for all n and compute $\lim_{n\to\infty} \mathbb{E}\left[Y_n^2\right]$.
(b) Find the range of the time step size δ so that the implicit Euler scheme

$$Y_{n+1} = Y_n + \lambda Y_{n+1} \delta + \sigma\sqrt{\delta}\xi_n; \quad Y_0 = 0;\ \text{where } \xi_n \sim \mathcal{N}(0,1)$$

is stable in the sense that $\mathbb{E}\left[Y_n^2\right] < \infty$ for all n and compute $\lim_{n\to\infty} \mathbb{E}\left[Y_n^2\right]$.