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# Statistics of automorphic representations through simplified trace formulas

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## Note on technical details

This subject has a lot of details that are distractions at the level of a 45 minute talk. Therefore,

- Feel free to ignore anything in gray if you aren't familiar with the subject.
- Anything in orange will be explained only intuitively and imprecisely.

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## Outline

- Background: Automorphic Representations
- Background: Trace Formulas
- Background: Simple Trace Formula
- Results of Shin-Templier
- New Work

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## **Unmotivated Defintion**

#### Definition

Let G be a reductive group over a number field F. A discrete automorphic representation for G is an irreducible subrepresentation of  $L^2(G(F)\setminus G(\mathbb{A}_F),\chi)$ .

- Reductive group: algeberaic group with nice representation theory (root and weight theory works).
  - ex.  $\operatorname{GL}_n, \operatorname{SL}_n, \operatorname{U}_n, \operatorname{SO}_n, \operatorname{Sp}_n$ .
  - Non ex. Upper triangular matrices.
- $L^2$ : square-integrable functions as a unitary representation of  $G(\mathbb{A}_F)$  under right-translation.
- subrepresentation: analysis issue—infinite-dimensional representations can be direct integrals instead of direct sums
- discrete: There is a definition for non-discrete

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#### Motivation

Why do we care about such bizarre objects?

- They have a lot of handles to grab onto when studying
  - representation theory of reductive groups
  - Fourier analysis
- They mysteriously encode information about so much else:
  - Number Theory: Galois representations (Langlands conjectures)
  - Computer Science: expander graphs/higher-dimensional expanders
  - Differential Geometry: spectra of Laplacians on locally symmetric spaces
  - Combinatorics: identities for the partition function
  - Finite Groups: representation theory of large sporadic simple groups (moonshine)
  - Mathematical Physics: Scattering amplitudes in string theory, black hole partition functions

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Example					

If  $G=\mathrm{GL}_2/\mathbb{Q}$ 

{aut. reps. for G}  $\approx$  {new, eigen modular/Maass forms}

- This is NOT obvious
- Key step: If  $K^{\infty}$  is a maximal compact subgroup at the finite places,

 $\mathrm{GL}_2(\mathbb{Q})\mathbb{R}^{\times}\backslash\mathrm{GL}_2(\mathbb{A})/\mathrm{SO}_2(\mathbb{R}){\mathcal K}^{\infty}=\Gamma_{{\mathcal K}^{\infty}}\backslash{\mathcal H}$ 

where  $\Gamma_{\mathcal{K}^\infty}$  is some arithmetic subgroup of  ${\rm SL}_2\mathbb{R}$  and  $\mathcal H$  is upper-half plane

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#### Flath Decomposition

#### Theorem

Let  $\pi$  be an automorphic representation for group G/F. Then  $\pi$  factors over places v of F:

$$\pi = \widehat{\bigotimes}' \pi_{\mathbf{v}}$$

where each  $\pi_v$  is an admissible, unitary representation of  $G(F_v)$ . For  $G = GL_2/\mathbb{Q}$ :

- $\pi_\infty$  is the qualitative "type" of  $\pi$ : modular vs. Maass, weight
- $\pi_p$  relates to the  $p^n$ th Fourier coefficients of  $\pi$ .

Key Question: Which combinations of  $\pi_v$  actually appear in  $L^2$ ?



#### Motivation

First trick to try for decomposing a representation: look at traces.

• Assume for a moment

$$L^{2}(G(F)\setminus G(\mathbb{A}_{F}),\chi) = \bigoplus_{\pi \text{ d.a.}} \pi$$

• Then if R is an operator on  $L^2$ 

$$\operatorname{tr}_{L^2} R = \sum_{\pi \text{ d.a.}} \operatorname{tr}_{\pi} R$$

• Choose R cleverly  $\implies$  information towards key question: distribution of fixed component  $\pi_v$  "in families"

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## Test Functions Example

Idea:  $f = \prod_{\nu} f_{\nu}$  so  $\operatorname{tr}_{\pi}(f) = \prod_{\nu} \operatorname{tr}_{\pi_{\nu}}(f_{\nu})$ .

- Choose one test place v. Everything else is condition places.
- For lots of reasonable conditions on  $\pi_w$ , can find

$${\sf tr}_{\pi_w}(f_w) = egin{cases} 1 & {\sf condition at } w {\sf satisfied} \ 0 & {\sf else} \end{cases}$$

(In general: more complicated weights  $tr_{\pi_w}(f_w) = a_w(\pi_w)$ )

- Set family weight:  $a_{\mathcal{F}}(\pi) = \prod_{w \neq v} a_w(\pi_w)$
- Choose probe function f<sub>v</sub>

$$\sum \operatorname{tr}_{\pi}(f) = \sum a_{\mathcal{F}}(\pi) \operatorname{tr}_{\pi_{\nu}}(f_{\nu})$$

average over harmonic family of local statistic



#### Fantasy

How do we compute these traces?

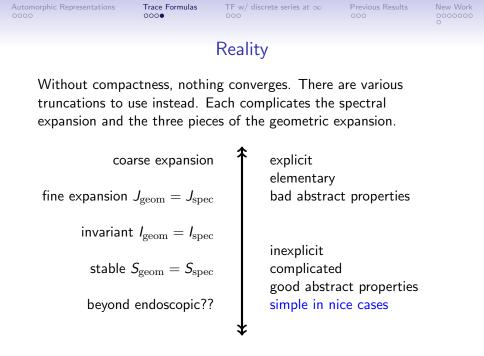
• Convolution operators: f compactly supported smooth on  $G(\mathbb{A}_F)$ :

$$R_f: \mathsf{v} \mapsto \int_{G(\mathbb{A}_F)} f(g) \mathsf{g} \mathsf{v} \, dg$$

• If 
$$G(F) \setminus G(\mathbb{A}_F)$$
 is compact,

$$\operatorname{tr}_{L^{2}} R_{f} = \sum_{[\gamma] \in [G(F)]} \operatorname{Vol}(G(F)_{\gamma} \setminus G(\mathbb{A}_{F})_{\gamma}) \int_{G(\mathbb{A}_{F})_{\gamma} \setminus G(\mathbb{A}_{F})} f(g^{-1}\gamma g) \, dg$$

• conjugacy classes, volume term, orbital integral  $O_{\gamma}(f)$ 





- Restrict attention to the nicest "qualitative type" of automorphic representations ↔ the nicest real representations
- Discrete series: appear discretely in  $L^2(G(F_{\infty}))$ .
- Classified into *L*-packets  $\Pi_{\lambda}$
- $G = \operatorname{GL}_2/\mathbb{Q}$ 
  - L-packets singletons parameterized by  $k \ge 2$ .
  - $\pi_{\infty} \in \Pi_k$  means  $\pi$  a holomorphic modular form of weight k.
- The invariant trace formula dramatically simplifies when restricted to representations with discrete series at infinity.

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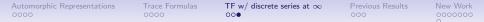
## "Simple" trace formula

#### Theorem ([Art89])

Let G/F be a cuspidal reductive group and let  $\Pi_{\lambda}$  be a regular discrete series L-packet. Let  $\mathcal{A}_{\lambda}$  be the set of automorphic representations  $\pi$  of G with  $\pi_{\infty} \in \Pi_{\lambda}$ . Then for any compactly supported smooth test function f on  $G(\mathbb{A}^{\infty})$ 

$$\sum_{\pi \in \mathcal{A}_{\lambda}} \operatorname{tr}_{\pi^{\infty}} f = \sum_{M \text{ std. Levi}} (-1)^{[G:M]} \frac{|\Omega_{M}|}{|\Omega_{G}|} \sum_{\gamma \in [M(F)]_{ell}} a_{\gamma} \Phi_{M}^{G}(\gamma) O_{\gamma}^{M,\infty}(f_{M})$$

- "Conjugacy classes" counted with principle of inclusion-exclusion
- "Volume term"
- "Orbital integral" factored into infinite and finite places



#### Some Ideas in Proof that May Come Up Later

- Discrete Series π come with pseudocoefficients φ<sub>π</sub>. For ρ a basic representation, tr<sub>ρ</sub>(φ<sub>π</sub>) = 1<sub>π=ρ</sub>
- $\eta_{\lambda}$  Euler-Poincaré function

$$\eta_{\lambda} = rac{1}{|\Pi_{ ext{disc}}(\lambda)|} \sum_{\pi \in \Pi_{\lambda}} arphi_{\pi}$$

- When  $\lambda$  regular, for  $\rho$  any unitary representation:  $\operatorname{tr}_{\rho}(\eta_{\lambda}) = |\Pi_{\operatorname{disc}}(\lambda)|^{-1} \mathbf{1}_{\pi \in \Pi_{\lambda}}$
- Use Euler-Poincaré's as infinite component of test function:  $\eta_{\lambda} f^{\infty}$ , the above computes spectral side
- Much more!

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## Prototypical Result

This allows us to get good enough error bounds on statistics over these families for applications. The shape of the result is:

#### Theorem (prototype)

Fix G. Let  $\mathcal{F}_k$  be a sequence of increasing-size families of aut. reps. of G with regular discrete series at infinity. Then for any unramified test function  $\varphi_v$  at large enough place v:

$$\frac{1}{|\mathcal{F}_k|} \sum_{\pi \in \mathcal{F}_k} \mathsf{a}_{\mathcal{F}_k} \operatorname{tr}_{\pi_v}(\varphi_v) = \mu^{\operatorname{pl}}(\varphi_v) + O(\|\varphi_v\|_{\infty} q_v^{\mathcal{A}+\mathcal{B}_{\mathcal{K}}} |\mathcal{F}_k|^{-\mathcal{C}}).$$

- Families: set of aut. reps weighted by  $a_{\mathcal{F}}$ , total weight is  $|\mathcal{F}|$ .
- $\mu^{\mathrm{pl}}$ : average over space of representations
- $\kappa$ : measure of size of support of unramified  $\varphi$ .

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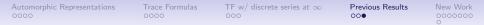
## Which Families?

#### Shin-Templier '16:

•  $\mathcal{F}_k$ : Level condition on  $\pi^\infty$  and  $\pi_\infty$  in a fixed discrete series L-packet

$$a_{\mathcal{F}_k}(\pi) = \mathbf{1}_{\pi_\infty \in \Pi_{\mathrm{disc}}(\lambda)} \mathrm{dim}((\pi^{v,\infty})^{U^{v,\infty}})$$

- $k \to \infty$ : level  $\to \infty$  or if G has trivial center, weight of L-packet  $\to \infty$
- Applications:
  - Automorphic Sato-Tate—equidstribution of unramified  $\pi_v$  over all v.
  - Distributions of low-lying zeros of L-functions in families
- Error bound essential for applications!
- Proof-of-concept that detailed information can be computed



#### Computing terms

Reasonably general reductive groups, but terms somewhat explicit!

$$\sum_{\pi \in \mathcal{A}_{\lambda}} \operatorname{tr}_{\pi^{\infty}} f = \sum_{M \text{ std. Levi}} (-1)^{[G:M]} \frac{|\Omega_{M}|}{|\Omega_{G}|} \sum_{\gamma \in [M(F)]_{ell}} a_{\gamma} \Phi_{M}^{G}(\gamma) O_{\gamma}^{M,\infty}(f_{M})$$

- Sums: reductive group theory, Steinberg-Hitchen base?
- $\Phi$ : Weyl character formula + more root combinatorics
- O<sub>γ</sub>: Counting points moved some amount by automorphisms of Bruhat-Tits buildings
- *f<sub>M</sub>*: The *p*-adic integrals are easier OR branching laws + Kato-Lusztig formula
- $a_{\gamma}$ : *L*-functions of Gross motives for red. groups



We want to compute detailed information about families that distinguish representations with  $\pi_{\infty}$  in the same *L*-packet:

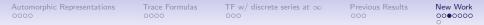
$$a_{\mathcal{F}_{\pi_0}}(\pi) = \mathbf{1}_{\pi_{\infty} = \pi_0} \dim((\pi^{\nu,\infty})^{U^{\nu,\infty}})$$

[Dal19]: Develop the necessary techniques and generalize Shin-Templier's results as a proof-of-concept.



Different members of the *L*-packet are the same for Galois Representation applications so why do we care about distinguishing them?

- Ex.  $G = SL_2$ : one member holomorphic, one antiholomorphic
- Ex.  $G = \operatorname{Sp}_{2n}$ : one member holomorphic
- Ex. G exceptional: one member quaternionic
- When  $\lambda$  non-regular, some members may be "entangled" with non-tempered reps at infinity when trying to pick them out with the trace formula



## Spectral Side

Strategy: plug  $arphi_{\pi_0} f^\infty$  into the trace formula

#### Lemma

If  $\pi_0$  is a regular discrete series representation of  $G_{\infty}$ , then for all unitary representations of  $G_{\infty}$ , tr<sub> $\rho$ </sub>  $\varphi_{\pi_0} = \mathbf{1}_{\rho=\pi_0}$ .

#### Proof.

(Idea) Expand  $\rho$  as a sum of basic representations in the Grothendieck group. All of  $\pi_0$ 's *L*-packet has the same sign.

#### Corollary

$$I_{\mathrm{spec}}(\varphi_{\pi_0}f^{\infty}) = \sum_{\pi \in \mathcal{AR}_{\mathrm{disc}}(\mathcal{G})} m_{\mathrm{disc}}(\pi) a_{\mathcal{F}_{\pi_0}}(\pi) \operatorname{tr}_{\pi_{\mathcal{S}}}(f_{\mathcal{S}})$$



# Geometric Side: Endoscopy and Stabilization Goal:

- Rational conjugacy is too complicated, work with stable conjugacy instead
- $\implies$  want a trace formula with stably-invariant terms: *SO*'s

How?

- G has elliptic endoscopic groups  $H \in \mathcal{E}_{ell}(G)$  if  $G^{der}$  simply connected
  - $(H, s, \eta)$ :  $\widehat{H} = Z_{\widehat{G}}(s), \eta : {}^{L}H \hookrightarrow {}^{L}G$
- f on G has a transfer  $f^H$  on H
  - $\kappa$ -orbital integral identity locally:  $O_{\gamma_G}^{\kappa_H}(f) = SO_{\gamma_H}(f^H)$
- For *S*<sub>\*</sub> stably-invariant:

$$I^{G}_{\star}(f) = \sum_{H \in \mathcal{E}_{\rm ell}(G)} \iota(G, H) S^{H}_{\star}(f^{H})$$

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#### How to compute $I_{\text{geom}}$ ?

- $I_{\text{geom}}(f_{\infty}f^{\infty})$  simplifies if  $f_{\infty}$  linear combination of  $\eta_{\lambda}$ 's.
- Try: write  $I_{
  m geom}(arphi_{\pi_0} f^\infty)$  in terms of  $I_{
  m geom}(\eta_\lambda f^\infty)$ 's

#### Lemma

If  $\pi_0 \in \Pi_{\text{disc}}(\lambda)$ ,  $\varphi_{\pi_0}$  has the same stable orbital integrals as  $\eta_{\lambda}$ . Furthermore, all endoscopic transfers  $(\varphi_{\pi_0})^H$ 's can be taken to be linear combinations of  $\eta_{\lambda}$ 's.

• Therefore stabilization will help!

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# Hyperendscopy Outline

Trick from [Fer07]:

• Rearrange the stabilization of the spectral side

$$S^{G^*}_{\mathrm{disc}}(f^{G^*}) = I^G_{\mathrm{disc}}(f) + \sum_{H \in \mathcal{E}_{\mathrm{ell}}(G)} (-\iota(G,H)) S^H_{\mathrm{disc}}(f^H)$$

- Inductively continue expanding each of the  $S^H_{\rm disc} \to$  a linear combination of  $I^H_{\rm disc}$ 's that is stable
- Forward substitution terminates at tori:  $I_{\star}^{T} = S_{\star}^{T}$
- Set this equal for f, f' with the same stable orbital integrals:

$$I_{\text{disc}}^{\mathcal{G}}(f) = I_{\text{disc}}^{\mathcal{G}}(f') + \sum_{\mathcal{H} \in \mathcal{HE}_{\text{ell}}(\mathcal{G})} \iota(\mathcal{G}, \mathcal{H}) I_{\text{disc}}^{\mathcal{H}}((f' - f)^{\mathcal{H}})$$

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# Hyperendoscopy Application Outline

We use this with  $f = \varphi_{\pi_0} f^\infty$  and  $f' = \eta_\lambda f^\infty$ :

 $I_{\rm disc}^{G}(f) = I_{\rm disc}^{G}(f') + \sum_{\mathcal{H} \in \mathcal{HE}_{\rm ell}(G)} \iota(G, \mathcal{H}) I_{\rm disc}^{\mathcal{H}}((f'-f)^{\mathcal{H}})$ 

- These are linear combinations of  $\eta_{\lambda}$ 's
- I<sub>disc</sub> should therefore be computable by just applying Shin-Templier

Issues:

- Sum: which terms appear depend on  $f^\infty$
- This: needs to be bounded
- $\mathcal{HE}_{ell}$ : Major technical issue coming from precise definition, extend Arthur/Shin-Templier to arbitrary center

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#### Papers Mentioned

- James Arthur, *The L<sup>2</sup>-Lefschetz numbers of Hecke operators*, Invent. Math. **97** (1989), no. 2, 257–290. MR 1001841
- Rahul Dalal, Sato-tate equidistribution for families of automorphic representations through the stable trace formula, arXiv preprint arXiv:1910.10800 (2019).
- Axel Ferrari, Théorème de l'indice et formule des traces, Manuscripta Math. 124 (2007), no. 3, 363–390. MR 2350551
- Sug Woo Shin and Nicolas Templier, Sato-Tate theorem for families and low-lying zeros of automorphic L-functions, Invent. Math. 203 (2016), no. 1, 1–177, Appendix A by Robert Kottwitz, and Appendix B by Raf Cluckers, Julia Gordon and Immanuel Halupczok. MR 3437869
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