SOLUTIONS: FINAL FOR 110 405
FALL 2004

Closed book, no calculators. Fully justify your answer for all six questions.

Question 1. (10 points). Prove that

$$(a/2 + b/2)^{10} \leq a^{10}/2 + b^{10}/2$$

for any real numbers a and b.

You can do this with the binomial theorem, but that is messy. The easy way is to observe that the function $f(x) = x^{10}$ has $f'' \geq 0$ and, hence, sits above any of its secant lines.

Question 2. (10 points). Prove that the function $g(x) = 1/(1+x^2)$ is uniformly continuous on all of **R**.

Notice first that $|g'|$ is bounded:

$$|g'(x)| = 2|x|/(1 + x^2)^2 \leq 1 \, .$$

By the MVT, this bound implies that g is Lipshitz:

$$|g(x) - g(y)| = |x - y| \, |g'(z)| \leq |x - y| \, .$$

This implies uniform continuity (we can even take $\delta = \epsilon$).

Question 3. (20 points). Let $f(x)$ be the function defined for $x > 0$ by

$$f(x) = \int_1^x \frac{dt}{t} \, .$$

(A) Compute the Taylor series for f about the point $x = 1$.
(B) Compute the radius of convergence of this Taylor series.

For (A): Just start differentiating:

- By the FTC, we have $f'(x) = 1/x$.
- Then $f''(x) = -x^{-2}$.
- Iterating this gives $f^{(n)}(x) = (-1)^{n-1} \, (n-1)! \, x^{-n}$.

Plugging this in gives the power series

$$(x - 1) - (x - 1)^2/2 + (x - 1)^3/3 - \dots$$

For (B):
The general term a_n is given by $a_n = (-1)^{n-1}/n$. Using that $n^{1/n} \to 1$ (we proved this in class), we see that

$$1/R = \limsup_{n \to \infty} |a_n|^{1/n} = 1 \, .$$

Hence, the radius of convergence R is one.

Question 4. (20 points). Suppose that f is a continuous function defined on $[0,1]$. Show that for any $\epsilon > 0$, there exists a constant M so that for every x and y we have

$$|f(x) - f(y)| \leq \epsilon + M\,|x - y|\,.$$

Fix some $\epsilon > 0$. We have to find an M that works.

Since f is continuous on the compact set $[0,1]$, we know that

- f is uniformly cts: There exists $\delta > 0$ so that $|f(x) - f(y)| \leq \epsilon$ if $|x - y| \leq \delta$.
- f is bounded: There exists B so that $|f(x) - f(y)| \leq B$ for <u>any</u> x and y in $[0,1]$.

It now follows that for <u>any</u> x and y we have

$$|f(x) - f(y)| \leq \epsilon + (B/\delta)\,|x - y|\,.$$

To see this, use the first dot when $|x - y| \leq \delta$ and use the second dot when $|x - y| > \delta$.

Question 5. (20 points). Suppose that $f : \mathbf{R} \to \mathbf{R}$ is a C^1 function satisfying $f(0) = 0$ and $f'(x) > f(x)$ for every $x \in \mathbf{R}$. Prove that $f(x) > 0$ for every $x > 0$.

We will argue by contradiction. Choose $x > 0$ to be the first zero of f; i.e., set

$$x = \inf\{y > 0 \,|\, f(y) = 0\}\,.$$

This inf exists because the set is bounded below by 0. Moreover, since f is cts, the set of zeros is closed and, hence, the inf is in the set — i.e., $f(x) = 0$.

Note that $x > 0$. To see this, use $f'(0) > 0$ to show that f is strictly increasing at 0.

We have arranged that $f(0) = 0$ and $f(x) = 0$, but f - and hence also f' - is positive on $(0, x)$. This gives a contradiction by the MVT: There must be some y in $(0, x)$ with

$$0 = \frac{f(x) - f(0)}{x - 0} = f'(y) > 0\,.$$

Question 6. (20 points). Let f_j be a sequence of C^1 functions on $[0,1]$. Suppose that $\lim_{j\to\infty} f_j(x) = 0$ for every $x \in [0,1]$ and $|f_j'(x)| \leq 1$ for every $x \in [0,1]$. Prove that

$$\lim_{j\to\infty} \int_0^1 f_j(x)\,dx = 0\,.$$

We proved that we can interchange limits and integrals if the functions converge uniformly and the integral is over a compact set. Therefore, we must show uniform convergence of the f_j's to zero.

Pick a big N. Since $f_j(k/N) \to 0$ for each k, there exists M_k so that

$$|f_j(k/N)| < 1/(2N) \text{ for every } j \geq M_k\,.$$

Set $M = \max M_k$ (this is finite since there are only $N + 1$ of them). For $j \geq M$, we know that $f_j \leq 1/(2N)$ at each of the points k/N. We will use the derivative bound to control

f_j at nearby points. Namely, if x is <u>any</u> point in $[0,1]$, then we can choose a k so that $|x - k/N| \leq 1/(2N)$. The MVT then gives

$$|f_j(x) - f_j(k/N)| = |x - k/N|\,|f_j'(y)| \leq 1/(2N)\,.$$

The triangle inequality then gives

$$|f_j(x)| \leq |f_j(k/N)| + |f_j(x) - f_j(k/N)| =\leq 1/(2N) + 1/(2N) = 1/N\,.$$

Since this holds for any x and for any $j \geq M$, this gives uniform convergence.