FINAL FOR 110 405
FALL 2004

Closed book, no calculators. Fully justify your answer for all six questions.

Question 1. (10 points). Prove that

$$(a/2 + b/2)^{10} \leq a^{10}/2 + b^{10}/2$$

for any real numbers a and b.

Question 2. (10 points). Prove that the function $g(x) = 1/(1+x^2)$ is uniformly continuous on all of $\mathbf{R}$.

Question 3. (20 points). Let $f(x)$ be the function defined for $x > 0$ by

$$f(x) = \int_1^x \frac{dt}{t}\,.$$

(A) Compute the Taylor series for f about the point $x = 1$.
(B) Compute the radius of convergence of this Taylor series.

Question 4. (20 points). Suppose that f is a continuous function defined on $[0,1]$. Show that for any $\epsilon > 0$, there exists a constant M so that for every x and y we have

$$|f(x) - f(y)| \leq \epsilon + M\,|x - y|\,.$$

Question 5. (20 points). Suppose that $f : \mathbf{R} \to \mathbf{R}$ is a C^1 function satisfying $f(0) = 0$ and $f'(x) > f(x)$ for every $x \in \mathbf{R}$. Prove that $f(x) > 0$ for every $x > 0$.

Question 6. (20 points). Let f_j be a sequence of C^1 functions on $[0,1]$. Suppose that $\lim_{j\to\infty} f_j(x) = 0$ for every $x \in [0,1]$ and $|f_j'(x)| \leq 1$ for every $x \in [0,1]$. Prove that

$$\lim_{j\to\infty} \int_0^1 f_j(x)\,dx = 0\,.$$